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# ON THE DETERMINATION OF 3D AUTONOMOUS FORCE FIELDS PRODUCING TRAJECTORIES THAT ARE SOLUTIONS OF A SYSTEM OF ODES

THOMAS KOTOULAS and NICOLAS CARANICOLAS

**Abstract.** This study is a generalization of a recent work by Bozis and Borghero (2008) establishing connections between autonomous planar force fields and the entire two-parametric set of solutions of a given linear second order ODE (solvable or not). In this paper we find 3D force fields which give rise to a *three-parametric family of spatial orbits*. It is shown that the three-parametric set of all solutions of any system of *linear* ordinary differential equations of the type $y''(x) = f_0(x)+yf_1(x)+zf_2(x)+y'f_3(x)$, $z'(x) = g_0(x)+yg_1(x)+zg_2(x)+y'g_3(x)$ (which may be solvable by quadratures or not) represents a set of regular orbits traced by a material point of unit mass, in the presence of *at least* one autonomous force field $\bar{F}(X, Y, Z)$, for adequate initial conditions. The corresponding force field is determined by quadratures on the grounds of the eight functions $f_i(x)$, $g_k(x)$ $(i, k$=0,1,2,3) which specify the above system of ODEs. Subcases are also studied and pertinent examples are offered.

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**Key words.** Inverse Problems, 3D autonomous force fields, three-parametric families of orbits, ODEs and PDEs.

## 1. INTRODUCTION

The inverse problem of dynamics in a broad sense aims at the determination of forces, parameters and constraints, which are required for the realization of the motion of a mechanical system with some properties given in advance (Galiulin, 1984). Three-dimensional versions of the inverse problem were studied for a two-parametric family of orbits by Bozis(1983) (for general force fields), and by Váradi and Érdi (1983). Other results have been obtained by Bozis and Nakhla (1986), Shorokov (1988) and Puel (1992). These results are summarized in the review paper of Bozis (1995). Recently, Bozis and Kotoulas (2004) studied the case of two-parametric families of straight lines (FSL) produced by genuine three-dimensional potentials. Moreover, the same authors dealt with the construction of 3D homogeneous potentials which give rise to two-parametric families of homogeneous orbits in space. Several examples were given there (Bozis and Kotoulas, 2005). At the same time, Anisiu (2005) produced in a direct way the *two* energy-free PDEs of the three-dimensional

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inverse problem and the region where real motion is allowed, presenting also several families of orbits compatible with 3D potentials.

In all the above mentioned papers the authors studied the following version of the inverse problem of Dynamics: *"A two-parametric family of spatial curves is given in the form: $f(x, y, z) = c_1$, $g(x, y, z) = c_2$. Find all the potentials which generate these curves as trajectories"*. As it was shown by Bozis and Kotoulas (2004), there is an one-to-one correspondence between the curves and the "*slope functions*" $\alpha$ and $\beta$ defined by

$$\alpha = \frac{\delta_2}{\delta_1}, \quad \beta = \frac{\delta_3}{\delta_1}, \quad \bar{\delta} = (\delta_1, \delta_2, \delta_3) = \nabla f \times \nabla g. \tag{1}$$

Furthermore, Bozis (1983) and Anisiu (2005) extended their study to include 3D autonomous force fields.

In a recent paper, Bozis and Borghero (2008) studied an inverse problem of different character: They showed that the two-parametric set of all solutions of any linear ODE of the second order $y'' + a(x)y' + b(x)y = f(x)$ can be considered as a set of orbits traced by a material point of unit mass, in the presence of at least one autonomous force field $\bar{F}(X, Y)$, for adequate initial conditions, and they investigated also the conditions for the existence of potentials producing such set of orbits. In the present article we extend the above work in the 3D-space: Neither the analytic expressions of spatial curves are given in advance, nor the *slope functions* $\alpha$ and $\beta$. We are given only the "mother system of ODEs" from which these orbits arise. In Section 2 we present the basic facts for autonomous force fields. We have two equations: one of them is algebraic and the other one is a PDE (see the equations (4) and (5) of Section 2), which involve the components $X,\ Y,\ Z$ of the corresponding 3D force field and the slope functions. In Section 3 we analyze the system of two ODEs, which represents the families of spatial curves and we explain how we establish two algebraic equations, (14) and (16), the first quadratic in $\alpha$, and the second cubic in $\alpha$. Finally, we find a set of seven conditions on the coefficients of the previous algebraic equations, i.e., a system of PDEs in the unknown components $X,\ Y,\ Z$ of the force field, which must be fulfilled simultaneously. From these relations we derive results for the components of the autonomous force field and the slope functions. Pertinent examples are offered in each case. Finally, conclusions are presented in Section 4.

