MATHEMATICA, Tome 53 (76), N° 1, 2011, pp. 65–72

INTERVAL-VALUED (α, β)-FUZZY SUBGROUPS II

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Abstract. In continuation of [13] we provide characterizations of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup. We show that a proper interval-valued $(\in, \in)$-fuzzy subgroup $\hat{\mu}_F$ of group G such that $\#\ \{\text{Im}(\hat{\mu}_F)\} \geq 3$ can be expressed as the union of two proper non-equivalent interval-valued $(\in, \in)$- fuzzy subgroup of group G. Finally, we also prove that if $\hat{\mu}_F$ is a proper interval-valued $(\in, \in \vee q)$-fuzzy subgroup of group G such that $\#\ \{\hat{\mu}_F(x) | \hat{\mu}_F(x) < [0.5, 0.5]\} \geq 2$, then there exist two proper non-equivalent interval-valued $(\in, \in \vee q)$-fuzzy subgroup of group G such that $\hat{\mu}_F$ can be expressed as the union of them.

MSC 2010. 20K27, 04A72, 08A72, 20N25.

Key words. Interval-valued $(\in, \in \vee q)$-fuzzy subgroup, interval-valued $(\in, \in)$-fuzzy subgroup, interval-valued $(\in, \in \vee q)$-fuzzy subgroup, interval-valued (α, β)-fuzzy subgroup.

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [22]. Since then, this idea has been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies; for a detailed study see [1, 2, 3, 4, 7, 14, 16, 17, 19]. Later on, Zadeh [23] also introduced the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). The interval-valued fuzzy subgroups were first defined and studied by Biswas [6], and they are subgroups of the same nature as the fuzzy subgroups defined by Rosenfeld. Zeng et al. [24] gave a kind of method to describe the entropy of interval-valued fuzzy set based on its similarity measure, and discussed their relationship between the similarity measure and the entropy of the interval-valued fuzzy sets in detail. However, the obtained results can still be applied in many fields such as pattern recognition, image processing and fuzzy reasoning etc. The definition of a fuzzy subgroup with thresholds, which is a generalization of Rosenfeld's fuzzy subgroup and Bhakat and Das's fuzzy subgroup was introduced by Xuehai Yuan et al. [21]. Saeid [20] introduced the notion of interval-valued fuzzy BG-algebras and determined the relationship between these notions and BG-subalgebras. A new type of fuzzy subgroup (that is, the $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [5] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which

The authors are extremely grateful to the learned referees for their valuable comments and suggestions which helped us a lot for improving the standard of this paper.

was introduced by Pu and Liu [18]. In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. With this objective in view, Davvaz [8] applied this theory to near-rings and obtained some useful results. The notion of interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras was discussed by Jianming Zhan et al. [25]. Later Xueling Ma et al. [15] introduced the notion of interval-valued $(\in, \in \vee q)$-fuzzy implicative ideals of pseudo-MV algebras is considered and some characterization theorems of these generalized fuzzy implicative ideals are discussed. Using the "belongs to" relation $(\in)$ and quasi-coincidence with the relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subalgebra on BCK-algebra, where (α, β) are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced, and related properties were investigated in [11]. As a continuation of paper [11], relations between a fuzzy subalgebra with thresholds and an $(\in, \in \vee q)$-fuzzy subalgebra were discussed in [9, 10]. The notion of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup, interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds by using their level subgroups, T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup, the direct product and T_G-product of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup and T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds by using their level subgroups was introduced and some of their related properties were investigated in [12].

In this paper, using the "belongs to" relation $(\in)$ and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets, a new concept of interval-valued (α, β)-fuzzy subgroup is considered, where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and related properties are investigated. We provide characterizations of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup. We show that a proper interval-valued $(\in, \in)$- fuzzy subgroup $\hat{\mu}_F$ of group G such that $\#\ \{\mathrm{Im}(\hat{\mu}_F)\} \geq 3$ can be expressed as the union of two proper non-equivalent interval-valued $(\in, \in)$-fuzzy subgroup of group G. Finally, we also prove that if $\hat{\mu}_F$ is a proper interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G such that $\#\ \{\hat{\mu}_F(x)/\hat{\mu}_F(x) < [0.5, 0.5]\} \geq 2$, then there exist two proper non-equivalent interval-valued $(\in, \in \vee q)$-fuzzy subgroup of group G such that $\hat{\mu}_F$ can be expressed as the union of them.

