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IMPROVED RESULTS FOR CONTINUOUS MODIFIED NEWTON-TYPE METHODS

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Abstract. We provide semilocal convergence results for continuous modified Newton-type methods to solve nonlinear operator equations in a real Hilbert space setting. Using a combination of Lipschitz and center Lipschitz continuous conditions, we provide a finer convergence analysis than before under weaker conditions, and the same hypotheses and computational cost [1]-[4], [11]-[15]. In this way we expand the applicability of Newton-type continuous methods under the same computational cost as before.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution of equation

$$F(x) = 0, \tag{1.1}$$

where F is defined on a closed subset D of a real Hilbert space X with values in a real Hilbert space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges

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to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Newton-type methods (NTM) are undoubtedly the most popular methods for generating a sequence approximating a solution of equation (1.1). Recent results on local as well as semilocal convergence results for (NTM) on a Banach space setting can be found in [5], [6], [7], and the references there. In a series of our works [5], [6], [7], we developed a technique which combines a Lipschitz and a center Lipschitz condition (instead of only a Lipschitz condition used by researchers) in the computation of the sufficient convergence conditions, and error estimates for (NTM). This approach leads under the same hypotheses and computational cost to the following advantages. Semilocal case: Weaker sufficient convergence conditions, and tighter error bounds on the distances involved, and an at least as precise information on the location of the solution. That is the applicability of (NTM) is expanded. Note also that, fewer steps are required to achieve a given error tolerance.

In this article we show that the above advantages are transferred in a Hilbert space setting. In Section 2, we first consider the continuous analog of a modified Newton's method (MNM)

$$\dot{x}(t) = -[F'(x_0)]^{-1}F(x(t)), \quad x(0) = x_0 \in X \tag{1.2}$$

along the lines of the elegant works by Airapetyan ([1]), Ram, Smirnora, Favini [14], [2]-[4], [11]-[13], [15]. We associate (1.2) with the autonomous dynamical system

$$\dot{x}(t) = \Phi(x(t)), \quad 0 \leq t < +\infty, \quad x(0) = x_0, \tag{1.3}$$

where x_0 is an initial approximation to $x^\star$, and $x(t)$ is the trajectory. Then we improve the results in Section 2 of [14], by simply showing that we can replace the Lipschitz condition

$$\|F'(x) - F'(y)\| \leq M\|x - y\|, \forall x, y \in U(x_0, r) \subseteq D, \tag{1.4}$$

where $U(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$, by the actually needed center-Lipschitz condition

$$\|F'(x) - F'(x_0)\| \leq M_0\|x - x_0\|, \forall x \in U(x_0, r) \subseteq D, \tag{1.5}$$

where r is a known radius. The results of [14] for the local case are improved in Section 3 following an analogous way. Note that if condition (1.4) holds, then M_0 exists, $M_0 \leq M$ and $\frac{M}{M_0}$ can be arbitrarily large [5], [6], [7]. The importance of (NTM) for solving well and ill posed problems, and a short history of their continuous analogs can be found in [14]. Finally, we provide numerical examples where (1.5) holds but (1.4) does not, and also cases where both conditions hold but our results are finer than the ones in [14] as already stated in the semilocal case above.

2. SEMILOCAL CONVERGENCE OF (MNM): WELL-POSED CASE

We need a slightly modified existence, and uniqueness result for the semilocal convergence of (MNM) [14, p.40].

LEMMA 2.1. *Let X, Y be real Hilbert spaces, D a closed subset of X, $F : D \to Y$, and $\Phi : D \to Y$. Assume that there exist $c_1, c_2 > 0$ such that for*

$$r = \frac{c_2 \|F(x_0)\|}{c_1}, \tag{2.1}$$

operators F and Φ are Fréchet differentiable on $U(x_0, r)$. Also assume that for all $y \in U(x_0, r)$, the following conditions hold:

$$(F'(y)\Phi(y), F(y)) \le -c_1 \|F(y)\|^2, \tag{2.2}$$

$$\|\Phi(y)\| \le c_2 \|F(y)\|, \tag{2.3}$$

$$U(x_0, r) \subseteq D. \tag{2.4}$$

Then there is a global solution $x = x(t)$ to system (1.3) *in $U(x_0, r)$, such that*

$$\lim_{t \to +\infty} x(t) = x^\star, \tag{2.5}$$

where $x^\star$ is a solution of equation $F(x) = 0$ in $U(x_0, r)$.

