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HARMONIC FUNCTIONS WHICH ARE STARLIKE OF COMPLEX ORDER WITH RESPECT TO CONJUGATE POINTS

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Abstract. Let H denote the class of functions f which are harmonic, orientation preserving and univalent in the open unit disc $D = \{z : |z| < 1\}$. This paper defines and investigates a family of complex-valued harmonic functions that are orientation preserving and univalent in D and are related to the functions starlike of complex order with respect to conjugate points. The authors obtain coefficient conditions and growth result.

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1. INTRODUCTION

Let $f = u + iv$ be a continuous complex-valued harmonic function in a complex domain E if both u and v are real harmonic in the domain E. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = \text{Re }(U)$ and $v = Im\ (V)$. Then, we can write $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic in E. The mapping $z \mapsto f(z)$ is orientation preserving and locally univalent in E if and only if the Jacobian of f given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive in E. The function $f = h + \overline{g}$ is said to be harmonic univalent in E if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one-to-one in E. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let H denote the family of functions $f = h+\overline{g}$ that are harmonic, orientation preserving and univalent in the open unit disc $D = \{z : |z| < 1\}$ with the normalisation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

In [3], Clunie and Sheil-Small investigated the class H plus some of it geometric subclasses and obtained some coefficient bounds. Since then, there have been many authors which looked at related subclasses, see [9] and [10] to name a few. In particular, refer to Duren [4], which provides comprehensive reference in the theory of harmonic functions. Furthermore, Jahangiri in [5] considered

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a subclass of H consisting of functions which are starlike of α, for $0 \le \alpha < 1$. We denote such class as $HS^\star(\alpha)$. Specifically, a function f of the form (1) is harmonic starlike of order α, $0 \le \alpha < 1$, for $z \in D$ if (see Sheil-Small [7]) $\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) \ge \alpha$, $|z| = r < 1$. Next, we denote further the class $\overline{H}$, a subclass of H such that the functions h and g in $f = h + \overline{g}$ are of the form:

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{2}$$

Also let $\overline{H}S^\star(\alpha) = HS^\star(\alpha) \cap \overline{H}$.

In [6], Nasr and Aouf introduced the class of starlike functions of complex order b. Denote $S^*(b)$ to be the class consisting of functions which are analytic and starlike of complex order b(b is a non-zero complex number) and satisfying the following condition:

$$\text{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0,\ z \in D.$$

In [1], Abdul Halim and Janteng were motivated to form a new subclass of H based on Nasr and Aouf's class. A function f of the form (1) is harmonic starlike of complex order, for $0 \le \alpha < 1$, b non-zero complex number with $|b| \le 1$ and $z \in D$, if and only if

$$\text{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{z'f(z)} - 1\right)\right\} \ge \alpha,\ |z| = r < 1,$$

where $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \le r < 1$ and $0 \le \theta < 2\pi$. The class of these functions is denoted by $HS^\star(b,\alpha)$.

If f takes the form (2) then we denote it as $\overline{H}S^\star(b,\alpha)$. The constraint $|b| \le 1$ is to ensure $J_f(z) > 0$ so that f is univalent.

Now, we define new class of functions as follows:

Definition. Let $f \in H$. Then $f \in HS_c^\star(b,\alpha)$ is said to be *harmonic starlike of complex order with respect to conjugate points* if for $0 \le \alpha < 1$ and b non-zero complex number with $|b| \le 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \le r < 1$ and $0 \le \theta < 2\pi$,

$$\text{Re}\left\{1 + \frac{1}{b}\left(\frac{2zf'(z)}{z'(f(z) + \overline{f(\overline{z})})} - 1\right)\right\} \ge \alpha,\ |z| = r < 1. \tag{3}$$

Also, we let $\overline{H}S_c^\star(b,\alpha) = HS_c^\star(b,\alpha) \cap \overline{H}$.

