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MULTIPLE SOLUTIONS FOR A NON-HOMOGENEOUS NEUMANN BOUNDARY-VALUE PROBLEM

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Abstract. In this paper we obtain multiple solutions in double weighted Sobolev spaces for a non-homogeneous elliptic semilinear eigenvalue problem on unbounded domain. We use a very recent Ricceri type critical points theorem proved by Kristály, Marzantowicz, Varga in [4].

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1. INTRODUCTION

For $\lambda, \mu > 0$, we consider the following elliptic eigenvalue problem with non-homogeneous boundary condition:

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta u + b(x)u = \lambda f(x,u) \text{ in } \Omega \\ \partial_n u = \mu g(x,u) \text{ on } \Gamma, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ is an unbounded domain with smooth boundary Γ, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, n denotes the unit outward normal on Γ and ∂_n is the outer normal derivative on Γ.

Problems of this type were studied by several authors in the last years. We mention here Kristály [3], Lisei, Horváth, Varga [5], Pflüger [9], Montefusco and Rădulescu [8], Mezei and Varga [7] and others. The problems studied in these papers involve the p-Laplacian and the nonlinear term defined on the boundary of Ω is *subcritical* and either *sublinear* or *superlinear* in the second variable in the origin and at the infinity. Mezei in [6] proved the existence of an open interval of eigenvalues, for which the eigenvalue problem $(P_{\lambda,\mu})$ has two distinct, nontrivial solutions. We remark that in [6] an important assumption is that g is *subcritical* and *sublinear* in the second variable in 0 and at infinity. In present paper we drop the sublinearity conditions for g, we assume only the subcriticality of g and we obtain multiple solutions of the problem $(P_{\lambda,\mu})$. Hence, this paper provides a more general multiplicity result than the earlier ones.

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The main tool used in this proof is a three critical points theorem proved in 2003 by G. Bonanno [1], which is actually a consequence of the three critical points theorem of B. Ricceri [12]. Later, in 2008, B. Ricceri "revisited" this theorem and reached a much more precise conclusion under an additional condition (which is always satisfied in the applications). In 2009, Kristály, Marzantowicz, Varga extended this result of Ricceri to locally Lipschitz functions in [4]. We are going to use this latter result, therefore we recall it.

For every $\tau > 0$, we introduce the following class of functions:
$(\mathcal{G}_\tau) : g \in C^1(\mathbb{R}, \mathbb{R})$ is bounded and $g(t) = t$, for any $t \in [-\tau, \tau]$.

THEOREM 1. [4, Theorem 2.1] *Let $(X, ||\cdot||)$ be a real reflexive Banach space and $\tilde{X}_i (i = 1, 2)$ be two Banach spaces such that the embeddings $X \hookrightarrow \tilde{X}_i$ are compact. Let Λ be a real interval, $h : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function and let $\Phi_i : \tilde{X}_i \to \mathbb{R} (i = 1, 2)$ be two locally Lipschitz functions such that $E_{\lambda,\mu} = h(||\cdot||) + \lambda\Phi_1 + \mu(g \circ \Phi_2)$ restricted to X satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, $\lambda \in \Lambda$, $\mu \in [0, |\lambda| + 1]$ and $g \in \mathcal{G}_\tau$, $\tau \geq 0$. Assume that $h(||\cdot||) + \lambda\Phi_1$ is coercive on X for all $\lambda \in \Lambda$ and that there exists $\rho \in \mathbb{R}$ such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(||x||) + \lambda(\Phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(||x||) + \lambda(\Phi_1(x) + \rho)].$$

Then, there exist a non-empty open set $A \subset \Lambda$ and $r > 0$ with the property that for every $\lambda \in A$ there exists $\mu_0 \in]0, |\lambda| + 1[$ such that, for each $\mu \in [0, \mu_0]$ the functional $\mathcal{E}_{\lambda,\mu} = h(||\cdot||) + \lambda\Phi_1 + \mu\Phi_2$ has at least three critical points in X whose norms are less than r.