Moreover, the following estimates hold:

$$\|x(t) - x^\star\| \le r e^{-c_1 t}, \tag{2.6}$$

$$\|F(x(t))\| \le \|F(x_0)\| e^{-c_1 t}. \tag{2.7}$$

Note that if $D = X = Y$, Lemma 2.1 reduces to the corresponding one in [14, p. 40].

We shall show the following semilocal convergence theorem for (MNM) which relates the asymptotic behavior of a solution $x(t)$ to (1.2), and solutions to (1.1).

THEOREM 2.2. *Let X, Y be real Hilbert spaces, D be a closed subset of X, $F : D \to Y$. Assume that the operator F is Fréchet-differentiable, its Fréchet derivative satisfies* (1.5) *on $U(x_0, r)$,*

$$r \in \{r_1, r_2\}, \tag{2.8}$$

where provided that

$$h_A = 4M_0 m_1^2 \|F(x_0)\| \le 1, \quad m_1 = \left\|F'(x_0)^{-1}\right\|, \tag{2.9}$$

$$r_1 = \frac{1 - \sqrt{1 - 4M_0 m_1^2 \|F(x_0)\|}}{2M_0 m_1}, \tag{2.10}$$

$$r_2 = \frac{1 + \sqrt{1 - 4M_0 m_1^2 \|F(x_0)\|}}{2M_0 m_1}, \tag{2.11}$$

(2.4) *holds, and $h_A \ne 0$, when $r = r_2$.*

Then there exists a unique solution $x = x(t)$, $t \in [0, +\infty)$, to system (1.3) *in* $U(x_0, r)$, *and* $\lim_{t\to+\infty} x(t) = x^\star$, *where $x^\star$ is a solution of equation $F(x) = 0$. Moreover, the following estimates hold:*

$$\|x(t) - x^\star\| \le r\mathrm{e}^{-c_1(r)t}, \quad c_1(r) = 1 - M_0 m_1 r, \tag{2.12}$$

$$\|F(x(t))\| \le \|F(x_0)\|\mathrm{e}^{-c_1(r)t}. \tag{2.13}$$

Proof. As in [14], set

$$\Phi(x(t)) = -F'(x_0)^{-1}F(x(t)). \tag{2.14}$$

Using (1.5) we can obtain in turn for all $y \in U(x_0, r)$:

$$\begin{aligned}
&(F'(y)\Phi(y), F(y)) = -(F'(y)[F'(x_0)]^{-1}F(y), F(y)) \\
&= -\|F(y)\|^2 + (\{I - F'(y)[F'(x_0)]^{-1}\}F(y), F(y)) \\
&= -\|F(y)\|^2 + (\{(F'(x_0) - F'(y))F'(x_0)^{-1}\}F(y), F(y)) \\
&\le -\|F(y)\|^2 + M_0 m_1 r\|F(y)\|^2 \\
&= -(1 - M_0 m_1 r)\|F(y)\|^2 = -c_1(r)\|F(y)\|^2,
\end{aligned} \tag{2.15}$$

$$\|\Phi(y)\| \le m_1\|F(y)\|. \tag{2.16}$$

Choose

$$c_1 = c_1(r) \quad \text{and} \quad c_2 = m_1. \tag{2.17}$$

We have:

$$\frac{c_2\|F(x_0)\|}{c_1} = \frac{m_1\|F(x_0)\|}{c_1(r)} = r, \tag{2.18}$$

which holds true by (2.9), and the choices of r_1 and r_2. The result now follows by Lemma 2.1. That completes the proof of Theorem 2.2. □

Remark 2.3. (a) If $D = X = Y$, and $M_0 = M$, then Theorem 2.2 reduces to Theorem 2.3 in [14].

(b) The conditions used in [14] under (1.4), and $D = X = Y$ are

$$h_R = 4Mm_1^2\|F(x_0)\| \le 1, \tag{2.19}$$

whereas the corresponding error estimates are:

$$\|x(t) - x^\star\| \le 2m_1\|F(x_0)\|\mathrm{e}^{-\frac{t}{2}}, \tag{2.20}$$

$$\|F(x(t))\| \le \|F(x_0)\|\mathrm{e}^{-\frac{t}{2}}. \tag{2.21}$$

By comparing (2.9) to (2.19), we see that

$$h_R \le 1 \quad \Rightarrow \quad h_A \le 1, \tag{2.22}$$

but not necessarily viceversa unless $M_0 = M$. In view of (2.22), the applicability of (MNM) has been extended. Moreover, in view of (2.12), (2.13), (2.20), and (2.21), our error estimates are tighter, and the information on the location of the solution at least as precise.