## 2. BASIC FACTS FOR GENERAL AUTONOMOUS FORCE FIELDS

We suppose that a test particle of unit mass moves in the 3D space. In the present paper we deal with three-parametric families of spatial curves given in the solved form:

$$f(x, y, z, c_3) = c_1, \quad g(x, y, z, c_3) = c_2. \tag{2}$$

Let us consider the equations of motion of a test particle:

$$\ddot{x} = X, \quad \ddot{y} = Y, \quad \ddot{z} = Z, \tag{3}$$

where the force components $X,\ Y,\ Z$ are of $C^1$-class on an open domain in $\mathbb{R}^3$. The equations of the trajectories of (3) read as (Anisiu, 2005):

$$(\alpha B - \beta A)X - BY + AZ = 0 \tag{4}$$

and

$$-X_x + \frac{1}{\alpha}Y_x - \alpha X_y + Y_y + qX_z + pY_z = lX + mY. \tag{5}$$

The coefficients $p, q, l, m$ are given by

$$\begin{aligned} &p = \tfrac{\beta}{\alpha}, \quad q = -\beta, \\ &l = \tfrac{3A}{\alpha} - \alpha m, \quad m = \tfrac{A_x + \alpha A_y + \beta A_z}{\alpha A}, \end{aligned} \tag{6}$$

and

$$A = \alpha_x + \alpha\alpha_y + \beta\alpha_z, \quad B = \beta_x + \alpha\beta_y + \beta\beta_z. \tag{7}$$

REMARK 1. In the above equations we have to assume that $A \neq 0$. Otherwise, we have to make the transformation $(\alpha \to \beta \to \alpha)$ in order to avoid the zeroing of denominators (Bozis and Kotoulas, 2005).

The above equations (4), (5) are necessary and sufficient conditions for the autonomous force field $\bar{F}(X, Y, Z)$ to be compatible with the family (2). For any admissible triplet $< X, Y, Z >$, these equations determine also the region in the 3D-Cartesian space where *real* motion is allowed to take place. This is so when the following inequality holds (Anisiu, 2005):

$$\frac{Y - \alpha X}{A} \geq 0 \tag{8}$$

## 3. ANALYSIS FOR FIRST AND SECOND ORDER ODES

We consider the following system consisting of two linear ordinary differential equations

$$\begin{aligned} y'' &= f_0(x) + yf_1(x) + zf_2(x) + y'f_3(x), \\ z' &= g_0(x) + yg_1(x) + zg_2(x) + y'g_3(x). \end{aligned} \tag{9}$$

The system (9) is then identified with the eight sufficiently smooth functions $f_i,\ g_k,\ i, k \in \{0, 1, 2, 3\}$. The totality of its solutions constitutes a set of spatial orbits depending on three parameters:

$$y = y(x, c_1, c_2, c_3), \quad z = z(x, c_1, c_2, c_3). \tag{10}$$

Now, we set the following question: *Is there any* $3D$ *autonomous force field* $X = X(x, y, z),\ Y = Y(x, y, z),\ Z = Z(x, y, z)$ *which can produce as real orbits, traced by a unit-mass material point, all the members of the three-parametric set of families* (10)?

We shall show that the answer is always *positive*, although it is not generally expected to find an appropriate force field which produces these regular curves as orbits.