2. PRELIMINARIES

In this section we recall some basic definitions for the sake of completeness.

Definition 1 ([19]). Let μ be a fuzzy set in a group G. Then μ is called a *fuzzy subgroup* of G if

$(FS1)$ $(\forall x, y \in G)$ $(\mu(xy) \geq \min\{\mu(x), \mu(y)\})$,

$(FS2)$ $(\forall x \in G)$ $(\mu(x^{-1}) \geq \mu(x))$.

An interval-valued fuzzy set F defined on G is given by the set $F = \{(x, [\mu_F^L(x), \mu_F^U(x)])\}$ for all $x \in G$. Briefly denote $F = [\mu_F^L, \mu_F^U]$, where μ_F^L and μ_F^U are any two fuzzy sets in G such that $\mu_F^L(x) \leq \mu_F^U(x)$ for all $x \in G$. For simplicity $\hat{\mu}_F(x) = [\mu_F^L(x), \mu_F^U(x)]$, for all $x \in G$.

Let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$. It is clear that if $\mu_F^L(x) = \mu_F^U(x) = c$ where $0 \leq c \leq 1$ then $\hat{\mu}_F(x) = [c,c]$ is in $D[0,1]$. Thus $\hat{\mu}_F(x) \in D[0,1]$ for all $x \in G$. Therefore the interval-valued fuzzy set F is given by $F = \{x, \hat{\mu}_F(x)\}$ for all $x \in G$, where $\hat{\mu}_F : G \to D[0,1]$. Now we define the notion of refined minimum (briefly, $r \min$) and the order "$\leq$" on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as: $r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$, and $D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2 \wedge b_1 \leq b_2$ respectively. Similarly we can define $\geq$ and $=$.

First we can extend the concept of fuzzy subgroup to the concept of interval-valued fuzzy subgroup of G as follows:

Definition 2 ([6]). An interval-valued fuzzy set F in G is called an *interval-valued fuzzy subgroup* of G if

(i) $(\forall x, y \in G)$ $(\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\})$,
(ii) $(\forall x \in G)$ $(\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x))$.

An interval-valued fuzzy set $F = \{x, \hat{\mu}_F(x) | \, x \in G\}$ of the form:

$$\hat{\mu}_F(x) = \begin{cases} \hat{t}(\neq [0,0]) & \text{if} \quad x = y \\ [0,0] & \text{if} \quad x \neq y \end{cases}$$

is said to be *an interval-valued fuzzy point* with support y and the interval-valued $\hat{t}$ and is denoted by $x_{\hat{t}}$. For an interval-valued fuzzy point $x_{\hat{t}}$ and an interval-valued fuzzy set $\hat{\mu}_F$ in a set G, Pu and Liu [18] gave meaning to the symbol $x_{\hat{t}} \alpha \hat{\mu}_F$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

To say that $x_{\hat{t}} \in \hat{\mu}_F$ (resp. $x_{\hat{t}} q \hat{\mu}_F$) means that $\hat{\mu}_F(x) \geq \hat{t}$ $(resp. \, \hat{\mu}_F(x) + \hat{t} > [1,1])$, and in this case $x_{\hat{t}}$ is said to *belong to* (respectively, is *quasi-coincident with*) an interval-valued fuzzy set $\hat{\mu}_F$. To say that $x_{\hat{t}} \in \vee q \hat{\mu}_F$ (respectively, $x_{\hat{t}} \in \wedge q \hat{\mu}_F$) means that $x_{\hat{t}} \in \hat{\mu}_F$ or $x_{\hat{t}} q \hat{\mu}_F$ (respectively, $x_{\hat{t}} \in \hat{\mu}_F$ and $x_{\hat{t}} q \hat{\mu}_F$).

For all $\hat{t}_1, \hat{t}_2 \in [0,1], r\min\{\hat{t}_1, \hat{t}_2\}$ will be denoted by $m(\hat{t}_1, \hat{t}_2)$.

