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INTERVAL-VALUED (α, β)-FUZZY SUBGROUPS I

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Abstract. Using the "belongs to" relation $(\in)$ and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets, the new concept of interval-valued (α, β)-fuzzy subgroup is introduced, where α and β are any of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and related properties are investigated. We provide characterizations of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup and study their related properties.

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1. INTRODUCTION

Following the introduction of fuzzy sets by L.A. Zadeh [15], the fuzzy set theory developed by Zadeh himself and others have found many applications in the domain of mathematics and elsewhere. The study of the fuzzy algebraic structures has started with the introduction of the concepts of fuzzy (subgroupoids) subgroups and fuzzy (left, right) ideals in the pioneering paper of Rosenfeld [12]. In 1979, Anthony and Sherwood [1] redefined fuzzy (subgroupoids) subgroups using the concept of triangular norm. Das [5] defined level subgroups and used them to study fuzzy groups. Antony and Sherwood [2] began to investigate the structure of fuzzy subgroups. Several mathematicians have followed the Rosenfeld - Antony - Sherwood approach in investigating the fuzzy subgroup theory.

Later on, Zadeh [1] also introduced the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). The interval-valued fuzzy subgroups were first defined and studied by Biswas [4], and are the subgroups of the same nature of the fuzzy subgroups defined by Rosenfeld. In [12] Zeng et al. gave a kind of method to describe the entropy of interval-valued fuzzy set based on its similarity measure and discussed their relationship between the similarity measure and the entropy of the interval-valued fuzzy sets in detail. However, the obtained results can still be applied in many fields such as pattern recognition, image processing and fuzzy reasoning etc. The definition of a fuzzy subgroup with thresholds, which is a generalization of Rosenfeld's fuzzy subgroup and Bhakat and Das's fuzzy

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subgroup, was introduced by Xuehai Yuan et al. Saeid [1] introduced interval-valued fuzzy BG-algebras and determined the relationship between them and BG-subalgebras. A new type of fuzzy subgroup (that is, the $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets. In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. With this objective in view, Davvaz [6] applied this theory to near-rings and obtained some useful results. The notion of interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras was discussed by Jianming Zhan et al. [15]. Later Xueling Ma et al. [12] introduced the notion of interval-valued $(\in, \in \vee q)$-fuzzy implicative ideals of pseudo-MV algebras and discussed some characterization theorems of these generalized fuzzy implicative ideals. Using the "belongs to" relation $(\in)$ and quasi-coincidence with the relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subalgebra on BCK-algebra was introduced, where (α, β) are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and related properties were investigated in [1]. As a continuation of paper [1], relations between a fuzzy subalgebra with thresholds and an $(\in, \in \vee q)$-fuzzy subalgebra were discussed in [2, 3]. The following notions were introduced and some of their related properties were investigated in [2]: interval-valued $(\in, \in \vee q)$-fuzzy subgroup, interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds by using their level subgroups, T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup, the direct product and T_G-product of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup and T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds by using their level subgroups.

In this paper, using the "belongs to" relation $(\in)$ and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets, a new concept of interval-valued (α, β)-fuzzy subgroup is introduced, where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and related properties are investigated. We provide characterizations of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup.

2. PRELIMINARIES

In this section, we recall some basic definitions for the sake of completeness.

Definition 1. ([12]) Let μ be a fuzzy set in a group G. Then μ is called a *fuzzy subgroup* of G if
$(FS1)$ $(\forall x,\, y \in G)$ $(\mu(xy) \geq \min\{\mu(x),\, \mu(y)\})$,
$(FS2)$ $(\forall x \in G)$ $(\mu(x^{-1}) \geq \mu(x))$.

An interval-valued fuzzy set F defined on G is given by

$$F = \{(x, [\mu_F^L(x), \mu_F^U(x)])\},$$

for all $x \in G$. Briefly denote F by $F = [\mu_F^L, \mu_F^U]$, where μ_F^L and μ_F^U are any two fuzzy sets in G such that $\mu_F^L(x) \leq \mu_F^U(x)$ for all $x \in G$. For simplicity let $\hat{\mu}_F(x) = [\mu_F^L(x), \mu_F^U(x)]$, for all $x \in G$.