We proceed as follows: Firstly, we rewrite the system of (9) in terms of $\alpha$ and $\beta$. More precisely, as we had shown in a previous paper (Bozis and Kotoulas, 2004), for one-dimensional curves $\vec{r} = x\vec{\mathrm{i}} + y(x)\vec{\mathrm{j}} + z(x)\vec{\mathrm{k}}$ parametrized by the coordinate $x$ and defined by $f(x,y,z) = c_1,\ g(x,y,z) = c_2$, we have

$$y' = \alpha(x,y,z), \quad z' = \beta(x,y,z). \tag{11}$$

In a similar way we obtain $y'' = A(x,y,z)$. Thus the system (9) is rewritten as:

$$\begin{aligned} A &= f_0(x) + yf_1(x) + zf_2(x) + \alpha f_3(x), \\ \beta &= g_0(x) + yg_1(x) + zg_2(x) + \alpha g_3(x). \end{aligned} \tag{12}$$

From (7) and (12) we calculate $B$ and we have:

$$B = g_0'(x) + g_1'(x)y + g_2'(x)z + g_3'(x)a + g_1(x)\alpha + g_2(x)\beta + g_3(x)A. \tag{13}$$

We substitute the relations (12) and (13) into (4) and (5) successively and we obtain two algebraic equations in $\alpha$. The first one, quadratic in $\alpha$, reads as:

$$r_2\alpha^2 + r_1\alpha + r_0 = 0, \tag{14}$$

where

$$\begin{aligned} r_2 &= (g_3' + g_1 + g_2g_3)X, \\ r_1 &= (\mathcal{B}_x - f_3\mathcal{B} + g_2\mathcal{B})X - (g_3' + g_1 + g_2g_3 + f_3g_3)Y + f_3Z, \\ r_0 &= \mathcal{A}\mathcal{B}X + (\mathcal{B}_x + g_2\mathcal{B} + g_3\mathcal{A})Y - \mathcal{A}Z, \end{aligned} \tag{15}$$

and the second one, cubic in $\alpha$, is

$$s_3\alpha^3 + s_2\alpha^2 + s_1\alpha + s_0 = 0, \tag{16}$$

where

$$\begin{aligned} s_3 &= f_3(X_y + g_3X_z), \\ s_2 &= (f_3' + f_1 - 2f_3^2 + f_2g_3)X - f_3X_x - \mathcal{A}X_y - \\ &- (f_3\mathcal{B} + g_3\mathcal{A})X_z + f_3(Y_y + g_3Y_z), \\ s_1 &= (\mathcal{A}_x + f_2\mathcal{B} - 5f_3\mathcal{A})X - \mathcal{A}X_x - \mathcal{A}\mathcal{B}X_z - \\ &- (f_3' + f_1 + f_3^2 + f_2g_3)Y + f_3Y_x + \mathcal{A}Y_y + (f_3\mathcal{B} + g_3\mathcal{A})Y_z, \\ s_0 &= -3\mathcal{A}^2X - (\mathcal{A}_x + f_3\mathcal{A} + f_2\mathcal{B})Y + \mathcal{A}Y_x + \mathcal{A}\mathcal{B}Y_z, \end{aligned} \tag{17}$$

and

$$\mathcal{A} = f_0(x) + yf_1(x) + zf_2(x), \quad \mathcal{B} = g_0(x) + yg_1(x) + zg_2(x). \tag{18}$$

The slope functions ($\alpha$, $\beta$) depend on *one of the three* integration constants $c_1,\ c_2,\ c_3$ introduced by the general solution (10) of the system (9). This is so because, in the case of three-parametric families of orbits (see (2)), we firstly calculate the vector $\bar{\delta}$ from (1) and we see that only one integration constant, i.e. $c_3$, is included in it. Then we estimate ($\alpha$, $\beta$) from (1) and we find them as a function of one of the integration constants. Moreover, if the "*slope functions*" ($\alpha$, $\beta$) do depend on one of the integration constants, i.e. $c_3$, then we can find the analytic expression of surfaces in the solved form $f(x,y,z,c_3) = c_1$, $g(x,y,z,c_3) = c_2$ integrating the system of ODEs (11).

On the other hand, the above coefficients *must be independent* of the integration constants. They are expressed in terms of the functions $f_i$, $g_k$, $i, k \in \{0, 1, 2, 3\}$, and the components of the force field $(X,\ Y,\ Z)$, which depend *only* on the Cartesian coordinates $x,\ y,\ z$. In order to satisfy the algebraic equations (14) and (16) for *all* solutions of the system (9), we must have:

$$r_2 = r_1 = r_0 = 0 \ \textit{and} \ s_3 = s_2 = s_1 = s_0 = 0. \tag{19}$$

Our problem then is: "*Given the system of ODEs* (9)*, examine if the seven PDEs* (19) *in the three unknown functions* $X$, $Y$, $Z$ *are compatible*". To this end, we start with the equations $r_2$=0 and $s_3$=0 which seem to be the simplest ones. We determine the quantity

$$D_0 = f_3^2 + (g_3' + g_1 + g_2 g_3)^2 \tag{20}$$

and we distinguish two cases: $D_0$=0 and $D_0 \neq 0$.