(c) In order to obtain a result for the continuous Newton method

$$\dot{x}(t) = -[F'(x(t))]^{-1}F(x(t)), \quad x(0) = x_0 \in X, \tag{2.23}$$

set $\Phi(y) = -[F'(y)]^{-1}F(y)$,

$$c_2 = \frac{m_1}{1 - m_1 r M_0} \quad \text{and} \quad c_1 = 1. \tag{2.24}$$

Indeed, in view of (1.5), we have:

$$\|F'(x_0)^{-1}\|\|F'(x_0) - F'(x)\| \le m_1 M_0 \|x - x_0\| \le m_1 M_0 r < 1. \tag{2.25}$$

It follows from (2.25) and the Banach lemma on invertible operators [5], [6], [7] that $F'(x)^{-1}$ exists and

$$\|F'(x)^{-1}\| \le \frac{m_1}{1 - m_1 M_0 r}, \tag{2.26}$$

from which the definition of c_2 follows.

(d) A simple iteration method is obtained for $\Phi(y) = -F(y)$. Simply, choose $F' \ge c_1 > 0$, and $c_2 = 1$.

(e) The gradient method is obtained for $\Phi(y) = -[F'(y)]^\star F(y)$. Choose $c_2 = M_1 = \sup_{x \in U(x_0,r)} \|F'(x)\|$, and $c_1 = \mu_1^{-2}$. Here $\mu_1 = \sup_{x \in U(x_0,r)} \|F'(x)^{-1}\|$.

(f) The continuous Gauss-Newton's method is obtained for

$$\Phi(y) = -[F'^\star(y)F'(y)]^{-1}F'^\star(y)F(y).$$

Set $c_1 = 1$, and $c_2 = \mu_1^2 M_1$, where μ_1 and M_1 are the same as in (e) above.

(g) In order to make the comparison easier with the results in [14], we also provided the results in non-affine invariant form. However, we note that the results can be immediately obtained in affine invariant form if the operator F is replaced by $F'(x_0)^{-1}F$. In this case one should set $m_1 = 1$.

The advantages of presenting results in affine instead of non-affine invariant form are well-known in the literatures (see e.g. [7]).

3. LOCAL CONVERGENCE: ILL-POSED CASE

A problem is ill-posed when the Fréchet derivative is not boundedly invertible. It was then suggested in [14] the regularized version of (MNM):

$$\dot{x}(t) = -[F'(x_0) + \varepsilon(t)I]^{-1}(F(x(t)) + \varepsilon(t)(x(t) - x_0)),\ x(0) = x_0 \in H,\ 0 < \varepsilon(t),$$

where x_0 is chosen so that $(F'(x_0)y, y) \ge 0$ for all $y \in D$.

By simply exchanging hypothesis (1.5) in Theorem 3.1 [14] by the analog of (1.4) for the local case, we arrive at:

THEOREM 3.1. *Let F, D, X, Y be as in Theorem 2.2. Assume that:*

(i) *$x^\star$ is a solution of equation $F(x) = 0$;*

(ii) *There exists a positive function $\varepsilon(t) \in C^1[0, +\infty)$ converging monotonically to zero as $t \to +\infty$, such that $\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}$ is nondecreasing, and $\varepsilon(0) > |\dot{\varepsilon}(0)|$;*

(iii) *The operator* F *is Fréchet-differentiable, and* M_0*-center-Lipschitz:*

$$\|F'(x) - F'(x^\star)\| \leq M_0 \|x - x^\star\|,$$

with $F'(x) = F'(x_0)G(x_0, x)$, $G(x_0, x) \in L(X)$, $\|G(x_0, x) - I\| \leq C(G)\|x - x_0\|$ *for all* $x, x_0 \in U(x^\star, \rho)$, $C(G) > 0$,

$$\rho = \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{M_0 + C(G)\varepsilon(0)};$$

(iv) $F'(x_0)$ *is non-negative definite:* $(F'(x_0)y, y) \geq 0$ *for all* $y \in X$;
(v) *there exists* $v \in X$ *such that* $x^\star - x_0 = F'(x_0)v$,

$$\varepsilon(0) - |\dot{\varepsilon}(0)| \geq [M_0 + C(G)\varepsilon(0)]\varepsilon(0)\sqrt{\frac{2\|v\|}{M_0}},$$

and $U(x_0, \rho) \subseteq D$.