Let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$. It is clear that if $\mu_F^L(x) = \mu_F^U(x) = c$, where $0 \le c \le 1$, then $\hat{\mu}_F(x) = [c, c]$ is in $D[0,1]$. Thus $\hat{\mu}_F(x) \in D[0,1]$, for all $x \in G$. Therefore the interval-valued fuzzy set F is given by $F = \{x, \hat{\mu}_F(x)\}$, for all $x \in G$, where $\hat{\mu}_F : G \to D[0,1]$. Now we define the notion of refined minimum (briefly, $r\min$) and the order "$\le$" on two elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as :

$$r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}],$$

respectively,

$$D_1 \le D_2 \Leftrightarrow a_1 \le a_2 \wedge b_1 \le b_2.$$

Similarly one can define $\ge$ and $=$.

One can extend the concept of fuzzy subgroup to the concept of interval-valued fuzzy subgroup of G as follows:

Definition 2. ([4]) An interval-valued fuzzy set F in G is called an *interval-valued fuzzy subgroup* of G if

(i) $(\forall x,\, y \in G)$ $(\hat{\mu}_F(xy) \ge r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\})$,
(ii) $(\forall x \in G)$ $(\hat{\mu}_F(x^{-1}) \ge \hat{\mu}_F(x))$.

An interval-valued fuzzy set $F = \{x, \hat{\mu}_F(x) | \, x \in G\}$ of the form

$$\hat{\mu}_F(x) = \begin{cases} \hat{t}(\ne [0,0]) & \text{if} \quad x = y \\ [0,0] & \text{if} \quad x \ne y \end{cases}$$

is said to be *an interval-valued fuzzy point* with support y and the interval-valued $\hat{t}$, and is denoted by $x_{\hat{t}}$. For an interval-valued fuzzy point $x_{\hat{t}}$ and an interval-valued fuzzy set $\hat{\mu}_F$ in a set G, Pu and Liu [1] gave meaning to the symbol $x_{\hat{t}} \alpha \hat{\mu}_F$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

To say that $x_{\hat{t}} \in \hat{\mu}_F$ (respectively, $x_{\hat{t}} q \hat{\mu}_F$) means that $\hat{\mu}_F(x) \ge \hat{t}$ (respectively, $\hat{\mu}_F(x) + \hat{t} > [1,1]$), and in this case $x_{\hat{t}}$ is said to *belong to* (respectively, is *quasi-coincident with*) an interval-valued fuzzy set $\hat{\mu}_F$.

To say that $x_{\hat{t}} \in \vee q \hat{\mu}_F$ (respectively, $x_{\hat{t}} \in \wedge q \hat{\mu}_F$) means that $x_{\hat{t}} \in \hat{\mu}_F$ or $x_{\hat{t}} q \hat{\mu}_F$ (respectively, $x_{\hat{t}} \in \hat{\mu}_F$ and $x_{\hat{t}} q \hat{\mu}_F$).

For all $\hat{t}_1, \hat{t}_2 \in [0,1]$, $r\min\{\hat{t}_1, \hat{t}_2\}$ will be denoted by $m(\hat{t}_1, \hat{t}_2)$.

3. (α, β)- FUZZY SUBGROUPS

In what follows, let α and β denote anyone of the symbols $\in$, q, $\in \vee q$, or $\in \wedge q$, and let G be a group, unless otherwise specified. To say that $x_{\hat{t}} \bar{\alpha} \hat{\mu_F}$ means that $x_{\hat{t}} \alpha \hat{\mu_F}$ does not hold.

Proposition 1. *For any interval-valued fuzzy set $\hat{\mu}_F$ in G, the condition* (i) *of* Definition 2 *is equivalent to the following condition* (iii)*, while* (ii) *is equivalent to* (iv)*, where*:

(iii) $(\forall x, y \in G$ *and* $\forall \hat{t}_1, \hat{t}_2 \in (0,1])$ $(x_{\hat{t}_1}, y_{\hat{t}_2} \in \hat{\mu}_F \Rightarrow (xy)_{m(\hat{t}_1, \hat{t}_2)} \in \hat{\mu}_F)$.
(iv) $(\forall x \in G$ *and* $\forall \hat{t} \in (0,1])$ $(x_{\hat{t}} \in \hat{\mu}_F \Rightarrow x_{\hat{t}}^{-1} \in \hat{\mu}_F)$.