**3.1. The case $D_0$=0.** If $D_0$=0, then $r_2$=0 and the coefficient $r_1$ reads

$$(\mathcal{B}_x + g_2\mathcal{B})X = 0. \tag{21}$$

Then we check the function

$$D_1 = (g_0' + g_0 g_2)^2 + (g_1' + g_1 g_2)^2 + (g_2' + g_1 g_2)^2. \tag{22}$$

If $D_1$=0, then the coefficient $r_0$ reads

$$\mathcal{A}Z = g_3\mathcal{A}Y + \mathcal{A}\mathcal{B}X. \tag{23}$$

Before simplifying the last equation with $\mathcal{A}$, we check if, for the given functions $f_0(x)$, $f_1(x)$ and $f_2(x)$, $\mathcal{A}$ is zero or not. Equivalently we can check if the expression

$$D_2 = f_0^2 + f_1^2 + f_2^2 \tag{24}$$

becomes zero or not. If the answer is positive, then we have $\mathcal{A}$=0 and we have to change the pair of the "*slope functions*" $(\alpha,\ \beta)$. If the answer is negative, then we can proceed as follows: We eliminate the quantity $\mathcal{A}$ from (23) and we consider the equation

$$Z = \mathcal{B}X + g_3 Y. \tag{25}$$

The expression of $s_3$ becomes identically zero and the other three remaining equations provide us with the results:

$$\begin{aligned} &\mathcal{A}(X_y + g_3 X_z) = (f_1 + f_2 g_3)X, \\ &(\mathcal{A}_x + f_2\mathcal{B})X - \mathcal{A}(X_x + \mathcal{B}X_z) + \mathcal{A}(Y_y + g_3 Y_z) = (f_1 + f_2 g_3)Y, \\ &\mathcal{A}(Y_x + \mathcal{B}Y_z) = 3\mathcal{A}^2 X + (\mathcal{A}_x + f_2\mathcal{B})Y. \end{aligned} \tag{26}$$

Example 1. We select the functions

$$\begin{aligned} &f_0(x) = k_0,\ \ f_1(x) = -g_3(x)f_2(x) = k_1,\ \ f_2(x) = k_2,\ \ f_3(x) = 0, \\ &g_0(x) = g_1(x) = g_2(x) = 0,\ \ g_3(x) = g_{30} = const., \end{aligned} \tag{27}$$

where $k_0,\ k_1,\ k_2$ = const. This selection leads to $\mathcal{B}$=0. From the system of equations (25) and (26) we determine the force field

$$\begin{aligned} X &= R(u), \\ Y &= 3x(k_0 + k_1 y + k_2 z)R(u) + W(u), \\ Z &= g_{30} Y, \end{aligned} \tag{28}$$

where $R$, $W$ are arbitrary functions of their common argument $u = z - g_{30}y$. The system of ODEs (9) becomes

$$y''(x) = k_0 + (z - g_{30}y)k_2, \ \ z'(x) = g_{30}y'(x), \tag{29}$$

and its solution is

$$y - \frac{1}{2}(k_0 + (z - g_{30}y)k_2)x^2 - c_3 x = c_1, \ \ z - g_{30}y = c_2 \tag{30}$$

which is compatible with the force field (28).