In the last section of this paper we use a particular case of Theorem 1, when Φ_i, $(i = 1, 2)$ are functions of class C^1.

2. MAIN RESULT

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be an unbounded domain with smooth boundary Γ. For the positive measurable functions u and w, both defined in Ω, we define the weighted p-norm $(1 \leq p < \infty)$ as

$$||u||_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) \mathrm{d}x \right)^{\frac{1}{p}},$$

and denote by $L^p(\Omega; w)$ the space of all measurable functions u such that $||u||_{p,\Omega,w}$ is finite.

The double weighted Sobolev space $W^{1,p}(\Omega; v_0, v_1)$ is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0(x) \mathrm{d}x \right)^{\frac{1}{p}}.$$

We are choosing our weight functions from the so-called Muckenhoupt class A_p, which is defined as the set of all positive functions v in $\mathbb{R}^N$ satisfying

$$\frac{1}{|Q|}\left(\int_\Omega v\ \mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega v^{-\frac{1}{p-1}}\ \mathrm{d}x\right)^{\frac{p-1}{p}} \le \bar{C},\ \text{if } 1<p<\infty$$

$$\frac{1}{|Q|}\int_\Omega v\ \mathrm{d}x \le \bar{C}\ \operatorname{ess}\inf_{x\in Q} v(x),\ \text{if } p=1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this paper we always assume that the weight functions v_0, v_1, w are defined on Ω, belong to A_p and are chosen such that the embeddings

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w), \tag{1}$$

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w) \tag{2}$$

are compact for $p \in]2, 2^*[$, $q \in]2, \bar{2}^*[$ and continuous for $p \in [2, 2^*]$, $q \in [2, \bar{2}^*]$ respectively, where $2^* = \dfrac{2N}{N-2}$ and $\bar{2}^* = \dfrac{2(N-1)}{N-2}$ are the critical exponents. Such weight functions there exist, see e.g. [10], [11].

Therefore, there exist the best embedding constants denoted by $C_{p,\Omega}, C_{q,\Gamma}$ such that:

$$||u||_{p,\Omega,w} \le C_{p,\Omega}||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1), \tag{3}$$

$$||u||_{q,\Gamma,w} \le C_{q,\Gamma}||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1), \tag{4}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.

First, we define an operator $A : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ by $A(u) = -\Delta u + b(x)u$ for a positive measurable function b, then a continuous bilinear form associated with this operator as

$$\langle u, v\rangle_A = \int_\Omega (\nabla u \nabla v + b(x)uv)\mathrm{d}x \tag{5}$$

and the corresponding norm with

$$||u||_A^2 = \langle u, u\rangle_A = \int_\Omega (|\nabla u(x)|^2 + b(x)|u(x)|^2)\mathrm{d}x. \tag{6}$$

Now, we define the Banach space

$$X_A = \{u \in W^{1,2}(\Omega; v_0, v_1) : ||u||_A < \infty\}, \tag{7}$$

endowed with the norm $||\cdot||_A$.

We say that $u \in X_A$ is a *weak solution* of the problem $(P_{\lambda,\mu})$, if

$$\langle u, v\rangle_A - \lambda\int_\Omega f(x, u(x))v(x)\mathrm{d}x - \mu\int_\Gamma g(x, u(x))v(x)\mathrm{d}\Gamma = 0,\ \text{for every } v \in X_A.$$

The relation between the spaces $W^{1,2}(\Omega; v_0, v_1)$ and X_A is given by the ellipticity condition

$$(A) \qquad ||u||_A^2 \geq 2K||u||_{v_0,v_1}^2 \text{ for every } u \in X_A,$$

with some positive constant $K > 0$;

Furthermore we consider the following assumptions on f, g:

(F1) $f(\cdot, 0) = 0$ and $|f(x,s)| \leq f_0(x) + f_1(x)|s|^{p-1}$ for $x \in \Omega$, $s \in \mathbb{R}$, where $p \in]2, 2^*[$ and f_0, f_1 are positive measurable functions satisfying $f_0 \in L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $f_0(x) \leq C_f w(x)$ and $f_1(x) \leq C_f w(x)$ for a.e. $x \in \Omega$, with an appropiate constant C_f;

(F2) $\lim\limits_{s\to 0} \dfrac{f(x,s)}{f_0(x)|s|} = 0$, uniformly in $x \in \Omega$;

(F3) $\limsup\limits_{s\to\infty} \dfrac{F(x,s)}{f_0(x)|s|^2} \leq 0$ uniformly in $x \in \Omega$ and $\max\limits_{|s|\leq M} F(\cdot, s) \in L^1(\Omega)$, for all $M > 0$, where $F(x,u) = \displaystyle\int_0^u f(x,s)\mathrm{d}s$;

(F4) there exist $x_0 \in \Omega$, $s_0 \in \mathbb{R}$ and $R_0 > 0$ such that $\min\limits_{|x-x_0|<R} F(x, s_0) > 0$.

(G) $g(\cdot, 0) = 0$ and $|g(x,s)| \leq g_0(x) + g_1(x)|s|^{q-1}$, for $x \in \Gamma$, $s \in \mathbb{R}$, where $q \in]2, \bar{2}^*[$ and g_0, g_1 are positive measurable functions satisfying $g_0 \in L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$, $g_0(x) \leq C_g w(x)$ and $g_1(x) \leq C_g w(x)$, a.e. $x \in \Gamma$, with an appropiate constant C_g.

The main result of this paper is the following

Theorem 2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the conditions* (F1)-(F4) *and* (G). *Then there exists a nondegenerate interval $[a,b] \subset]0, +\infty[$ and a number $r > 0$, such that for every $\lambda \in [a,b]$ there exists $\mu_0 \in]0, \lambda + 1[$ such that for each $\mu \in [0, \mu_0]$, the problem $(P_{\lambda,\mu})$ has at least three distinct solutions with X_A-norms less than r.*

3. AUXILIARY RESULTS AND PROOF OF THEOREM 2

In this section first we present some auxiliary results. These properties guarantee that all the assumptions of Theorem 1 are satisfied, so we can apply it obtaining our main result.

First, we define the functionals $J_F, J_G : X_A \to \mathbb{R}$ by

$$J_F(u) = \int_\Omega F(x, u(x))\mathrm{d}x, \qquad J_G(u) = \int_\Gamma G(x, u(x))\mathrm{d}\Gamma,$$

where $G(x,u) = \displaystyle\int_0^u g(x,s)\mathrm{d}\Gamma$, then the energy functional $\mathcal{E}_{\lambda,\mu} : X_A \to \mathbb{R}$ associated to $(P_{\lambda,\mu})$ by $\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2}||u||_A^2 - \lambda J_F(u) - \mu J_G(u)$.

In the next result we use the Nemytskii operator of a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, defined by $N_h(u) = h(x, u(x))$.

Lemma 1. [6, Lemma 2.1] *Assume that the conditions* (F1), (G) *are satisfied. Then, the Nemytskii operators* $N_f : L^p(\Omega; w) \to L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $N_F : L^p(\Omega; w) \to L^1(\Omega)$, $N_g : L^q(\Gamma; w) \to L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$ *and* $N_G : L^q(\Gamma; w) \to L^1(\Gamma)$ *are bounded and continuous.*

Lemma 2. [10, Lemma 8] *The energy functional* $\mathcal{E}_{\lambda,\mu} : X_A \to \mathbb{R}$ *is Fréchet differentiable and its derivative is given by*

$$\langle \mathcal{E}'_{\lambda,\mu}(u), v\rangle = \langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)\mathrm{d}x - \mu \int_\Gamma g(x, u(x))v(x)\mathrm{d}\Gamma. \tag{8}$$

for every $v \in X_A$.

