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# INVESTIGATION OF ORIGIN OF BOUNDARY LAYER ON A PROBLEM FOR A FIRST ORDER ORDINARY LINEAR PERTURBED DIFFERENTIAL EQUATION UNDER GENERAL NON-LOCAL BOUNDARY CONDITION

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**Abstract.** The investigation of the forming of the boundary layer is usually carried out by Euler scheme. The limit value is calculated for the obtained solution by tending small quantity to zero. If this value does not satisfy to the boundary conditions the boundary layer is formed in this point.

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**Key words.** Boundary layer,general non-local boundary condition, perturbed differential equation, Euler scheme.

## 1. INVESTIGATION OF A BOUNDARY LAYER FOR AN EQUATION WITH INHOMOGENEOUS BOUNDARY CONDITIONS

It is known that appearance of a boundary layer under local boundary conditions is investigated by the scheme of the paper [1]. That is, the given problem is solved by the ordinary method (by Euler's scheme). Limit value for the obtained solution is calculated when a small parameter tends to zero. If this limit function does not satisfy any of these boundary conditions, a boundary layer appears at the point where this condition is not satisfied. Here, we consider a problem under non local boundary condition, that is suggested as an open problem in the paper [1].

Consider the following problem:

$$\ell_\varepsilon y_\varepsilon \equiv \varepsilon y_\varepsilon'(x) + y_\varepsilon(x) = 0, \quad x \in (0,1), \tag{1}$$

$$y_\varepsilon(0) + y_\varepsilon(1) = a. \tag{2}$$

At first we solve this problem by the ordinary method. The general solution of the given equation (1) is the function:

$$y_\varepsilon(x) = C\,\mathrm{e}^{-\frac{x}{\varepsilon}}. \tag{3}$$

The constant $C$ is determined from condition (2)

$$C + C\,\mathrm{e}^{-\frac{1}{\varepsilon}} = a, \quad C = \frac{a}{1+\mathrm{e}^{-\frac{1}{\varepsilon}}}.$$

Then for problem (1), (2) from (3) we get the solution:

$$y_\varepsilon(x) = \frac{a\,\mathrm{e}^{-\frac{x}{\varepsilon}}}{1+\mathrm{e}^{-\frac{1}{\varepsilon}}}, \quad x \in (0,1). \tag{4}$$

Here, if we pass to limit as $\varepsilon \to 0$ we get:

$$y_0(x) = 0. \tag{5}$$

Function (5) satisfies the equation obtained from equation (1) as $\varepsilon \to 0$, but does not satisfy boundary condition (2), if $a \neq 0$. So, there appears a boundary layer.

For that we act as in [2]. From the relation

$$\begin{aligned} 0 = (\ell_\varepsilon y_\varepsilon, z_\varepsilon) &= \varepsilon \int_0^1 y'_\varepsilon(x) z_\varepsilon(x) \mathrm{d}x + \int_0^1 y_\varepsilon(x) z_\varepsilon(x) \mathrm{d}x \\ &= \varepsilon\, y_\varepsilon(x) z_\varepsilon(x)|_0^1 - \varepsilon \int_0^1 y_\varepsilon(x) z'_\varepsilon(x) \mathrm{d}x + \int_0^1 y_\varepsilon(x) z_\varepsilon(x) \mathrm{d}x, \end{aligned} \tag{6}$$

the equation adjoint to equation (1) will be of the form:

$$\ell^*_\varepsilon z_\varepsilon \equiv -\varepsilon\, z'_\varepsilon(x) + z_\varepsilon(x). \tag{7}$$

Now, construct the fundamental solution of operator (7)

$$-\varepsilon\, z'_\varepsilon(x) + z_\varepsilon(x) = g(x),$$

$$z'_\varepsilon(x) - \frac{z_\varepsilon(x)}{\varepsilon} = -\frac{g(x)}{\varepsilon},$$

then

$$z_\varepsilon(x) = C\, \mathrm{e}^{\frac{x}{\varepsilon}} - \frac{1}{\varepsilon} \int_{x_0}^x \mathrm{e}^{\frac{x-\eta}{\varepsilon}} g(\eta) \mathrm{d}\eta.$$

Hence, for the fundamental solution we get the expression:

$$z_\varepsilon(x - \eta) = \frac{\mathrm{e}^{-\frac{\eta - x}{\varepsilon}}}{\varepsilon} \theta(\eta - x), \tag{8}$$

with condition $x_0 = 1$.