Example 2. We select the functions

$$\begin{aligned} &f_0(x) = \tfrac{1}{x}, \ \ f_1(x) = f_2(x) = f_3(x) = 0, \\ &g_0(x) = g_1(x) = g_2(x) = 0, \ \ g_3(x) = g_{30} = const. \end{aligned} \tag{31}$$

This selection leads to $\mathcal{B}$=0. In this case we did not manage to solve analytically the system of equations (25) and (26). Thus we found a particular solution, namely the force field

$$\begin{aligned} X &= -\tfrac{d_1}{x} + d_2, \\ Y &= \tfrac{d_2 y + p_0 - 3d_1 \log x}{x} + 3d_2, \\ Z &= g_{30}Y, \end{aligned} \tag{32}$$

where $d_1, d_2, p_0$ are constants. The system of ODEs (9) becomes

$$y''(x) = \frac{1}{x}, \ \ z'(x) = g_{30}y'(x), \tag{33}$$

and its general solution is

$$y - x(\log x - 1 + c_3) = c_1, \ \ z - g_{30}y = c_2 \tag{34}$$

which is compatible with the force field (32). We note here that $c_1,\ c_2,\ c_3$ are the integration constants and they are different from the other constants $d_1,\ d_2,\ p_0$.

We shall now examine the case if $D_1 \neq 0$. Then, from (21), we obtain that

$$X = 0. \tag{35}$$

This result leads to the fact that the coefficients $r_2$, $s_3$ and $s_2$ are identically zero and the other three remaining equations, i.e., $r_0$=0, $s_1$=0, $s_0$=0, give rise to:

$$
\mathcal{A}Z = (\mathcal{B}_x + g_2\mathcal{B} + g_3\mathcal{A})Y,\\
\mathcal{A}(Y_y + g_3 Y_z) = (f_1 + f_2 g_3)Y,\\
\mathcal{A}(Y_x + \mathcal{B}Y_z) = (\mathcal{A}_x + f_2\mathcal{B})Y. \tag{36}
$$

Example 3. We are given the functions

$$
f_0(x) = x,\ \ f_1(x) = -g_3(x)f_2(x),\ \ f_2(x) = 1,\ \ f_3(x) = 0,\\
g_0(x) = x,\ \ g_1(x) = -1,\ \ g_2(x) = -1,\ \ g_3(x) = -1 + d_1\mathrm{e}^x, \tag{37}
$$

where $d_1$ =const. The function $g_3$ is selected so that the expression of $D_0$ in (20) becomes zero. From the system of equations (35) and (36) we determine the force field

$$
X = 0,\\
Y = x + z - g_3(x)y,\\
Z = 1 + (-2 + d_1\mathrm{e}^x)x - d_1^2\mathrm{e}^{2x}y + d_1\mathrm{e}^x(2y + z). \tag{38}
$$

The system of ODEs (9) becomes

$$
y''(x) = x - g_3(x)y + z,\ \ z'(x) = x - y - z + g_3(x)y'. \tag{39}
$$

Although we cannot find analytically the solution of the system (39), we can assert that all the unknown to us solutions represent orbits generated by the force field (38).

**3.2. The case $D_0 \neq$0.** Then we specify two cases: $f_3$ =0 and $f_3 \neq 0$.

**3.2.1.** *The case $f_3$=0.* Then the expression

$$
E_1 = g_3' + g_1 + g_2 g_3 \tag{40}
$$

is not zero. Thus, from the relation $r_2$=0, we obtain

$$
X = 0. \tag{41}
$$

Having fixed that $f_3$=0 and $X$=0, from $r_1$=0 we obtain

$$
Y = 0. \tag{42}
$$

With $X$=0 and $Y$=0, the equation $r_0$=0 gives

$$
\mathcal{A}Z = 0. \tag{43}
$$

Consequently, if $\mathcal{A}$=0, or equivalently $f_0^2$+$f_1^2$+$f_2^2$=0, then $Z$ is free. Otherwise, $Z$=0 which is trivial.

**3.2.2.** *The case $f_3 \neq$0.* If this is so, then we have to check the expression $E_1$ in (40). If $E_1 \neq 0$, then, from the equation $r_2$=0, we obtain

$$
X = 0. \tag{44}
$$

Proceeding in a similar way and using the result (44), the relation $r_1$=0 gives us

$$f_3 Z = (g_3' + g_1 + g_2 g_3 + f_3 g_3)Y, \tag{45}$$

and from the last one, i.e., $r_0$=0, we get

$$\mathcal{A}Z = (\mathcal{B}_x + g_2\mathcal{B} + g_3\mathcal{A})Y. \tag{46}$$

We examine now the other equations $s_i$=0, $i \in \{0,1,2,3\}$. The expression of $s_3$ becomes identically zero, since $X$=0 and $f_3 \neq 0$. The other three remaining equations take simpler forms as follows:

$$\begin{aligned} &Y_y + g_3 Y_z = 0,\\ &f_3 Y_x + \mathcal{A}Y_y + (g_3\mathcal{A} + f_3\mathcal{B})Y_z = (f_1 + f_3^2 + f_3' + f_2 g_3)Y,\\ &\mathcal{A}(Y_x + \mathcal{B}Y_z) = (\mathcal{A}_x + f_3\mathcal{A} + f_2\mathcal{B})Y. \end{aligned} \tag{47}$$