2. MAIN RESULTS

Avci and Zlotkiewicz [2] proved that the coefficient condition $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1$ is a sufficient condition for functions $f = h + \overline{g}$ to be in $HS^\star(1,0)$ with $b_1 = 0$. Silverman [8] also proved that this condition is also a necessary when a_n and b_n are negative, as well as $b_1 = 0$. In the following theorem, Jahangiri in 1999 [5], obtained analogue sufficient condition for $f \in HS^\star(1,\alpha)$, where b_1 is not necessarily 0.

Theorem 1 ([5]). *Let $f = h + \overline{g}$ be given by* (1). *Furthermore, let*

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n| \right) \leq 1,$$

where $a_1 = 1$ and $0 \leq \alpha < 1$. Then f is harmonic univalent in D and $f \in HS^\star(1,\alpha)$.

Jahangiri also proved that the condition in Theorem 1 is a necessary condition for $f = h + \overline{g}$ given by (2) and belongs to $\overline{H}S^\star(1,\alpha)$.

The following theorem proved by Abdul Halim and Janteng in [1] will be used throughout in this paper.

Theorem 2. *Let $f = h + \overline{g}$ be given by* (1). *If*

$$\sum_{n=2}^{\infty} \left(\frac{n-1+|b|-\alpha|b|}{(1-\alpha)|b|} \right) |a_n| + \sum_{n=1}^{\infty} \left(\frac{n+1-|b|+\alpha|b|}{(1-\alpha)|b|} \right) |b_n| \leq 1, \tag{4}$$

where $0 \leq \alpha < 1$ and b a non-zero complex number with $|b| \leq 1$ then f is harmonic univalent in D and $f \in HS^\star(b,\alpha)$. Condition (4) is also necessary if $f \in \overline{H}S^\star(b,\alpha)$.

In this paper we give sufficient coefficient conditions for functions $f = h+\overline{g}$ of the form (1) to be in $HS_c^\star(b,\alpha)$, where $0 \leq \alpha < 1$ and b is a non-zero complex number such that $|b| \leq 1$. We also show that these conditions are necessary when $f \in \overline{H}S_c^\star(b,\alpha)$.

Theorem 3. *Let $f = h + \overline{g}$ be given by* (1). *If*

$$\sum_{n=2}^{\infty} \left(\frac{n+(|b|-\alpha|b|-1)}{(1-\alpha)|b|} \right) |a_n| + \sum_{n=1}^{\infty} \left(\frac{n-(|b|-\alpha|b|-1)}{(1-\alpha)|b|} \right) |b_n| \leq 1, \tag{5}$$

where $0 \leq \alpha < 1$ and b a non-zero complex number with $|b| \leq 1$ then f is harmonic univalent in D and $f \in HS_c^\star(b,\alpha)$.

Proof. It follows from Theorem 2 that $f \in HS^\star(b,\alpha)$ and hence f is locally univalent and orientation preserving in D. Next, we show that $f \in HS_c^\star(b,\alpha)$. To do so, we need to show that when (4) holds, then (3) also holds true. Letting

$$\begin{aligned} w(z) &= 1 + \frac{1}{b}\left(\frac{2zf'(z)}{z'(f(z)+\overline{f(\overline{z})})} - 1 \right) \\ &= 1 + \frac{1}{b}\left(\frac{2(zh'(z) - \overline{zg'(z)}) - (h(z) + \overline{g(z)} + \overline{h(\bar{z})} + g(\bar{z}))}{h(z) + \overline{g(z)} + \overline{h(\bar{z})} + g(\bar{z})} \right), \end{aligned}$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq r < 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(re^{i\theta}))$, condition (3) is equivalent to Re $w(z) \geq \alpha$. And since Re $w(z) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ for $0 \leq \alpha < 1$, it suffices to show that

$$(6) \qquad |A(z) + ((2-\alpha)b-1)B(z)| - |(\alpha b+1)B(z) - A(z)| \geq 0,$$

where $A(z) = 2(zh'(z) - \overline{zg'(z)})$ and $B(z) = h(z) + \overline{g(z)} + \overline{h(\bar{z})} + g(\bar{z})$. Substituting for $A(z)$ and $B(z)$ in (6) gives:

$$\begin{aligned}
&|A(z) + (2b-\alpha b-1)B(z)| - |(\alpha b+1)B(z) - A(z)| \\
&= \left|(2b-\alpha b-1)\left(2z + \sum_{n=2}^{\infty} 2\, a_n\, z^n + \sum_{n=1}^{\infty} 2\bar{b}_n\, (\overline{z})^n\right)\right. \\
&\left. + \left(2z + \sum_{n=2}^{\infty} 2na_n\, z^n - \sum_{n=1}^{\infty} 2n\bar{b}_n\, (\overline{z})^n\right)\right| \\
&- \left|(\alpha b+1)\left(2z + \sum_{n=2}^{\infty} 2a_n\, z^n + \sum_{n=1}^{\infty} 2\bar{b}_n\, (\overline{z})^n\right)\right. \\
&\left. - \left(2z + \sum_{n=2}^{\infty} 2na_n\, z^n - \sum_{n=1}^{\infty} 2n\bar{b}_n\, (\overline{z})^n\right)\right| \\
&= \left|2(2-\alpha)bz + \sum_{n=2}^{\infty} 2[((2-\alpha)b-1)+n]\, a_n\, z^n\right. \\
&\left. + \sum_{n=1}^{\infty} 2[((2-\alpha)b-1)-n]\, \bar{b}_n\, (\overline{z})^n\right| \\
&- \left|2\alpha bz - \sum_{n=2}^{\infty} 2(n-(\alpha b+1))\, a_n\, z^n + \sum_{n=1}^{\infty} 2(n+(\alpha b+1))\, \bar{b}_n\, (\overline{z})^n\right| \\
&\geq 2(2-\alpha)|b||z| - \sum_{n=2}^{\infty} 2|n+((2-\alpha)b-1)|\, |a_n|\, |z|^n \\
&- \sum_{n=1}^{\infty} 2|n-((2-\alpha)b-1)|\, |b_n|\, |z|^n - 2\alpha|b||z| \\
&- \sum_{n=2}^{\infty} 2|n-(\alpha b+1)|\, |a_n|\, |z|^n - \sum_{n=1}^{\infty} 2|n+(\alpha b+1)|\, |b_n|\, |z|^n \\
&= 4(1-\alpha)|b||z| - \sum_{n=2}^{\infty} 2\Big(|n-(\alpha b+1)| + |n+((2-\alpha)b-1)|\Big)\, |a_n|\, |z|^n \\
&- \sum_{n=1}^{\infty} 2\Big(|n+(\alpha b+1)| + |n-((2-\alpha)b-1)|\Big)\, |b_n|\, |z|^n
\end{aligned}$$

$$= 4(1-\alpha)|b||z|\left[1-\sum_{n=2}^{\infty}\left(\frac{2|n-(\alpha b+1)|+2|n+((2-\alpha)b-1)|}{4(1-\alpha)|b|}\right)|a_n||z|^{n-1}\right.$$

$$\left.-\sum_{n=1}^{\infty}\left(\frac{2|n+(\alpha b+1)|+2|n-((2-\alpha)b-1)|}{4(1-\alpha)|b|}\right)|b_n||z|^{n-1}\right]$$

$$\geq 4(1-\alpha)|b||z|\left\{1-\sum_{n=2}^{\infty}\frac{(n+(|b|-\alpha|b|-1))}{(1-\alpha)|b|}|a_n|\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{(n-(|b|-\alpha|b|-1))}{(1-\alpha)|b|}|b_n|\right\}\geq 0$$

by (4). □

The functions

$$f(z) = z+\sum_{n=2}^{\infty}\left(\frac{(1-\alpha)|b|}{(n+(|b|-\alpha|b|-1))}\right)x_n z^n$$
$$+\sum_{n=1}^{\infty}\left(\frac{(1-\alpha)|b|}{(n-(|b|-\alpha|b|-1))}\right)\overline{y}_n\overline{z}^{\,n},$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound in (4) is sharp.