Proof. Assume that the condition (i) is valid. Let $x, y \in G$ and $\hat{t}_1, \hat{t}_2 \in (0,1]$ be such that $x_{\hat{t}_1}, y_{\hat{t}_2} \in \hat{\mu}_F$. Then $\hat{\mu}_F(x) \geq \hat{t}_1$ and $\hat{\mu}_F(y) \geq \hat{t}_2$ which imply from (i) that

$$\begin{aligned}\hat{\mu}_F(xy) &\geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} \\ &\geq m\{\hat{t}_1, \hat{t}_2\}.\end{aligned}$$

Hence $(xy)_{m(\hat{t}_1,\hat{t}_2)} \in \hat{\mu}_F$.

Let $x \in G$ and $\hat{t} \in (0,1]$ be so that $x_{\hat{t}} \in \hat{\mu}_F$. Then $\hat{\mu}_F(x) \geq \hat{t}$ which implies from (ii) that $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x) \geq \hat{t}$. Hence $x_{\hat{t}}^{-1} \in \hat{\mu}_F$.

Conversely, suppose that the condition (iii) is valid. Note that $x_{\hat{\mu}_F(x)} \in \hat{\mu}_F$ and $y_{\hat{\mu}_F(y)} \in \hat{\mu}_F$ for all $x, y \in G$. Thus, by (iii),

$$(xy)_{m(\hat{\mu}_F(x),\hat{\mu}_F(y))} \in \hat{\mu}_F,$$

so $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$. Suppose now that the condition (iv) is valid. Observe that $x_{\hat{\mu}_F(x)} \in \hat{\mu}_F$ for all $x \in G$. Thus $x^{-1}_{\hat{\mu}_F(x^{-1})} \in \hat{\mu}_F$, by (iv), and so $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x)$. This finishes the proof. $\square$

Note that if $\hat{\mu}_F$ is an interval-valued fuzzy set in G defined by $\hat{\mu}_F(x) \leq [0.5, 0.5]$ for all $x \in G$, then the set $\{x_{\hat{t}} | \, x_{\hat{t}} \in \wedge q\hat{\mu}_F\}$ is empty.

Definition 3. An interval-valued fuzzy set $\hat{\mu}_F$ in G is called an *interval-valued (α, β)-fuzzy subgroup* of G, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions:

(v) $(\forall x, y \in G)$ and $(\forall \hat{t}_1, \hat{t}_2 \in (0,1])$ $(x_{\hat{t}_1}\alpha\hat{\mu}_F, y_{\hat{t}_2}\alpha\hat{\mu}_F \Rightarrow (xy)_{m(\hat{t}_1,\hat{t}_2)}\beta\hat{\mu}_F)$,

(vi) $(\forall x \in G$ and $\forall \hat{t} \in (0,1])$ $(x_{\hat{t}}\alpha\hat{\mu}_F \Rightarrow x_{\hat{t}}^{-1}\beta\hat{\mu}_F)$.

Example 1. Let $G = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(G, *)$ is a group. Let $\hat{\mu}_F$ be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_F(x) = \begin{cases} [1,1] & \text{if } x = 0 \\ [0.2, 0.3] & \text{if } x \in [1,3] \end{cases}.$$

Then clearly $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G with the following properties:

(a) $\hat{\mu}_F$ is not an interval-valued $(\in, \in)$-fuzzy subgroup of G since $1_{[0.8,0.84]} \in \hat{\mu}_F$ and $1_{[0.86,0.88]} \in \hat{\mu}_F$, but $(1 * 1)_{m\{[0.8,0.84],[0.86,0.88]\}} = 0_{[0.8,0.84]} \overline{\in} \hat{\mu}_F$.

(b) $\hat{\mu}_F$ is not an interval-valued $(q, \in \vee q)$-fuzzy subgroup of G, since we have $1_{[0.41,0.43]} q \hat{\mu}_F$ and $2_{[0.77,0.86]} q \hat{\mu}_F$, but

$$(1 * 2)_{m\{[0.4,0.43],[0.77,0.86]\}} = 0_{[0.4,0.43]} \overline{\in \vee q} \hat{\mu}_F.$$

(c) $\hat{\mu}_F$ is not an interval-valued $(\in \vee q, \in \vee q)$-fuzzy subgroup of G, since $1_{[0.51,0.55]} \in \vee q \hat{\mu}_F$ and $3_{[0.67,0.77]} \in \vee q \hat{\mu}_F$, but

$$(1 * 3)_{m\{[0.51,0.55],[0.67,0.77]\}} = 3_{[0.51,0.55]} \overline{\in \vee q} \hat{\mu}_F.$$