Example 4. We consider the functions

$$\begin{aligned} &f_0(x) = \tfrac{1}{x^2},\ \ f_1(x) = f_0(x),\ \ f_2(x) = 0,\ \ f_3(x) = -x f_0(x),\\ &g_0(x) = \tfrac{1}{x},\ \ g_1(x) = g_0(x),\ \ g_2(x) = g_3(x) = 0. \end{aligned} \tag{48}$$

From the system of equations (44)–(47) we determine the force field

$$X = 0,\ \ Y = \frac{d_0}{x^3},\ \ Z = -\frac{d_0}{x^3},\ \ d_0 = const. \tag{49}$$

The system of ODEs (9) becomes

$$y''(x) = \frac{1}{x^2}(1 + y(x) - xy'(x)),\ \ z'(x) = \frac{1}{x}(1 + y'(x)). \tag{50}$$

In this case we can find analytically its solution. This is

$$\begin{aligned} &\tfrac{1}{2}(xy - xz + x + c_3 x) = c_1,\\ &\tfrac{y+z+1-c_3}{2x} = c_2, \end{aligned} \tag{51}$$

and we can check directly that the system (51) combining with the force field (49) satisfies the equations (4) and (5).

Example 5. We consider the functions

$$\begin{aligned} &f_0(x) = \tfrac{2x}{x^2+1},\ \ f_1(x) = f_2(x) = 0,\ \ f_3(x) = f_0(x),\\ &g_0(x) = 1 - x^2,\ \ g_1(x) = g_2(x) = 0,\ \ g_3(x) = g_0(x). \end{aligned} \tag{52}$$

From the system of equations (44)–(47) we determine the force field

$$X = 0,\ \ Y = d_0 x,\ \ Z = -2d_0 x^3,\ \ d_0 = const. \tag{53}$$

The system of ODEs (9) becomes

$$y''(x) = \frac{2x}{1+x^2}(1 + y'(x)),\ \ z'(x) = (1 - x^2)y'(x). \tag{54}$$

In this case we can find analytically its solution. This is

$$\begin{aligned} &y + x - c_3 x - c_3 \tfrac{x^3}{3} = c_1, \\ &z - c_3 x + c_3 \tfrac{x^5}{5} = c_2. \end{aligned} \tag{55}$$

**3.2.3.** *The subcase $E_1$=0.* Then the expression of $r_2$ becomes identically zero, and from the equation $s_3$=0 we obtain

$$X_y + g_3 X_z = 0. \tag{56}$$

We shall use this result in what follows. The equations $r_1$=0 and $r_0$=0 get a simpler form, more exactly

$$\begin{aligned} &(\mathcal{B}_x + (g_2 - f_3)\mathcal{B})X - f_3 g_3 Y + f_3 Z = 0, \\ &\mathcal{A}\mathcal{B}X + (\mathcal{B}_x + g_2\mathcal{B} + g_3\mathcal{A})Y - \mathcal{A}Z = 0. \end{aligned} \tag{57}$$

As we have explained above, we have $s_3$=0. Then we have to consider again three equations, namely, $s_2$=0, $s_1$=0, $s_0$=0, which are

$$\begin{aligned} (f_3' + f_1 - 2f_3^2 + f_2 g_3)X - f_3 X_x - f_3\mathcal{B}X_z + f_3(Y_y + g_3 Y_z) = 0, \\ (\mathcal{A}_x + f_2\mathcal{B} - 5f_3\mathcal{A})X - \mathcal{A}X_x - \mathcal{A}\mathcal{B}X_z \\ -(f_3' + f_1 + f_3^2 + f_2 g_3)Y + f_3 Y_x + \mathcal{A}Y_y + (f_3\mathcal{B} + g_3\mathcal{A})Y_z = 0, \\ -3\mathcal{A}^2 X - (\mathcal{A}_x + f_3\mathcal{A} + f_2\mathcal{B})Y + \mathcal{A}Y_x + \mathcal{A}\mathcal{B}Y_z = 0. \end{aligned}$$

We offer now the last example which completes this case.