The next theorem shows that condition (4) is necessary for $f \in \overline{H}S_c^{\star}(b,\alpha)$.

THEOREM 4. *Let $f = h+\overline{g}$ be given by (2). Then $f \in \overline{H}S_c^{\star}(b,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{n+(|b|-\alpha|b|-1)}{(1-\alpha)|b|}\right)|a_n|+\sum_{n=1}^{\infty}\left(\frac{2n-(|b|-\alpha|b|-1)}{(1-\alpha)|b|}\right)|b_n| \leq 1, \tag{7}$$

where $0 \leq \alpha < 1$, b a non-zero complex number such that $|b| \leq 1$.

Proof. Since $\overline{H}S_c^{\star}(b,\alpha) \subset HS_s^{\star}(b,\alpha)$, the “if” part follows from Theorem 2. To prove the “only if” part, we show that when (7) does not hold, f is not in $\overline{H}S_c^{\star}(b,\alpha)$. First, if $f \in \overline{H}S_c^{\star}(b,\alpha)$, then:

$$\text{Re}\left\{1+\frac{1}{b}\left(\frac{2zf'(z)}{z'(f(z)+\overline{f(\overline{z})})}-1\right)\right\}-\alpha$$

$$= \text{Re}\left\{(1-\alpha)+\frac{1}{b}\left(\frac{2(zh'(z)-\overline{zg'(z)})-(h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z}))}{h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z})}\right)\right\}$$

$$= \text{Re}\left\{\frac{(1-\alpha)b(h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z}))+2(zh'(z)-\overline{zg'(z)})}{b(h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z}))}\right.$$

$$-\frac{(h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z}))}{b(h(z)+\overline{g(z)}+\overline{h(\bar{z})}+g(\bar{z}))}\Bigg\}$$
$$= \operatorname{Re}\left\{\frac{2(1-\alpha)bz-\sum_{n=2}^{\infty}2(((1-\alpha)b-1)+n)\ |a_n|\ z^n}{2b\left(z-\sum_{n=2}^{\infty}\ |a_n|\ z^n+\sum_{n=1}^{\infty}\ |b_n|\ (\overline{z})^{\ n}\right)}\right.$$
$$\left.-\frac{\sum_{n=1}^{\infty}2(n-((1-\alpha)b-1))\ |b_n|\ (\overline{z})^n}{2b\left(z-\sum_{n=2}^{\infty}\ |a_n|\ z^n+\sum_{n=1}^{\infty}\ |b_n|\ (\overline{z})^{\ n}\right)}\right\}$$
$$= \operatorname{Re}\left\{\frac{(1-\alpha)|b|^2-\sum_{n=2}^{\infty}(n+((1-\alpha)b-1))\bar{b}\ |a_n|\ z^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\ |a_n|\ z^{n-1}+\sum_{n=1}^{\infty}\ |b_n|\ z^{\ n-1}\right)}\right.$$
$$\left.-\frac{\sum_{n=1}^{\infty}(n-((1-\alpha)b-1))\bar{b}\ |b_n|\ z^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\ |a_n|\ z^{n-1}+\sum_{n=1}^{\infty}\ |b_n|\ z^{\ n-1}\right)}\right\}\geq 0$$

for all values of z, $|z|=r<1$ and any b such that $0<|b|<1$. Choose z to be on the positive real axis, where $z=r<1$. Thus, the above condition becomes:

$$\begin{aligned}&\frac{(1-\alpha)|b|^2-\sum_{n=2}^{\infty}(n+((1-\alpha)b-1))\ \bar{b}\ |a_n|\ r^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\ |a_n|\ r^{n-1}+\sum_{n=1}^{\infty}\ |b_n|\ r^{n-1}\right)}\\ &-\frac{\sum_{n=1}^{\infty}(n-((1-\alpha)b-1))\ \bar{b}\ |b_n|\ r^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\ |a_n|\ r^{n-1}+\sum_{n=1}^{\infty}\ |b_n|\ r^{n-1}\right)}\geq 0.\end{aligned}\tag{8}$$