Definition 3 ([13]). An interval-valued fuzzy set $\hat{\mu}_F$in G is called an interval-valued (α, β)-fuzzy subgroup of G, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions:

(i) $(\forall x, y \in G)$ and $(\forall \hat{t}_1, \hat{t}_2 \in (0,1])$ $(x_{\hat{t}_1} \alpha \hat{\mu}_F, y_{\hat{t}_2} \alpha \hat{\mu}_F \Rightarrow (xy)_{m(\hat{t}_1, \hat{t}_2)} \beta \hat{\mu}_F)$,
(ii) $(\forall x \in G$ and $\forall \hat{t} \in (0,1])$, $(x_{\hat{t}} \alpha \hat{\mu}_F \Rightarrow x_{\hat{t}}^{-1} \beta \hat{\mu}_F)$.

3. CHARACTERIZATIONS OF (α, β)-FUZZY SUBGROUPS

In what follows let α and β denote any one of the symbols $\in, q, \in \vee q$, or $\in \wedge q$ and let G be a group, unless otherwise specified. To say that $x_{\hat{t}} \bar{\alpha} \hat{\mu_F}$ means that $x_{\hat{t}} \alpha \hat{\mu_F}$ does not hold.

Theorem 1. *Let $f : G \to G'$ be a group homomorphism, and let $\hat{\mu}_F$ and $\hat{\mu}_H$ be interval-valued $(\in, \in \vee q)$-fuzzy subgroups of G and G' respectively. Then:*

(i) *$f^{-1}(\hat{\mu}_H)$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.*

(ii) *If $\hat{\mu}_F$ satisfies the sup property, i.e., for any subset T of G there exists $x_0 \in T$ such that $\hat{\mu}_F(x_0) = \vee\{\hat{\mu}_F(x) | \, x \in T\}$, then $f(\hat{\mu}_F)$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G' when f is onto.*

Proof. (i) Let $x, y \in G$ and $\hat{t}_1, \hat{t}_2 \in (0,1]$ be such that $x_{\hat{t}_1} \in f^{-1}(\hat{\mu}_H)$ and $y_{\hat{t}_2} \in f^{-1}(\hat{\mu}_H)$. Then $(f(x))_{\hat{t}_1} \in (\hat{\mu}_H)$ and $(f(y))_{\hat{t}_2} \in (\hat{\mu}_H)$. Since $\hat{\mu}_H$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G', it follows that $(f(xy))_{m(\hat{t}_1,\hat{t}_2)} = (f(x)f(y))_{m(\hat{t}_1,\hat{t}_2)} \in \vee q\, (\hat{\mu}_H)$ so that $(xy)_{m(\hat{t}_1,\hat{t}_2)} \in \vee q\, f^{-1}(\hat{\mu}_H)$. The rest is obvious. Hence $f^{-1}(\hat{\mu}_H)$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

(ii) Let $a, b \in G'$ and $\hat{t}_1, \hat{t}_2 \in (0,1]$ be such that $a_{\hat{t}_1} \in f(\hat{\mu}_F)$ and $b_{\hat{t}_2} \in f(\hat{\mu}_F)$. Then $(f(\hat{\mu}_F))(a) \geq \hat{t}_1$ and $(f(\hat{\mu}_F))(b) \geq \hat{t}_2$. Since $\hat{\mu}_F$ has the sup property, there exist $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that $\hat{\mu}_F(x) = \vee\{\hat{\mu}_F(z) | \, z \in f^{-1}(a)\}$ and $\hat{\mu}_F(y) = \vee\{\hat{\mu}_F(w) | \, w \in f^{-1}(b)\}$. Then $x_{\hat{t}_1} \in (\hat{\mu}_F)$ and $y_{\hat{t}_2} \in (\hat{\mu}_F)$. Since $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G, we have $(xy)_{m(\hat{t}_1,\hat{t}_2)} \in \vee q \hat{\mu}_F$. Now $xy \in f^{-1}(ab)$, and so $(f(\hat{\mu}_F))(ab) \geq \hat{\mu}_F(xy)$. Thus $(f(\hat{\mu}_F))(ab) \geq m\{\hat{t}_1, \hat{t}_2\}$ or $(f(\hat{\mu}_F))(ab) + m\{\hat{t}_1, \hat{t}_2\} > [1,1]$ which means that $(ab)_{m(\hat{t}_1,\hat{t}_2)} \in \vee q\, f(\hat{\mu}_F)$. The rest is obvious. Consequently, $f(\hat{\mu}_F)$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G'. □

THEOREM 2. *Let $\hat{\mu}_F$ be an interval-valued $(q, \in \vee q)$-fuzzy subgroup of G such that $\hat{\mu}_F$ is not constant on G_0. If $\hat{\mu}_F(0) = \vee\{\hat{\mu}_F(x) | \, x \in G\}$, then $\hat{\mu}_F(x) \geq [0.5, 0.5]$ for all $x \in G_0$.*