Example 6. We consider the functions

$$\begin{aligned} &f_0(x) = f_1(x) = f_2(x) = 0, \quad f_3(x) = \tfrac{1}{x(1+x^2)}, \\ &g_0(x) = g_1(x) = g_2(x) = 0, \quad g_3(x) = g_{30} = const. \end{aligned} \tag{58}$$

From the system of equations (56)–(58) we determine the force field

$$X = \frac{R(u)}{x^3}, \quad Y = \frac{W(u)}{(1+x^2)^{3/2}}, \quad Z = g_{30} Y(x,y,z), \tag{59}$$

where $R$ and $W$ are arbitrary functions in the argument $u = z - g_{30}y$. The system of ODEs (9) becomes

$$y''(x) = f_3(x) y'(x), \quad z'(x) = g_{30} y'(x), \tag{60}$$

and has the general solution:

$$y - c_3\sqrt{1+x^2} = c_1, \quad z - g_{30} y = c_2. \tag{61}$$

In Figure 1 we present an orbit in 3D-space which is **one** member of the family (61). The values of parameters are: $c_1$=1, $c_2$=2, $c_3$=1.5, $g_{30}$=2.

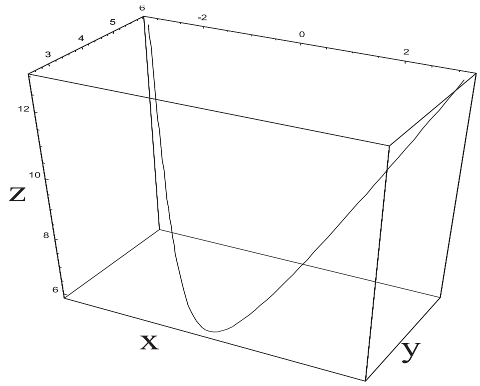


Fig. 3.1 – One representative orbit which belongs to the family (61) (Example 6). The values of parameters are: $c_1$=1, $c_2$=2, $c_3$=1.5, $g_{30}$=2.

## 4. CONCLUSIONS

Our work is based on a recent paper of Bozis and Borghero (2008). The authors considered the planar version of the problem and they showed that the two-parametric set of all solutions of any linear ODE of second order $y'' + a(x)y' + b(x)y = f(x)$ can be considered as a set of orbits traced by a material point of unit mass, in the presence of at least one autonomous force field $\bar{F}(X, Y)$, for adequate initial conditions. They found several interesting examples too. Following their idea, we have studied three-dimensional force fields which are compatible with three-parametric families of spatial orbits. We have proved that in any case we can find such a field although the equations of orbits is not given in advance; only the system of two ODEs which accept as solutions these orbits is given. This set of ODEs (see (9)) is not expected to can be solved analytically. So, a force field may be taught of as a "mechanical device" to produce solutions of solvable or non solvable ODE.

We have examined all the cases, starting with the simplest one and tested the other subcases, also presenting a suitable example. In the most cases we managed to solve analytically the system of equations (9) and to verify our examples. The only exception was Example 3 in which the general solution of the system (9) was not found although we had calculated the corresponding force-field analytically. In all cases the set of orbits is three-parametric; we have determined the components of the force field through differential equations. The generalized force fields which are solutions to this problem must be independent of any integration constants, because they produce the corresponding three-parametric family of orbits. In each example the three-parametric family of orbits and the corresponding force field verify the set of equations (4) and (5). We note here that an interesting point is the study of potentials which admit three-parametric families of orbits. Since the calculations are more complex than in the planar problem, we may look at this topic in a future work. For all calculations in our paper we have used *Mathematica 6.0*.

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*Aristotle University of Thessaloniki*
*Department of Physics*
*Section of Astrophysics, Astronomy and Mechanics*
*P.O. Box 102, 54124, Thessaloniki, Greece*
*E-mail:* `tkoto@skiathos.physics.auth.gr`