Theorem 1. *Every interval-valued* $(\in \vee q, \in \vee q)$*-fuzzy subgroup is an interval-valued* $(\in, \in \vee q)$*-fuzzy subgroup.*

Proof. Let $\hat{\mu}_F$ be an interval-valued $(\in \vee q, \in \vee q)$-fuzzy subgroup of G. Let $x, y \in G$ and $\hat{t}_1, \hat{t}_2 \in (0,1]$ be such that $x_{\hat{t}_1} \in \hat{\mu}_F$ and $y_{\hat{t}_2} \in \hat{\mu}_F$. Then $x_{\hat{t}_1} \in \vee q \hat{\mu}_F$ and $y_{\hat{t}_2} \in \vee q \hat{\mu}_F$, hence $(xy)_{m(\hat{t}_1,\hat{t}_2)} \in \vee q \hat{\mu}_F$. Let $x \in G$ and $\hat{t} \in (0,1]$ be such that $x_{\hat{t}} \in \hat{\mu}_F$. Then $x_{\hat{t}} \in \vee q \hat{\mu}_F$ which implies that $x^{-1}_{\hat{t}} \in \vee q \hat{\mu}_F$. Thus $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. □

Theorem 2. *Every interval-valued* $(\in, \in)$*-fuzzy subgroup is an interval-valued* $(\in, \in \vee q)$*-fuzzy subgroup.*

Proof. Let $\hat{\mu}_F$ be an interval-valued $(\in, \in)$-fuzzy subgroup of G. Let $x, y \in G$ and $\hat{t}_1, \hat{t}_2 \in (0,1]$ be such that $x_{\hat{t}_1} \in \hat{\mu}_F$ and $y_{\hat{t}_2} \in \hat{\mu}_F$. Then $\hat{\mu}_F(x) \geq \hat{t}_1$ and $\hat{\mu}_F(y) \geq \hat{t}_2$. Thus we have that the relations

$$\hat{\mu}_F(x) + \hat{t}_1 > [1,1], \quad \hat{t}_1 + \hat{t}_1 > [1,1], \quad 2\hat{t}_1 > [1,1]$$

are true for all $\hat{t}_1 \in (0,1]$. Hence $x_{\hat{t}_1} q \hat{\mu}_F$.

Since $x_{\hat{t}_1} \in \hat{\mu}_F$ or $x_{\hat{t}_1} q \hat{\mu}_F$, we get that $x_{\hat{t}_1} \in \vee q \hat{\mu}_F$. Similarly one obtains that $y_{\hat{t}_2} \in \vee q \hat{\mu}_F$.

Suppose now that $(xy)_{m(\hat{t}_1,\hat{t}_2)} \overline{\in \vee q} \hat{\mu}_F$. Then

$$\hat{\mu}_F(xy) < m\{\hat{\mu}_F(x), \hat{\mu}_F(y)\},$$

a contradiction with the fact that $x_{\hat{t}_1} \in \hat{\mu}_F$ and $y_{\hat{t}_2} \in \hat{\mu}_F$. Hence $(xy)_{m(\hat{t}_1,\hat{t}_2)} \in \vee q \hat{\mu}_F$.

Let $x \in G$ and $\hat{t} \in (0,1]$ be such that $x_{\hat{t}} \in \hat{\mu}_F$. Then $\hat{\mu}_F(x) \geq \hat{t}$ and

$$\hat{\mu}_F(x) + \hat{t} > [1,1], \quad \hat{t} + \hat{t} > [1,1], \quad 2\hat{t} > [1,1]$$

are true for all $\hat{t} \in (0,1]$. Hence $x_{\hat{t}} q \hat{\mu}_F$.

Since $x_{\hat{t}} \in \hat{\mu}_F$ or $x_{\hat{t}} q \hat{\mu}_F$ yields that $x_{\hat{t}} \in \vee q \hat{\mu}_F$, we obtain that $x^{-1}_{\hat{t}} \in \vee q \hat{\mu}_F$. This finishes the proof. □

Proposition 2. *If* $\hat{\mu}_F$ *is a non-zero interval-valued* (α, β)*-fuzzy subgroup of* G*, then* $\hat{\mu}_F(0) > [0,0]$.