It is easily established that the denominator is positive when $r\to 1$. We need to show that the numerator is also positive for any $z\in D$ and any $b\neq 0, |b|\leq 1$. In the case when $b=|b|$ (real and positive), the numerator becomes:

$$\begin{aligned}&\left((1-\alpha)|b|^2-\sum_{n=2}^{\infty}(n+((1-\alpha)b-1))\ |b|\ |a_n|\ r^{n-1}\right.\\ &\left.-\sum_{n=1}^{\infty}(n-((1-\alpha)b-1))|b|)\ |b|\ |b_n|\ r^{n-1}\right)\\ &=|b|\left((1-\alpha)|b|-\sum_{n=2}^{\infty}(n+((1-\alpha)b-1))\ |a_n|\ r^{n-1}\right.\\ &\left.-\sum_{n=1}^{\infty}(n-((1-\alpha)b-1))\ |b_n|\ r^{n-1}\right),\end{aligned}$$

which is negative if condition (7) does not hold. Thus, there exists some point $z_0=r_0$ in (0,1) and some b such that $b=|b|$ for which the quotient in (8) is negative, which contradicts the condition that $f\in\overline{H}S_c^{\star}(b,\alpha)$. Hence, the proof is complete. □

Next, growth estimates of $\overline{H}S_c^{\star}(b,\alpha)$ are determined.

THEOREM 5. *If* $f \in \overline{H}S_c^{\star}(b,\alpha)$ *then for* $|z| = r < 1$,

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{(1-\alpha)|b|}{|b|-\alpha|b|+1} - \frac{2-(|b|-\alpha|b|)}{|b|-\alpha|b|+1}|b_1|\right) r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{(1-\alpha)|b|}{|b|-\alpha|b|+1} - \frac{2-(|b|-\alpha|b|)}{|b|-\alpha|b|+1}|b_1|\right) r^2.$$

Proof. Let $f \in \overline{H}S_c^{\star}(b,\alpha)$. Taking the absolute value of f, by (7) we have:

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^2 \\
&= (1+|b_1|)r \\
&+ \frac{(1-\alpha)|b|}{|b|-\alpha|b|+1}\sum_{n=2}^{\infty}\left(\frac{|b|-\alpha|b|+1}{(1-\alpha)|b|}|a_n| + \frac{3-|b|+\alpha|b|}{(1-\alpha)|b|}|b_n|\right)\ r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\alpha)|b|}{|b|-\alpha|b|+1}\sum_{n=2}^{\infty}\Big(\frac{n+(|b|-\alpha|b|-1)}{(1-\alpha)|b|}|a_n| \\
&+ \frac{n-(|b|-\alpha|b|-1)}{(1-\alpha)|b|}|b_n|\Big)\ r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\alpha)|b|}{|b|-\alpha|b|+1}\left(1 - \frac{1-(|b|-\alpha|b|-1)}{(1-\alpha)|b|}|b_1|\right)\ r^2 \\
&= (1+|b_1|)r + \left(\frac{(1-\alpha)|b|}{|b|-\alpha|b|+1} - \frac{2-(|b|-\alpha|b|)}{|b|-\alpha|b|+1}|b_1|\right)\ r^2.
\end{aligned}$$

The second inequality follows similarly. □

The upper and lower bounds given in Theorem 5 are respectively attained for the following functions:

$$f(z) = z + |b_1|\overline{z} + \left(\frac{(1-\alpha)|b|}{|b|-\alpha|b|+1} - \frac{2-(|b|-\alpha|b|)}{|b|-\alpha|b|+1}|b_1|\right) \overline{z}^{\,2}$$

and

$$f(z) = (1-|b_1|)z - \left(\frac{(1-\alpha)|b|}{|b|-\alpha|b|+1} - \frac{2-(|b|-\alpha|b|)}{|b|-\alpha|b|+1}|b_1|\right) z^{\,2}.$$

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