Due to this result, one can see that the critical points of $\mathcal{E}_{\lambda,\mu}$ are exactly the weak solutions of $(P_{\lambda,\mu})$. Therefore, instead of looking for weak solutions of problem $(P_{\lambda,\mu})$, we are seeking for the critical points of $\mathcal{E}_{\lambda,\mu}$.

Lemma 3. *Assume that the conditions* (F1)-(F3) *are satisfied. Then, for every* $\lambda > 0$ *the functional* $\alpha : X_A \to \mathbb{R}$ *defined by* $\alpha(u) = \frac{1}{2}||u||_A^2 - \lambda J_F(u)$ *is coercive.*

Proof. Let us fix $\lambda > 0$ arbitrarily and $\eta > 0$ such that

$$\frac{K}{\lambda C_f C_{2,\Omega}^2} > \eta.$$

By the conditions (F2),(F3) there exist a positive function $k \in L^1(\Omega; w)$ such that

$$|F(x, s)| \le \eta f_0(x)|s|^2 + k(x)w(x), \quad \forall (x, s) \in \Omega \times \mathbb{R}. \tag{9}$$

Thus, using the relation (9), the embedding (1) and the (A) ellipticity conditions, for every $u \in X_A$ we obtain:

$$\begin{aligned}
\alpha(u) &\ge \frac{1}{2}||u||_A^2 - \lambda \int_\Omega \eta f_0(x)|u(x)|^2\mathrm{d}x - \lambda \int_\Omega k(x)w(x)\mathrm{d}x \\
&\ge \frac{1}{2}||u||_A^2 - \lambda\eta C_f||u||_{2,\Omega,w}^2 - \lambda||k||_{1,\Omega,w} \\
&\ge \frac{1}{2}||u||_A^2 - \lambda\eta C_f C_{2,\Omega,w}^2||u||_{v_0,v_1}^2 - \lambda||k||_{1,\Omega,w} \\
&\ge \frac{1}{2}\left(1 - \lambda\eta C_f C_{2,\Omega}^2 \frac{1}{K}\right)||u||_A^2 - \lambda||k||_{1,\Omega,w}.
\end{aligned}$$

Since $k \in L^1(\Omega; w)$, we have that $||k||_{1,\Omega,w}$is finite. Therefore, by the choice of η it follows, that $\mathcal{E}_{\lambda,\mu}(u) \to \infty$ as $||u||_A \to \infty$. Hence $\mathcal{E}_{\lambda,\mu}$ is coercive. □

Lemma 4. *Let the conditions* (F1) *and* (F2) *be satisfied. Then*

$$\lim_{t\to 0^+} \frac{\sup\{J_F(u) : u \in X_A, ||u||_A^2 < 2t\}}{t} = 0.$$

Proof. From the assumptions (F1), (F2) it results the existence of $\hat{c}(\varepsilon) > 0$, such that, for every $\hat{\varepsilon} > 0$ we have:

$$|f(x,s)| \leq \hat{\varepsilon} f_0(x)|s| + \hat{c}(\varepsilon) f_1(x)|s|^{p-1}, \text{ for } p \in]2, 2^*[. \tag{10}$$

Then integrating with respect to the second variable, from 0 to $u(x)$, we get the existence of $c(\varepsilon) > 0$, such that, for every $\varepsilon > 0$ we have:

$$|F(x,u(x))| \leq \varepsilon f_0(x)|u(x)|^2 + c(\varepsilon) f_1(x)|u(x)|^p, \text{ for } p \in]2, 2^*[. \tag{11}$$

Now, fix $\varepsilon > 0$ and $p \in]2, 2^*[$ arbitrarily. Then from (11) and the ellipticity condition (A), it follows that:

$$\begin{aligned} J_F(u) &\leq \varepsilon C_f C_{2,\Omega}^2 ||u||_{v_0,v_1}^2 + c(\varepsilon) C_f C_{p,\Omega}^p ||u||_{v_0,v_1}^p \\ &\leq \varepsilon C_f C_{2,\Omega}^2 \frac{||u||_A^2}{2K} + c(\varepsilon) C_f C_{p,\Omega}^p \left(\frac{||u||_A^2}{2K} \right)^{\frac{p}{2}}. \end{aligned}$$