Proof. Assume that $\hat{\mu}_F(0) = [0,0]$. Since $\hat{\mu}_F$ is non-zero, there exists $x \in G$ such that $\hat{\mu}_F(x) = \hat{t} > [0,0]$. If $\alpha = \in$ or $\alpha = \in \vee q$, then $x_{\hat{t}} \alpha \hat{\mu}_F$. On the other hand, $(x.x)_{m(\hat{t},\hat{t})} = 0_{\hat{t}}\overline{\beta}\hat{\mu}_F$, a contradiction. Also, if $\alpha = \in$ or $\alpha = \in \vee q$, then $x_{\hat{t}}\alpha\hat{\mu}_F$. But we have $x_{\hat{t}}^{-1}\overline{\beta}\hat{\mu}_F$, which is a contradiction.

If $\alpha = q$, then $x_1\alpha\hat{\mu}_F$, because $\hat{\mu}_F(x) + [1,1] = \hat{t} + [1,1] > [1,1]$. On the other hand, $(x.x)_{m([1,1],[1,1])} = 0_{[1,1]}\overline{\beta}\hat{\mu}_F$, which leads to a contradiction. Also, if $\alpha = q$, then $x_1\alpha\hat{\mu}_F$, because $\hat{\mu}_F(x) + [1,1] = \hat{t} + [1,1] > [1,1]$. But we have $x_{\hat{t}}^{-1}\bar{\beta}\hat{\mu}_F$, which is a contradiction. Therefore, $\hat{\mu}_F(0) > [0,0]$. This finishes the proof. □

For an interval-valued fuzzy set $\hat{\mu}_F$ in G, we denote

$$G_0 = \{x \in G |\, \hat{\mu}_F(x) > [0,0]\}.$$

THEOREM 3. *The following assertions hold for the set G_0:*

(i) *If $\hat{\mu}_F$ is a non-zero interval-valued $(\in,\in)$-fuzzy subgroup of G, then the set G_0 is a subgroup of G.*
(ii) *If $\hat{\mu}_F$ is a non-zero interval-valued $(\in,q)$-fuzzy subgroup of G, then the set G_0 is a subgroup of G.*
(iii) *If $\hat{\mu}_F$ is a non-zero interval-valued $(q,\in)$-fuzzy subgroup of G, then the set G_0 is a subgroup of G.*
(iv) *If $\hat{\mu}_F$ is a non-zero interval-valued (q,q)-fuzzy subgroup of G, then the set G_0 is a subgroup of G.*

Proof. (i) Let $x,y \in G_0$. Then $\hat{\mu}_F(x) > [0,0]$ and $\hat{\mu}_F(y) > [0,0]$. Suppose that $\hat{\mu}_F(xy) = [0,0]$. Note that $x_{\hat{\mu}_F(x)} \in \hat{\mu}_F$ and $y_{\hat{\mu}_F(y)} \in \hat{\mu}_F$. But $(xy)_{m(\hat{\mu}_F(x),\hat{\mu}_F(y))}\bar{\in}\hat{\mu}_F$, because $\hat{\mu}_F(xy) = [0,0] < m\{\hat{\mu}_F(x),\hat{\mu}_F(y)\}$. This is a contradiction, and thus $\hat{\mu}_F(xy) > [0,0]$, which shows that $xy \in G_0$.

Let $x \in G_0$. Then $\hat{\mu}_F(x) > [0,0]$. Suppose that $\hat{\mu}_F(x^{-1}) = [0,0]$. Note that $x_{\hat{\mu}_F(x)} \in \hat{\mu}_F$. But $x^{-1}_{\hat{\mu}_F(x^{-1})} \in \hat{\mu}_F$, because $\hat{\mu}_F(x^{-1}) = [0,0] < \hat{\mu}_F(x)$. This is a contradiction and thus $\hat{\mu}_F(x^{-1}) > [0,0]$, which shows that $x^{-1} \in G_0$. Therefore G_0 is a subgroup of G.

The assertions (ii), (iii), and (iv) can be proved similarly. □

COROLLARY 1. *If $\hat{\mu}_F$ is one of the following sets*:

(i) *a non-zero interval-valued $(\in,\in \wedge q)$-fuzzy subgroup of G,*
(ii) *a non-zero interval-valued $(\in,\in \vee q)$-fuzzy subgroup of G,*
(iii) *a non-zero interval-valued $(\in \vee q, q)$-fuzzy subgroup of G,*
(iv) *a non-zero interval-valued $(\in \vee q,\in)$-fuzzy subgroup of G,*
(v) *a non-zero interval-valued $(\in \vee q,\in \wedge q)$-fuzzy subgroup of G,*
(vi) *a non-zero interval-valued $(q,\in \wedge q)$-fuzzy subgroup of G,*
(vii) *a non-zero interval-valued $(q,\in \vee q)$-fuzzy subgroup of G,*

then the set G_0 is a subgroup of G.