Then there exists a unique solution $x = x(t)$ *of* (3.1) *for all* $t \in [0, +\infty)$ *such that*

$$\|x(t) - x^\star\| \leq \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{\varepsilon(0)[M_0 + C(G)\varepsilon(0)]}\varepsilon(t).$$

Corollary 3.2. *Let the operator* F *in* (1.1) *have the form* $F(x) = \Psi(x) - z$. *Assume that* Ψ *is given, and instead of* z, *we have a* δ*-approximation* z_δ: $\|z - z_\delta\| \leq \delta$. *Then the following estimates hold:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - x^\star\|^2 \leq -(1 - C(G)\rho)\|x(t) - x^\star\|^2 + (\varepsilon(t)\|v\| + \frac{\delta}{\varepsilon(t)})\|x(t) - x^\star\| + \frac{M_0}{2\varepsilon(t)}\|x(t) - x^\star\|^3,$$

$$\dot{f}(t) \leq -(1 - C(G)\rho - \frac{|\dot{\varepsilon}(0)|}{\varepsilon(0)})f(t) + 2\|v\| + \frac{M_0}{2}f^2(t), \quad f(0) = \frac{\|x_0 - x^\star\|}{\varepsilon(0)},$$

$$\|x(\tau_\delta) - x^\star\| \leq \frac{\rho}{\varepsilon(0)\|v\|^{\frac{1}{2}}}\delta^{\frac{1}{2}},$$

provided that the hypotheses of Theorem 3.1 *and the following hold:*

$$\varepsilon(0) - |\dot{\varepsilon}(0)| \geq 2[M_0 + C(G)\varepsilon(0)]\varepsilon(0)\sqrt{\frac{\|v\|}{M_0}}.$$

Remark 3.3. If $D = X = Y$, and $M_0 = M$, then Theorem 3.1 and Corollary 3.2 reduce to the corresponding ones in [14]. Otherwise these results constitute an improvement with advantages as stated in the local case of the introduction of this study.

4. APPLICATIONS

We provide two examples in a Hilbert space setting where Lipschitz condition (1.4) does not hold but center-Lipschitz condition (1.5) does.

EXAMPLE 4.1. Let $X = Y = \mathbb{R}$. $\mathbb{R}$ is a Hilbert space with the inner product

$$< x, y >= xy. \tag{4.1}$$

In fact, from (4.1) we obtain:

$$\|x\| =< x, x >^{\frac{1}{2}}= (x^2)^{\frac{1}{2}} = |x|. \tag{4.2}$$

Using (4.2) the Euclidean metric is defined by

$$\|x - y\| =< x - y, x - y >^{\frac{1}{2}}= |x - y|. \tag{4.3}$$

Let $D = [0, +\infty)$, $x_0 = 1$, and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1 + \frac{1}{i}} + c_1 x + c_2, \tag{4.4}$$

where $c_1, c_2 \in \mathbb{R}$, and $i > 2$ is an integer. Then using (4.4), we get:

$$F'(x) = x^{\frac{1}{i}} + c_1, \tag{4.5}$$

which is not Lipschitz in any neighborhood of 0. That is, the operator F' is not Lipschitz on D, i.e. (1.4) is not satisfied. However, center-Lipschitz condition (1.5) holds for $M_0 = 1$. Indeed, using (4.2) we have in turn:

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{\frac{1}{i}} - x_0^{\frac{1}{i}}| \\ &= \frac{|x-x_0|}{x_0^{\frac{i-1}{i}} + x_0^{\frac{i-2}{i}} x^{\frac{1}{i}} + \ldots + x_0^{\frac{1}{i}} x^{\frac{i-2}{i}} + x^{\frac{i-1}{i}}} \le \|x - x_0\|, \end{aligned} \tag{4.6}$$

which shows (1.5).