Proof. Assume that $\hat{\mu}_F(x) < [0.5, 0.5]$ for all $x \in G$. Since $\hat{\mu}_F$ is not constant on G_0, there exists $y \in G_0$ such that $\hat{t}_y = \hat{\mu}_F(y) \neq \hat{\mu}_F(0) = \hat{t}_0$. Then $\hat{t}_y < \hat{t}_0$. Choose $\hat{t}_1 > [0.5, 0.5]$ such that $\hat{t}_y + \hat{t}_1 < [1,1] < \hat{t}_0 + \hat{t}_1$. Then $0_{\hat{t}_1} q \hat{\mu}_F$ and $y_{\hat{t}_1} q \hat{\mu}_F$. Since $\hat{\mu}_F(x) + \hat{t}_1 = \hat{t}_y + \hat{t}_1 < [1,1]$, we get $y_{\hat{t}_1} \overline{q} \hat{\mu}_F$, and so $(y.0)_{m([1,1],\hat{t}_1)} = y_{\hat{t}_1} \overline{\in \vee q} \hat{\mu}_F$. This contradicts the fact that $\hat{\mu}_F$ is an interval-valued $(q, \in \vee q)$-fuzzy subgroup of G. Therefore $\hat{\mu}_F(x) \geq [0.5, 0.5]$ for some $x \in G$. Now if possible, let $\hat{t}_0 = \hat{\mu}_F(0) < [0.5, 0.5]$. Then there exists $x \in G$ such that $\hat{t}_x = \hat{\mu}_F(x) \geq [0.5, 0.5]$. Thus $\hat{t}_0 < \hat{t}_x$. Take $\hat{t}_1 > \hat{t}_0$ such that $\hat{t}_0 + \hat{t}_1 < [1,1] < \hat{t}_x + \hat{t}_1$. Then $x_{\hat{t}_1} q \hat{\mu}_F$ and $0_{\hat{t}_1} q \hat{\mu}_F$, but $(0.x)_{m([1,1],\hat{t}_1)} = 0_{\hat{t}_1} \overline{\in \vee q} \hat{\mu}_F$, a contradiction. Hence $\hat{\mu}_F(x) \geq [0.5, 0.5]$. Finally let $\hat{t}_x = \hat{\mu}_F(x) < [0.5, 0.5]$ for some $x \in G_0$. Taking $\hat{t}_1 > [0,0]$ such that $\hat{t}_x + \hat{t}_1 < [0.5, 0.5]$, then $x_{[1,1]} q \hat{\mu}_F$ and $0_{[0.5,0.5]+\hat{t}_1} q \hat{\mu}_F$ since $\hat{\mu}_F(0) \geq [0.5, 0.5]$. But $\hat{\mu}_F(x) + [0.5, 0.5] + \hat{t}_1 = \hat{t}_x + [0.5, 0.5] + \hat{t}_1 < [0.5, 0.5] + [0.5, 0.5] = [1,1]$, which implies that $x_{[0.5,0.5]+\hat{t}_1} \overline{q} \hat{\mu}_F$. Thus $(x.0)_{m([1,1],[0.5,0.5]+\hat{t}_1)} = x_{[0.5,0.5]+\hat{t}_1} \overline{\in \vee q} \hat{\mu}_F$, a contradiction. Hence $\hat{\mu}_F(x) \geq [0.5, 0.5]$ for all $x \in G_0$. The rest is obvious. This completes the proof. □

An interval-valued fuzzy set $\hat{\mu}_F$ in G is said to be *proper* if $\mathrm{Im}(\hat{\mu}_F)$ has at least two elements. Two interval-valued fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

THEOREM 3. *Let G be a fuzzy group. Then a proper interval-valued $(\in, \in)$-fuzzy subgroup $\hat{\mu}_F$ of G such that $\# \{Im(\hat{\mu}_F)\} \geq 3$ can be expressed as the union of two proper non-equivalent interval-valued $(\in, \in)$-fuzzy subgroup of G.*