Proof. It is similar to the proof of the Theorem 3. □

THEOREM 4. *Every non-zero interval-valued (q,q)-fuzzy subgroup of G is constant on G_0.*

Proof. Let $\hat{\mu}_F$ be an non-zero interval-valued (q,q)-fuzzy subgroup of G. Assume that $\hat{\mu}_F$ is not constant on G_0. Then there exists $y \in G_0$ such that $\hat{t}_y = \hat{\mu}_F(y) \neq \hat{\mu}_F(0) = \hat{t}_0$. Then either $\hat{t}_y > \hat{t}_0$ or $\hat{t}_y < \hat{t}_0$. Suppose that $\hat{t}_y < \hat{t}_0$ and choose $\hat{t}_1, \hat{t}_2 \in (0,1]$ such that $[1,1]-\hat{t}_0 < \hat{t}_1 < [1,1]-\hat{t}_y < \hat{t}_2$. Then

$$\hat{\mu}_F(0) + \hat{t}_1 = \hat{t}_0 + \hat{t}_1 > [1,1] \text{ and } \hat{\mu}_F(y) + \hat{t}_2 = \hat{t}_y + \hat{t}_2 > [1,1],$$

so $0_{\hat{t}_1} q \hat{\mu}_F$ and $y_{\hat{t}_2} q \hat{\mu}_F$. Since $\hat{\mu}_F((y.0)) + m(\hat{t}_1,\hat{t}_2) = \hat{\mu}_F(y) + \hat{t}_1 = \hat{t}_y + \hat{t}_1 < [1,1]$, we have $((y.0))_{m(\hat{t}_1,\hat{t}_2)} \bar{q} \hat{\mu}_F$, which is a contradiction.

Next assume that $\hat{t}_y > \hat{t}_0$. Then $\hat{\mu}_F(y) + ([1,1]-\hat{t}_0) = \hat{t}_y + [1,1] - \hat{t}_0 > [1,1]$ and so $y_{([1,1]-\hat{t}_0)} q \hat{\mu}_F$. Since

$$\hat{\mu}_F(y.y) + ([1,1]-\hat{t}_0) = \hat{\mu}_F(0) + [1,1] - \hat{t}_0 = \hat{t}_0 + [1,1] - \hat{t}_0 = [1,1],$$

we get $(y.y)_{m([1,1]-\hat{t}_0,[1,1]-\hat{t}_0)} \bar{q} \hat{\mu}_F$. But this is impossible. Also, condition (vi) is obvious. Therefore $\hat{\mu}_F$ is constant on G_0. □

Theorem 5. ([1]) *An interval-valued fuzzy set $\hat{\mu}_F$ of G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if and only if for all $x,y \in G$ the following two conditions are satisfied*:

(vii) $(\forall x,y \in G)$ $(\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5,0.5]\})$,

(viii) $(\forall x \in G)$ $(\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x))$.

Theorem 6. *Let G_0 be a subgroup of G. For any $\hat{t} \in (0,0.5]$, there exists an interval-valued $(\in, \in \vee q)$-fuzzy subgroup $\hat{\mu}_F$ of G such that $U(\hat{\mu}_F, \hat{t}) = G_0$.*

Proof. Fix $\hat{t} \in (0,0.5]$ and consider $\hat{\mu}_F$ the interval-valued fuzzy set in G defined, for $x \in G$, by

$$\hat{\mu}_F(x) = \begin{cases} \hat{t} & \text{if } x \in G_0 \\ [0,0] & \text{otherwise} \end{cases}.$$

Obviously $U(\hat{\mu}_F, \hat{t}) = G_0$. Assume that $\hat{\mu}_F(xy) < m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5,0.5]\}$ for some $x,y \in G$. Since $\#\ \{\mathrm{Im}(\hat{\mu}_F)\} = 2$, it follows that $\hat{\mu}_F(xy) = [0,0]$ and $m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5,0.5]\} = \hat{t}$. Hence $\hat{\mu}_F(x) = \hat{t} = \hat{\mu}_F(y)$, so that $x,y \in G_0$. But $xy \notin G_0$, a contradiction. So $\hat{\mu}_F(xy) \geq m\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5,0.5]\}$. Condition (vi) is obvious. Using Theorem 3.4 of [1] we conclude that $\hat{\mu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. □

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