Therefore, we have:

$$\sup\{J_\mu(u) : \frac{||u||_A^2}{2} < \rho\} \leq \varepsilon \frac{C_f C_{2,\Omega}^2}{K} \rho + \frac{c(\varepsilon) C_f C_{p,\Omega}^p}{K^{\frac{p}{2}}} \rho^{\frac{p}{2}}.$$

Since $p > 2$ and ε is chosen arbitrarily, by dividing this last inequality with ρ and taking the limit whenever $\rho \to 0$, we get the required equality. □

The next lemma can be proved arguing as in [7, Lemma 3.2].

Lemma 5. *Assume that* (F4) *is satisfied. Then there exists a function* $u_0 \in X_A$ *such that* $J_F(u_0) > 0$.

The result of Lemma 5 is deeply employed in the next lemma.

Lemma 6. *There exists* $\rho_0 \in \mathbb{R}$ *such that*

$$\sup_{\lambda > 0} \inf_{u \in X_A} \left(\frac{1}{2} ||u||_A^2 - \lambda (J_F(u) - \rho_0) \right) < \inf_{u \in X_A} \sup_{\lambda > 0} \left(\frac{1}{2} ||u||_A^2 - \lambda (J_F(u) - \rho_0) \right).$$

Proof. Let us define the function $\beta :]0, \infty) \to \mathbb{R}$ by

$$\beta(t) = \sup\{J_F(u) : u \in X_A, ||u||_A^2 < 2t\}.$$

Then, from the assumption (F2), we have that $\beta(t) \geq 0$, for every $t > 0$ and Lemma 4 yields that

$$\lim_{t \searrow 0} \frac{\beta(t)}{t} = 0. \tag{12}$$

We consider the function $u_0 \in X_A$ provided from Lemma 5, i.e. $J_F(u_0) > 0$. Therefore we can choose a number $\gamma > 0$ such that

$$0 < \gamma < J_F(u_0) \frac{2}{||u_0||_A^2}. \tag{13}$$

By (12) we get the existence of a number $t_0 \in (0, \frac{||u_0||_A^2}{2})$ such that $\beta(t_0) < \gamma t_0$. Thus by (13) we have:

$$\beta(t_0) < J_F(u_0)\frac{2}{||u_0||_A^2}t_0. \tag{14}$$

Then, we can find a number $\rho_0 > 0$ such that

$$\beta(t_0) < \rho_0 < J_F(u_0)\frac{2}{||u_0||_A^2}t_0. \tag{15}$$

Hence, by the choice of t_0 we have:

$$\beta(t_0) < \rho_0 < J_F(u_0). \tag{16}$$

Now, we define the function $\varphi : X_A \times [0, \infty[\to \mathbb{R}$ by

$$\varphi(u, \lambda) = \frac{||u||_A^2}{2} + \lambda(\rho_0 - J_F(u))$$

and we claim that

$$\sup_{\lambda>0} \inf_{u\in X_A} \varphi(u, \lambda) < \inf_{u\in X_A} \sup_{\lambda>0} \varphi(u, \lambda). \tag{17}$$

The function $[0, \infty[\ni \lambda \mapsto \inf_{u\in X_A} \left(\frac{||u||_A^2}{2} + \lambda(\rho_0 - J_F(u))\right)$ is upper semi-continuous on $[0, \infty[$. By the choice of ρ_0 in (16) it follows that:

$$\lim_{\lambda\to\infty} \inf_{u\in X_A} \varphi(u, \lambda) \le \lim_{\lambda\to\infty} \left(\frac{||u_0||_A^2}{2} + \lambda(\rho_0 - J_F(u_0))\right) = -\infty.$$