Proof. Let $\hat{\mu}_F$ be a interval-valued $(\in, \in)$-fuzzy subgroup of G with $\mathrm{Im}(\hat{\mu}_F) = \{\hat{t}_0, \hat{t}_1, \ldots \hat{t}_n\}$, where $\hat{t}_0 > \hat{t}_1 > \ldots > \hat{t}_n$ and $n \geq 2$. Then $U(\hat{\mu}_F; \hat{t}_0) \subseteq U(\hat{\mu}_F; \hat{t}_1) \subseteq \ldots \subseteq U(\hat{\mu}_F; \hat{t}_n) = G$ is the chain of $\in$-level subgroup of $\hat{\mu}_F$. Define an interval-valued fuzzy sets $\hat{\nu}_F$ and $\hat{\delta}_F$ in G by

$$\hat{\nu}_F(x) = \begin{cases} \hat{r}_1 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_1) \\ \hat{t}_2 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_2) \backslash U(\hat{\mu}_F; \hat{t}_1) \\ \ldots & \\ \hat{t}_n & \text{if } x \in U(\hat{\mu}_F; \hat{t}_n) \backslash U(\hat{\mu}_F; \hat{t_{n-1}}) \end{cases}$$

and

$$\hat{\delta}_F(x) = \begin{cases} \hat{t}_0 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_0) \\ \hat{t}_1 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_1) \backslash U(\hat{\mu}_F; \hat{t}_0) \\ \hat{r}_2 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_3) \backslash U(\hat{\mu}_F; \hat{t}_1) \\ \hat{t}_4 & \text{if } x \in U(\hat{\mu}_F; \hat{t}_4) \backslash U(\hat{\mu}_F; \hat{t}_3) \\ \ldots & \\ \hat{t}_n & \text{if } x \in U(\hat{\mu}_F; \hat{t}_n) \backslash U(\hat{\mu}_F; \hat{t}_{n-1}) \end{cases}$$

respectively, where $\hat{t}_2 < \hat{r}_1 < \hat{t}_1$ and $\hat{t}_4 < \hat{r}_2 < \hat{t}_2$. Then $\hat{\nu}_F$ and $\hat{\delta}_F$ are interval-valued $(\in, \in)$-fuzzy subgroups of G with $U(\hat{\mu}_F; \hat{t}_1) \subseteq U(\hat{\mu}_F; \hat{t}_2) \subseteq \ldots \subseteq U(\hat{\mu}_F; \hat{t}_n) = G$ and $U(\hat{\mu}_F; \hat{t}_0) \subseteq U(\hat{\mu}_F; \hat{t}_1) \subseteq \ldots \subseteq (\hat{\mu}_F; \hat{t}_n) = G$ as respective chains of $\in$-level subgroup, and $\hat{\mu}_F, \hat{\delta}_F \leq \hat{\mu}_F$. Thus $\hat{\nu}_F$ and $\hat{\delta}_F$ are non-equivalent, and obviously $\hat{\nu}_F \cup \hat{\delta}_F = \hat{\mu}_F$. This completes the proof. □

THEOREM 4. *Let $\hat{\mu}_F$ be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G such that $\hat{\mu}_F(x) < [0.5, 0.5]$ for all $x \in G$. Then $\hat{\mu}_F$ is an interval-valued $(\in, \in)$-fuzzy subgroup of G.*

Proof. Let $x, y \in G$ and $\hat{t}_1, \hat{t}_2 \in (0, 1]$ be such that $x_{\hat{t}_1} \in \hat{\mu}_F$ and $y_{\hat{t}_2} \in \hat{\mu}_F$. Then $\hat{\mu}_F(x) \geq \hat{t}_1$ and $\hat{\mu}_F(y) \geq \hat{t}_2$. It follows from [12, Theorem 3.4] that

$$\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} = m\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} \geq m\{\hat{t}_1, \hat{t}_2\}$$

so that $(xy)_{m(\hat{t}_1, \hat{t}_2)} \in \hat{\mu}_F$. The rest is obvious. Hence $\hat{\mu}_F$ is an interval-valued $(\in, \in)$-fuzzy subgroup of G. □

THEOREM 5. *An interval-valued fuzzy set $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if and only if the set $U(\hat{\mu}_F; \hat{t}) = \{x \in G | \, \hat{\mu}_F(x) \geq \hat{t}\}$ is a subgroup of G for all $\hat{t} \in (0, 0.5]$.*

Proof. Assume that $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. Let $x, y \in U(\hat{\mu}_F; \hat{t})$ for $\hat{t} \in (0, 0.5]$. Then $\hat{\mu}_F(x) \geq \hat{t}$ and $\hat{\mu}_F(y) \geq \hat{t}$. It follows from [12, Theorem 3.4] that $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} =$

$m\{\hat{t}, [0.5, 0.5]\} = \hat{t}$, so that $xy \in U(\hat{\mu}_F; \hat{t})$. The rest is obvious. Therefore $U(\hat{\mu}_F; \hat{t})$ is a subgroup of G.