EXAMPLE 4.2. Let $X = H^1[a, b]$, $Y = L^2[a, b]$, where Y is the completion of the normed space which consists of all continuous real-valued functions on $[a, b]$ with the norm defined by

$$\|x\| = \left(\int_a^b |x(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}}. \tag{4.7}$$

The norm in (4.7) can be obtained from the inner product defined by

$$< x, y >= \int_a^b x(t)y(t)\mathrm{d}t. \tag{4.8}$$

Hence X, Y so defined are Hilbert spaces. Let us consider the integral equation:

$$u(s) = f(s) + \lambda \int_a^b G(s, t)u(t)^{1+\frac{1}{n}} \mathrm{d}t, \quad n \in N. \tag{4.9}$$

Here, f is a given continuous function satisfying $f(s) > 0$, $s \in [a, b]$, λ is a real number, and the kernel $G \in L^\infty([a, b]^2)$.

For example, when $G(s,t)$ is the Green's kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \lambda u^{1+\frac{1}{n}}, \quad u(a) = f(a), \quad u(b) = f(b). \tag{4.10}$$

These type of problems have been considered in [7], [8]. Equations of the form (4.9) generalize equations of the type

$$u(s) = \int_a^b G(s,t)u(t)^n \mathrm{d}t \quad [8]. \tag{4.11}$$

Instead of (4.9) we solve equation $F(u) = 0$, where $F : D \subseteq X \to Y$, $D = \{u \in X : u(s) \geq 0, s \in [a,b]\}$, and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)u(t)^{1+\frac{1}{n}} \mathrm{d}t. \tag{4.12}$$

The derivative F' of the operator F is given by

$$F'(u)v(s) = v(s) - \lambda \left(1 + \frac{1}{n}\right) \int_a^b G(s,t)u(t)^{\frac{1}{n}} v(t)\mathrm{d}t, \quad v \in D. \tag{4.13}$$

We shall first show that the operator F' does not satisfy (1.4) in D. Let us consider for instance, $[a,b] = [0,1]$, $G(s,t) = 1$, and $y(t) = 0$. Then using (4.13), we have:

$$\|F'(x) - F'(y)\|_{L^2}^2 = \left(|\lambda| \left(1 + \frac{1}{n}\right)\right)^2 \int_0^1 \left(\int_0^1 x(\theta)^{\frac{1}{n}} \mathrm{d}\theta\right)^2 \mathrm{d}t. \tag{4.14}$$

If the operator F' satisfies (1.4), then:

$$\|F'(x) - F'(y)\|_{L^2} \leq A\|x - y\|_{L^\infty} \tag{4.15}$$

for some $A > 0$, or equivalently:

$$\left(\int_0^1 \left(\int_0^1 x(\theta)^{\frac{1}{n}} \mathrm{d}\theta\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} \leq B\|x\|_{L^\infty}, \tag{4.16}$$

would hold for all $x \in D$, and some constant $B > 0$. But this is not true. Consider, for example the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1]. \tag{4.17}$$

Note that $x_j(t) \in L^2[0,1]$ $j \geq 1$, $t \in [0,1]$. If these are substituted in (4.16), we have:

$$\frac{1}{j^{\frac{1}{n}}\left(1 + \frac{1}{n}\right)} \leq \frac{B}{j} \Leftrightarrow j^{1-\frac{1}{n}} \leq B\left(1 + \frac{1}{n}\right) \quad \text{for all } j \geq 1, \tag{4.18}$$

which does not hold when $j \to \infty$. Therefore, condition (1.4) fails in this case.

However, condition (1.5) holds. In order to show this, let $x_0 = f(t)$, $d = \min_{s\in[a,b]} f(s)$, $d > 0$, $d_1 = \|G\|_{L^\infty}$. Then we have using (4.7):

$$\begin{aligned}
&\|F'(x) - F'(x_0)\|_{L^2}^2 \\
&\leq \left(|\lambda|\left(1+\frac{1}{n}\right)\right)^2 \int_a^b \left[\int_a^b G(s,\theta)(x(\theta)^{\frac{1}{n}} - x_0(\theta)^{\frac{1}{n}})\mathrm{d}\theta\right]^2 \mathrm{d}t \\
&\leq d_1^2\left(|\lambda|\left(1+\frac{1}{n}\right)\right)^2 \int_a^b \int_a^b \frac{(x(\theta)-x_0(\theta))^2\mathrm{d}\theta\mathrm{d}t}{\left(x(\theta)^{\frac{n-1}{n}} + x(\theta)^{\frac{n-2}{n}}x_0(\theta)^{\frac{1}{n}} + \ldots + x_0(\theta)^{\frac{n-1}{n}}\right)^2} \\
&\leq d_1^2\left(|\lambda|\left(1+\frac{1}{n}\right)\right)^2 \int_a^b \int_a^b |x(\theta)-x_0(\theta)|^2\mathrm{d}\theta\mathrm{d}t = L_0\|x-x_0\|_{L^2}^2,
\end{aligned}$$

where

$$L_0 = \frac{\left(|\lambda|(1+\frac{1}{n})\right)^2 d_1^2}{d^{\frac{2(n-1)}{n}}}. \tag{4.19}$$

That is (1.5) holds for $M_0 = L_0^{\frac{1}{2}}$.