Therefore we can choose a number $\bar{\lambda} \in [0, \infty[$ such that

$$\sup_{\lambda>0} \inf_{u\in X_A} \varphi(u, \lambda) = \inf_{u\in X_A} \left(\frac{||u||_A^2}{2} + \bar{\lambda}(\rho_0 - J_F(u))\right). \tag{18}$$

From the definition of β we have that $J_F(u) \le \beta(t_0)$, for all $u \in X_A$ with $||u||_A^2 \le 2t_0$. Then by the choice of ρ_0, it follows that $\rho_0 > J_F(u)$, for every $u \in X_A$, with $||u||_A^2 \le 2t_0$. Hence $t_0 < \frac{||u||_A^2}{2}$, for $u \in X_A, \rho_0 \le J_F(u)$, therefore:

$$t_0 \le \inf\left\{\frac{||u||_A^2}{2} : u \in X_A, \rho_0 \le J_F(u)\right\}. \tag{19}$$

On the other hand,

$$\begin{aligned}\inf_{u\in X_A} \sup_{\lambda\in[0,\infty[} \varphi(u, \lambda) &= \inf_{u\in X_A} \left\{\frac{\|u\|_A^2}{2} + \sup_{\lambda\in[0,\infty[} \{\lambda(\rho_0 - J_F(u))\}\right\} \\ &= \inf_{u\in X_A} \left\{\frac{\|u\|_A^2}{2} : \rho_0 \le J_F(u)\right\}.\end{aligned}$$

Therefore, inequality (19) is equivalent to

$$t_0 \le \inf_{u\in X_A} \sup_{\lambda\in[0,\infty[} \varphi(u, \lambda). \tag{20}$$

Now, we consider two cases. First, when $0 \leq \bar{\lambda} < \frac{t_0}{\rho_0}$, then we have:

$$\inf_{u \in X_A} \left\{ \frac{||u||_A^2}{2} + \bar{\lambda}(\rho_0 - J_F(u)) \right\} = \inf_{u \in X_A} \varphi(u, \bar{\lambda}) \leq \varphi(0, \bar{\lambda}) = \bar{\lambda}\rho_0 < t_0.$$

Combining this inequality with (18) and (20) the claim follows.
Now, if $\bar{\lambda} \geq \frac{t_0}{\rho_0}$, applying the inequality (15) we have:

$$\inf_{u \in X_A} \left\{ \frac{||u||_A^2}{2} + \bar{\lambda}(\rho_0 - J_F(u)) \right\} \leq \frac{||u_0||_A^2}{2} + \bar{\lambda}(\rho_0 + J_F(u_0))$$
$$\leq \frac{||u_0||_A^2}{2} + \frac{t_0}{\rho_0}(\rho_0 - J_F(u_0)) = t_0 + \left(\frac{||u_0||_A^2}{2} - \frac{t_0}{\rho_0} J_F(u_0) \right) < t_0.$$

Using the relations (18) and (20), we obtain (20), which completes the proof. □

In the sequel, for a $\tau \geq 0$, let $h : \mathbb{R} \to \mathbb{R}$ be a bounded C^1 function from $\mathcal{G}_\tau$ and for $\lambda, \mu > 0$ let $E_{\lambda,\mu} : X_A \to \mathbb{R}$ be the functional defined by

$$E_{\lambda,\mu} = \frac{1}{2}||u||_A^2 - \lambda J_F(u) - \mu(h \circ J_G)(u), u \in X_A.$$

Lemma 7. *The functional $E_{\lambda,\mu}$ satisfies the Palais-Smale condition for every $\lambda \geq 0$ and $\mu \in [0, \lambda + 1]$.*