Conversely, let $\hat{\mu}_F$ be a fuzzy set in G with $U(\hat{\mu}_F; \hat{t}) = \{x \in G |\, \hat{\mu}_F(x) \geq \hat{t}\}$ a subgroup of G for all $\hat{t} \in (0, 0.5]$. If there are $x, y \in G$ such that $\hat{\mu}_F(xy) < m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$, then we take $\hat{t} \in (0, 1]$ such that $\hat{\mu}_F(xy) < \hat{t} < m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$. Thus $x, y \in U(\hat{\mu}_F; \hat{t})$ and $\hat{t} < [0.5, 0.5]$, and so $xy \in U(\hat{\mu}_F; \hat{t})$, i.e., $\hat{\mu}_F(xy) \geq \hat{t}$. This is a contradiction. Therefore $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$ for all $x, y \in G$. The rest is obvious. Using [12, Theorem 3.4], $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. □

For any interval-valued fuzzy set $\hat{\mu}_F$ in G and $\hat{t} \in (0, 1]$, denote $[\hat{\mu}_F]_{\hat{t}} = \{x \in G |\, x_{\hat{t}} q \hat{\mu}_F\}$ and $[\hat{\mu}_F]_{\hat{t}} = \{x \in G |\, x_{\hat{t}} \in \vee q \hat{\mu}_F\}$. Clearly, $[\hat{\mu}_F]_{\hat{t}} = U(\hat{\mu}_F; \hat{t}) \cup [\hat{\mu}_F]_{\hat{t}}$.

THEOREM 6. *An interval-valued fuzzy set $\hat{\mu}_F$ in G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if and only if $[\hat{\mu}_F]_{\hat{t}}$ is a subgroup of G for all $\hat{t} \in (0, 1]$.*

We call $[\hat{\mu}_F]_{\hat{t}}$ an interval-valued $(\in, \in \vee q)$-level subgroup of $\hat{\mu}_F$.

Proof. Let $\hat{\mu}_F$ be an interval-valued $(\in, \in \vee q)$- fuzzy subgroup of G and let $x, y \in [\hat{\mu}_F]_{\hat{t}}$ for $\hat{t} \in (0, 1]$. Then $x_{\hat{t}} \in \vee q \hat{\mu}_F$ and $y_{\hat{t}} \in \vee q \hat{\mu}_F$, that is, $\hat{\mu}_F(x) \geq \hat{t}$ or $\hat{\mu}_F(x) + \hat{t} > [1, 1]$ and $\hat{\mu}_F(y) \geq \hat{t}$ or $\hat{\mu}_F(y) + \hat{t} > [1, 1]$. Since $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$ by [12, Theorem 3.4], we have $\hat{\mu}_F(xy) \geq m\{\hat{t}, [0.5, 0.5]\}$. Otherwise, $x_{\hat{t}} \overline{\in \vee q} \hat{\mu}_F$ and $y_{\hat{t}} \overline{\in \vee q} \hat{\mu}_F$, a contradiction. If $\hat{t} \leq [0.5, 0.5]$, then $\hat{\mu}_F(xy) \geq m\{\hat{t}, [0.5, 0.5]\} = \hat{t}$ and so $xy \in U(\hat{\mu}_F; \hat{t}) \subseteq [\hat{\mu}_F]_{\hat{t}}$. If $\hat{t} > [0.5, 0.5]$, then $\hat{\mu}_F(xy) \geq m\{\hat{t}, [0.5, 0.5]\} = [0.5, 0.5]$ and thus $\hat{\mu}_F(xy) + \hat{t} > [0.5, 0.5] + [0.5, 0.5] = [1, 1]$. Hence $(xy)_{\hat{t}} q \hat{\mu}_F$, and so $xy \in [\hat{\mu}_F]_{\hat{t}} \subseteq [\hat{\mu}_F]_{\hat{t}}$. The rest is obvious. Therefore $[\hat{\mu}_F]_{\hat{t}}$ is a subgroup of G.