Example 4.3. Let $X = H^1[0,1]$, and $Y = L^2(0,1)$. Consider the operator F defined on X by $F(x)(s) = \int_0^1 K(s,t)P(t,x(t))\mathrm{d}t - v$, $s \in [0,1]$, where $K(s,t) \in L^\infty((0,1)^2)$, $P(t,u)$ is continuously differentiable with respect to u on $0 \leq t,s \leq 1$, $-\infty < u < +\infty$, and $v \in X$ is given. Then we get:

$$(F'(x)h)(s) = \int_0^1 K(s,t)P_x(t,x(t))h(t)\mathrm{d}t. \tag{4.20}$$

Moreover, assume

$$\|P_x(t,x(t)) - P_y(t,y(t))\|_{L^2} \leq \overline{M}\|x-y\|_{L^\infty} \quad \text{for all } x,y \in X. \tag{4.21}$$

Let $x_0 \in X$ be fixed. Then it follows from (4.21) that there exists $\overline{M_0}$ such that $\overline{M_0} \in [0,\overline{M}]$, and

$$\|P_x(t,x(t)) - P_{x_0}(t,x_0(t))\|_{L^2} \leq \overline{M_0}\|x-x_0\|_{L^\infty} \quad \text{for all } x \in X. \tag{4.22}$$

Using (4.20), and (4.21) we obtain in turn:

$$\begin{aligned}
\|(F'(x)-F'(y))h\|_{L^2} &= \left\{\int_0^1 \left[\int_0^1 K(t,\theta)P_x(t,x(\theta) - P_y(t,y(\theta))h(t)\mathrm{d}\theta\right]^2 \mathrm{d}t\right\}^{\frac{1}{2}} \\
&\leq \overline{M}\|K\|_{L^\infty}\|x-y\|_{L^\infty}\|h\|_{L^2}.
\end{aligned}$$

Set $M = \overline{M}\|K\|_{L^\infty}$. Similarly, using (4.20) and (4.22) we get:

$$\|(F'(x)-F'(x_0))h\|_{L^2} \leq \overline{M_0}\|K\|_{L^\infty}\|x-x_0\|_{L^\infty}\|h\|_{L^2}.$$

Set $M_0 = \overline{M_0}\|K\|_{L^\infty}$. Note that the $L^\infty(0,1)$-norms of $x-y$, $x-x_0$, and h can be estimated by their X-norms times some constants, due to Sobolev's embedding theorems. Conditions (1.4), (1.5) are now satisfied with the above choices of M and M_0. Clearly, one can now choose K, P, x_0 so that: $M_0 \leq M$

holds as a strict inequality, our condition (2.9) holds but the condition (2.19) given in [14] is violated.

These advantages extend in the case of the example given in Remark 3.3 [14] related to problem 3.1. Indeed, on top of the above choices of M_0 and M, consider an operator $T : H^1[0,1] \to H^1[0,1]$, and solve equation $T(x) = 0$. Clearly, $T'(x_0)$ is non-negative definite. Under the additional assumptions $|P_x(t, x_0)| \geq \gamma > 0$ for any $t \in (0,1)$, and $y(t,u) \in C^3((0,1) \times (-\infty, +\infty))$, we can set as in [14]: $(G(x_0, x)h)(t) = \frac{P_x(t,x(t))}{P_x(t,x_0(t))} h(t)$ to obtain $(T'(x)h)(t) = (T'(x_0)G(x_0,x)h)(t)$. Then for any $h \in H$ we get in turn [10], [14]:

$$\|(G(x_0,x) - I)h\|_{L^2} \leq \frac{\|P_{xx}\|_{L^\infty}}{\gamma} \|x - x_0\|_{L^\infty} \|h\|_{L^2},$$

$$\left\| \frac{d}{\mathrm{d}t}(G(x_0,x) - I)h \right\|_{L^2} \leq \frac{\|P_{xx}\|_{L^\infty}}{\gamma} \left[\|x' - x_0'\|_{L^2} \|h\|_{L^\infty} + \|x - x_0\|_{L^\infty} \|h'\|_{L^2} \right].$$

In the remaining examples we work on the more general Banach space setting to show that $M_0 < M$. One can replace, e.g. the space ($C[a,b]$, max norm) appropriately to obtain in particular examples in the special case of a Hilbert space (see, for instance Example 4.3).