If in Lagrange's formula (6), instead of $z_\varepsilon(x)$ we write the fundamental solution (8), we get [3]:

$$\begin{aligned} \varepsilon y_\varepsilon(0) z_\varepsilon(-\eta) - \varepsilon y_\varepsilon(1) z_\varepsilon(1 - \eta) &= \int_0^1 y_\varepsilon(x)[-\varepsilon z'_\varepsilon(x - \eta) + z_\varepsilon(x - \eta)] \mathrm{d}x \\ &= \begin{cases} y_\varepsilon(\eta), & \eta \in (0, 1), \\ \frac{1}{2} y_\varepsilon(\eta), & \eta = 0, \quad \eta = 1. \end{cases} \end{aligned} \tag{9}$$

The first expression in the obtained relation (9) gives the function $y_\varepsilon(\eta)$ $\eta \in (0, 1)$, the second one gives necessary conditions. Write these necessary conditions

$$\begin{cases} \frac{1}{2} y_\varepsilon(0) = \varepsilon\, y_\varepsilon(0) z_\varepsilon(0) - \varepsilon\, y_\varepsilon(1) z_\varepsilon(1), \\ \\ \frac{1}{2} y_\varepsilon(1) = \varepsilon\, y_\varepsilon(0) z_\varepsilon(-1) - \varepsilon\, y_\varepsilon(1) z_\varepsilon(0). \end{cases}$$

If in these conditions we take into account the values of the fundamental solution from (8), we get

$$\begin{cases} \frac{1}{2}y_\varepsilon(0) = \varepsilon\, y_\varepsilon(0)\frac{1}{2\varepsilon} - \varepsilon\, y_\varepsilon(1)\cdot 0, \\ \\ \frac{1}{2}y_\varepsilon(1) = \varepsilon\, y_\varepsilon(0)\frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon} - \varepsilon\, y_\varepsilon(1)\cdot \frac{1}{2\varepsilon}. \end{cases}$$

Since the first expression is identity, it gives nothing, but the second one gives the necessary condition:

$$y_\varepsilon(1) = y_\varepsilon(0)\mathrm{e}^{-\frac{1}{\varepsilon}}, \tag{10}$$

Solve this necessary condition together with boundary condition (2)

$$y_\varepsilon(0) + y_\varepsilon(0)\mathrm{e}^{-\frac{1}{\varepsilon}} = a,$$

$$y_\varepsilon(0) = \frac{a}{1+\mathrm{e}^{-\frac{1}{\varepsilon}}} \tag{11}$$

If we write (11) in (10), we get

$$y_\varepsilon(1) = \frac{a}{1+\mathrm{e}^{-\frac{1}{\varepsilon}}}\mathrm{e}^{-\frac{1}{\varepsilon}} \tag{12}$$

So, not solving problem (1), (2), we reduced non-local boundary condition (2) to local conditions (11), (12). Finally, having taken from (8) the solution of problem (1), (2) we should take into account (11) and (12) in the left hand side. But, as this problem has a unique solution, we can use the expression (4), obtained for the solution of problem (1), (2). Since (5) is the limit state of this solution, we must compare this function with limit states (11) and (12)

$$y_0(0) = a, \quad y_0(1) = 0. \tag{13}$$

It is seen from the obtained expressions that function (5) satisfies the second condition from conditions (13), it does not satisfy the first one if $a \neq 0$. So, a boundary layer appears at the point $x = 0$.

Theorem 1. *In the given non-local boundary value problem* (1), (2)*, if* $a \neq 0$*, a boundary layer appears at the point* $x = 0$*.*

Remark 1. By the scheme considered in the paper, we study appearance of a boundary layer for the boundary value problem.

$$\ell_\varepsilon y_\varepsilon \equiv \varepsilon y_\varepsilon'(x) + y_\varepsilon(x) = 0, \;\; x \in (0,1), \tag{14}$$

$$\alpha\, y_\varepsilon(0) + \beta\, y_\varepsilon(1) = a, \tag{15}$$

where $\alpha$, $\beta$ and $a$ are the given real constants.

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