Conversely, let $\hat{\mu}_F$ be a fuzzy set in G and $\hat{t} \in (0, 1]$ be such that $[\hat{\mu}_F]_{\hat{t}}$ is a subgroup of G. If possible, let $\hat{\mu}_F(xy) < \hat{t} < m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$ for some $\hat{t} \in (0, 0.5]$ and $x, y \in G$. Then $x, y \in U(\hat{\mu}_F; \hat{t}) \subseteq [\hat{\mu}_F]_{\hat{t}}$, which implies that $xy \in [\hat{\mu}_F]_{\hat{t}}$. Hence $\hat{\mu}_F(xy) \geq \hat{t}$ or $\hat{\mu}_F(xy) + \hat{t} > [1, 1]$, a contradiction. Therefore $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$ for all $x, y \in G$. The rest is obvious. Using [12, Theorem 3.4], we conclude that $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. □

THEOREM 7. *Let $\hat{\mu}_F$ be a proper interval-valued $(\in, \in \vee q)$- fuzzy subgroup of G such that $\#\ \{\hat{\mu}_F(x) |\, \hat{\mu}_F(x) < [0.5, 0.5]\} \geq 2$. Then there exist two proper non-equivalent interval-valued $(\in, \in \vee q)$- fuzzy subgroup of G such that $\hat{\mu}_F$ can be expressed as the union of them.*

Proof. Let $\{\hat{\mu}_F(x) |\, \hat{\mu}_F(x) < [0.5, 0.5]\} = \{\hat{t}_1, \hat{t}_2 \ldots \hat{t}_r\}$, where $\hat{t}_1 > \hat{t}_2 > \ldots > \hat{t}_r$ and $r \geq 2$. Then the chain of interval-valued $(\in \vee q)$-level subgroup of $\hat{\mu}_F$ is

$$[\hat{\mu}_F]_{[0.5, 0.5]} \subseteq [\hat{\mu}_F]_{\hat{t}_1} \subseteq [\hat{\mu}_F]_{\hat{t}_2} \subseteq \ldots \subseteq [\hat{\mu}_F]_{\hat{t}_r} = G.$$

Let $\hat{\nu}_F$ and $\hat{\delta}_F$ be interval-valued fuzzy sets in G defined by

$$\hat{\nu}_F(x) = \begin{cases} \hat{t}_1 & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_1} \\ \hat{t}_2 & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_2} \backslash [\hat{\mu}_F]_{\hat{t}_1} \\ \dots \\ \hat{t}_r & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_r} \backslash [\hat{\mu}_F]_{\hat{t}_{r-1}} \end{cases}$$

and

$$\hat{\delta}_F(x) = \begin{cases} \hat{\mu}_F(x) & \text{if } x \in [\hat{\mu}_F]_{[0.5,0.5]}, \\ k & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_2} \backslash [\hat{\mu}_F]_{[0.5,0.5]} \\ \hat{t}_3 & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_3} \backslash [\hat{\mu}_F]_{\hat{t}_2} \\ \dots \\ \hat{t}_r & \text{if } x \in [\hat{\mu}_F]_{\hat{t}_r} \backslash [\hat{\mu}_F]_{\hat{t}_{r-1}} \end{cases}$$

respectively, where $\hat{t}_3 < \hat{k} < \hat{t}_2$. Then $\hat{\nu}_F$ and $\hat{\delta}_F$ are interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G, and $\hat{\nu}_F, \hat{\delta}_F \leq \hat{\mu}_F$. The chains of interval-valued $(\in \vee q)$-level subgroup of $\hat{\nu}_F$ and $\hat{\delta}_F$ are given by $[\hat{\mu}_F]_{\hat{t}_1} \subseteq [\hat{\mu}_F]_{\hat{t}_2} \subseteq \ldots \subseteq [\hat{\mu}_F]_{\hat{t}_r}$ and $[\hat{\mu}_F]_{[0.5,0.5]} \subseteq [\hat{\mu}_F]_{\hat{t}_2} \subseteq \ldots \subseteq [\hat{\mu}_F]_{\hat{t}_r}$ respectively. Therefore, $\hat{\nu}_F$ and $\hat{\delta}_F$ are non-equivalent, and clearly $\hat{\nu}_F \cup \hat{\delta}_F = \hat{\mu}_F$. This completes the proof. □

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Received February 7, 2009
Accepted October 14, 2009

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