Proof. Let $\{u_n\} \subset X_A$ be an arbitrary Palais-Smale sequence for $E_{\lambda,\mu}$, i.e.
(a) $\{E_{\lambda,\mu}(u_n)\}$ is bounded;
(b) $E'_{\lambda,\mu}(u_n) \to 0$.
We have to prove that $\{u_n\}$ contains a strongly convergent subsequence in X_A.
By Lemma 3, we have that $\alpha(u) = \frac{1}{2}||u||_A^2 - \lambda J_F(u)$ is coercive. Then by the choice of h, we have that $E_{\lambda,\mu}$ is coercive as well. Therefore the sequence $\{u_n\}$ is bounded. X_A is a reflexive Banach space, so taking a subsequence if necessary (denoted in the same way), we get an element $u \in X_A$ such that $u_n \to u$ weakly in X_A.
Because the embeddings (1) and (2) are compact for $p \in]2, 2^*[$ and $q \in]2, \bar{2}^*[$, we have that $u_n \to u$ strongly in $L^p(\Omega; w)$ and $L^q(\Gamma; w)$, i.e.

$$(21) \qquad ||u_n - u||_{p,\Omega,w} \to 0 \text{ and } ||u_n - u||_{q,\Gamma,w} \to 0, \text{ whenever } n \to \infty.$$

From the condition (b) we have that $\left| \langle E'_{\lambda,\mu}(u_n), \frac{u_n}{||u_n||_A} \rangle \right| \leq \varepsilon$, for every $\varepsilon > 0$ and large $n \in \mathbb{N}$. Then

$$\langle u_n, u_n \rangle_A - \lambda \int_\Omega f(x, u_n(x)) u_n(x) \mathrm{d}x - \mu h'(J_G(u_n)) \int_\Gamma g(x, u_n(x)) u_n(x) \mathrm{d}\Gamma$$
$$\leq \varepsilon ||u_n||_A.$$

Rearranging this inequality and taking $u_n - u$ instead of u_n, we obtain:

$$
\begin{aligned}
\langle u_n - u, u_n - u\rangle_A &\le |\langle u_n, u_n - u\rangle_A| + |\langle u, u_n - u\rangle_A| \\
&\le 2\varepsilon ||u_n - u||_A + \lambda \left| \int_\Omega f(x, u_n(x))(u_n(x) - u(x))\mathrm{d}x \right| \\
&+ \lambda \left| \int_\Omega f(x, u(x))(u_n(x) - u(x))\mathrm{d}x \right| \\
&+ \mu \left| h'(J_G(u_n)) \int_\Gamma g(x, u_n(x))(u_n(x) - u(x))\mathrm{d}\Gamma \right| \\
&+ \mu \left| h'(J_G(u_n)) \int_\Gamma g(x, u(x))(u_n(x) - u(x))\mathrm{d}\Gamma \right| .
\end{aligned} \tag{22}
$$

Using Hölder's inequality we get:

$$
\begin{aligned}
&\left| \int_\Omega f(x, u_n(x))(u_n(x) - u(x))\mathrm{d}x \right| \\
&\le \int_\Omega \left| f(x, u_n(x)) w(x)^{-\frac{1}{p}} \right| \left| (u_n(x) - u(x)) w(x)^{\frac{1}{p}} \right| \mathrm{d}x \\
&\le \left(\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{-\frac{p'}{p}} \mathrm{d}x \right)^{\frac{1}{p'}} \left(\int_\Omega |u_n(x) - u(x)|^p w(x) \mathrm{d}x \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} \mathrm{d}x \right)^{\frac{1}{p'}} ||u_n - u||_{p,\Omega,w}
\end{aligned}
$$

and arguing in the same way for g, we obtain:

$$
\begin{aligned}
&\left| \int_\Gamma g(x, u_n(x))(u_n(x) - u(x))\mathrm{d}x \right| \\
&\le \left(\int_\Gamma |g(x, u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} \mathrm{d}\Gamma \right)^{\frac{1}{q'}} ||u_n - u||_{q,\Gamma,w}.
\end{aligned}
$$