Example 4.4. Let $X = Y = \mathbb{R}$, $x_0 = 0$, and define scalar functions F by

$$F(x) = c_0 x + c_1 + c_2 \sin \mathrm{e}^{c_3 x}, \tag{4.23}$$

where c_i, $i = 1,2,3$ are given parameters. Using (4.23), it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{M}{M_0}$ can be arbitrarily large.

Example 4.5. Let $X = Y = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x : |x - x_0| \leq 1 - \beta\}$, $\beta \in [0, \frac{1}{2})$, and define function F on U_0 by

$$F(x) = x^3 - \beta. \tag{4.24}$$

Using (4.24), (1.4), and (1.5), we get $M = 6(2-\beta)$, $M_0 = 3(3-\beta)$. Note that $M_0 < M$.

Example 4.6. Let $X = Y = C[0,1]$ be the space of real-valued continuous functions defined on the interval $[0,1]$ with norm $\|x\| = \max_{0 \leq s \leq 1} |x(s)|$.

Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s,t) u(t) \mathrm{d}t + y(s) - \theta. \tag{4.25}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $C[0,1]$. Equations of the form (4.25) arise in the kinetic theory of gasses [7]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \frac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s + t \neq 0$.

If we let $D = U(u_0, 1-\theta)$, and define the operator F on D by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s)\int_0^1 q(s,t)x(t)\mathrm{d}t + y(s) - \theta, \tag{4.26}$$

for all $s \in [0,1]$, then every zero of F satisfies (4.25). We have the estimate:

$$\max_{0\le s\le 1}\left|\int_0^1 \frac{s}{s+t}\mathrm{d}t\right| = \ln 2.$$

Hence it follows from (1.4), (1.5), and (4.26) that $M = 2(|\lambda|\ln 2 + 3(2-\theta))$ and $M_0 = 2|\lambda|\ln 2 + 3(3-\theta)$. Note also that $M_0 < M$ for all $\theta \in [0,1)$.

Example 4.7. Consider the nonlinear boundary value problem [1], [7], [8]:

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation:

$$u(s) = s + \int_0^1 Q(s,t)(u^3(t) + \gamma u^2(t))\mathrm{d}t, \tag{4.27}$$

where Q is the Green function $Q(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$ We observe that $\max_{0\le s\le 1}\int_0^1 |Q(s,t)|\mathrm{d}t = \frac{1}{8}$. Let $X = Y = C[0,1]$, with norm $\|x\| = \max_{0\le s\le 1}|x(s)|$. Then problem (4.27) is in the form (1.1), where $F : D \to Y$,

$$F(x)(s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\mathrm{d}t.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\mathrm{d}t.$$

If we set $u_0(s) = s$, and $D = U(x_0, R)$, where $R > 0$ is a real number, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$.

On the other hand, for $x, y \in D$, we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\mathrm{d}t.$$

Consequently,

$$\begin{aligned} \|F'(x) - F'(y)\| &\le \frac{\|x-y\|(2\gamma + 3(\|x\| + \|y\|))}{8} \\ &\le \frac{\|x-y\|(2\gamma + 6R + 6\|u_0\|)}{8} = \frac{\gamma + 3R + 3}{4}\|x-y\|, \end{aligned}$$

$$\begin{aligned} \|F'(x) - F'(u_0)\| &\le \frac{\|x-u_0\|(2\gamma + 3(\|x\| + \|u_0\|))}{8} \\ &\le \frac{\|x-u_0\|(2\gamma + 3R + 6\|u_0\|)}{8} = \frac{2\gamma + 3R + 6}{8}\|x-u_0\|. \end{aligned}$$

We have $M = \frac{\gamma+3R+3}{4}$ and $M_0 = \frac{2\gamma+3R+6}{8}$. Note also that $M_0 < M$.

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