Since $\{u_n\}$ is bounded, by Lemma 1 it follows the existence of the constants $M_f, M_g > 0$ such that

$$
\begin{aligned}
&\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} \mathrm{d}x \le M_f, \int_\Omega |f(x, u(x))|^{p'} w(x)^{\frac{1}{1-p}} \mathrm{d}x \le M_f, \\
&\int_\Gamma |g(x, u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} \mathrm{d}\Gamma \le M_g, \int_\Gamma |g(x, u(x))|^{q'} w(x)^{\frac{1}{1-q}} \mathrm{d}\Gamma \le M_g,
\end{aligned}
$$

and by the choice of the functional h, we obtain another positive constant M_h such that $|h'(J_G(u_n))| \le M_h$.

Therefore the inequality (22) becomes:

$$
||u_n - u||_A^2 \le 2\varepsilon ||u_n - u||_A + 2\lambda M_f ||u_n - u||_{p,\Omega,w} + 2\mu M_g M_h ||u_n - u||_{q,\Gamma,w}.
$$

By (21) we have that $||u_n - u||_{p,\Omega,w}$ and $||u_n - u||_{q,\Gamma,w}$ tend to zero and since $\varepsilon > 0$ is arbitrary, it follows that u_n converges strongly to u in X_A, whenever $n \to \infty$. □

Proof of Theorem 2. We choose $X = X_A$, $||\cdot|| = ||\cdot||_A$, $\tilde{X}_1 = L^p(\Omega; w)$, with $p \in]2, 2^*[$, $\tilde{X}_2 = L^q(\Gamma; w)$, with $q \in]2, \bar{2}^*[$,$\Lambda = [0, \infty[$ and $h(t) = t^2/2$, $t \geq 0$. Using the Lemmas from this section, all the assumptions of Theorem 1 are satisfied, so we can apply it, achieving the claimed result. □

REFERENCES

[1] Bonanno, G., *Some remarks on a three critical points theorem*, Nonlinear Anal., **54** (2003), 651–665.

[2] Brézis, H., *Analyse fonctionelle. Théorie et applications*, Masson, Paris, 1983.

[3] Kristály, A. and Varga, Cs., *On a class of quasilinear eigenvalue problems in* $\mathbb{R}^N$, Math. Nachr., **278** (2005), 1756–1765.

[4] Kristály, A., Marzantowicz, W. and Varga, Cs., *A non-smooth three critical points theorem with applications in differential inclusions*, J. Global Optim., **46** (2010), 49–62.

[5] Lisei, H., Horváth, A. and Varga, Cs., *Multiplicity results for a class of quasilinear eigenvalue problems on unbounded domain*, Arch. Math. (Basel), **90** (2008), 256–266.

[6] Mezei, I.I., *Multiple solutions for a double eigenvalue semilinear problem in double weighted Sobolev spaces*, Studia Univ. Babeş-Bolyai Math., **53** (2008), 33–48.

[7] Mezei, I.I. and Varga, Cs. *Multiplicity result for a double eigenvalue quasilinear problem on unbounded domain*, Nonlinear Anal., **69** (2008), 4099–4105.

[8] Montefusco, E. and Rădulescu, V., *Nonlinear eigenvalue problems for quasilinear operators on unbounded domains*, Nonlinear Differ. Equ. Appl., **8** (2001), 481–497.

[9] Pflüger, K., *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition*, Electron. J. Differ. Equ., **1998** (1998), 1–13.

[10] Pflüger, P., *Compact traces in weighted Sobolev space*, Analysis (Munich), **18** (1998), 65–83.

[11] Pflüger, K., *Semilinear Elliptic Problems in Unbounded Domains: Solutions in weighted Sobolev Spaces*, Institüt for Mathematik I, Freie Universität Berlin, Preprint no. 21, 1995.

[12] Ricceri, B., *On a three critical points theorem*, Arch. Math. (Basel), **75** (2000), 220–226.

[13] Ricceri, B., *A three critical points theorem revisited*, Nonlinear Anal., **70**(2008), 3084–3089.

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