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ON NON-NORMALIZED SUBORDINATION CHAINS IN $\mathbb{C}^n$

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Abstract. In this paper we consider non-normalized univalent subordination chains $f(z,t) = \exp(\int_0^t A(\tau)\mathrm{d}\tau)z + \cdots$ and we present the connection with the notion of generalized A-asymptotic spirallikeness on the Euclidean unit ball B^n in $\mathbb{C}^n$, where $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ is a measurable operator that satisfies certain natural conditions.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B^1 is the unit disc U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of linear and continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. If $\Omega \subseteq \mathbb{C}^n$ is a domain, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If Ω is a domain in $\mathbb{C}^n$ which contains the origin and $f \in H(\Omega)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n.

If $A \in L(\mathbb{C}^n, \mathbb{C}^n)$, let $m(A) = \min\{\Re\langle A(z), z\rangle : \|z\| = 1\}$ and $k(A) = \max\{\Re\langle A(z), z\rangle : \|z\| = 1\}$. Also let $k_+(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$ be the upper exponential index of A, where $\sigma(A)$ is the spectrum of A. It is known that $k_+(A) = \lim\limits_{t\to\infty} \dfrac{\ln\|\mathrm{e}^{tA}\|}{t}$ (see [2]; see also [24]).

In this paper we use measurable linear operators that satisfy the assumptions of Definition 1.1. We remark that the condition (1.1) is satisfied if $A(t)$ is constant or if $A(t)$ is diagonal (for further details see [9]).

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Definition 1.1. Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping such that $m(A(t)) > 0$ for $t \geq 0$ and $\int_0^\infty m(A(t))\mathrm{d}t = \infty$. Moreover, assume that $\|A(\cdot)\|$ is uniformly bounded on $[0,\infty)$ and

$$(1.1)\quad \int_s^t A(\tau)\mathrm{d}\tau \circ \int_r^s A(\tau)\mathrm{d}\tau = \int_r^s A(\tau)\mathrm{d}\tau \circ \int_s^t A(\tau)\mathrm{d}\tau, \quad t \geq s \geq r \geq 0.$$

Note that the condition (1.1) implies the following relation (see [9]):

$$A(t) \circ \int_0^t A(\tau)\mathrm{d}\tau = \int_0^t A(\tau)\mathrm{d}\tau \circ A(t), \qquad \text{a.e. } t \in [0,\infty).$$

We shall need the estimates (1.2) and (1.3) related to an operator A that satisfies the assumptions of Definition 1.1 (see [9, Remark 1.6 (v)]).

Lemma 1.2. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfy the conditions in Definition 1.1. Then*

$$(1.2)\quad \mathrm{e}^{\int_s^t m(A(\tau))\mathrm{d}\tau} \leq \|\mathrm{e}^{\int_s^t A(\tau)\mathrm{d}\tau}(u)\| \leq \mathrm{e}^{\int_s^t k(A(\tau))\mathrm{d}\tau},\ t \geq s \geq 0,\ \|u\| = 1,$$

$$(1.3)\quad \mathrm{e}^{-\int_s^t k(A(\tau))\mathrm{d}\tau} \leq \|\mathrm{e}^{-\int_s^t A(\tau)\mathrm{d}\tau}(u)\| \leq \mathrm{e}^{-\int_s^t m(A(\tau))\mathrm{d}\tau},\ t \geq s \geq 0,\ \|u\| = 1.$$

The following family of holomorphic mappings on B^n occurs in our discussion:

$$\mathcal{N} = \Big\{h \in H(B^n) : h(0) = 0,\ \Re\langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\Big\}.$$

For various applications of this family in the study of biholomorphic mappings on the unit ball B^n see [6], [10], [11], [17], [22], [25].

Definition 1.3. (i) If $f, g \in H(B^n)$, we say that f is subordinate to g ($f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$ for $z \in B^n$) such that $f = g \circ v$.

(ii) A mapping $f : B^n \times [0,\infty) \to \mathbb{C}^n$ is called a subordination chain if $f(\cdot,t)$ is holomorphic on B^n, $f(0,t) = 0$ for $t \geq 0$, and $f(\cdot,s) \prec f(\cdot,t)$, $0 \leq s \leq t < \infty$. In addition, if $f(\cdot,t)$ is biholomorphic on B^n for $t \geq 0$, we say that $f(z,t)$ is a univalent subordination chain. If $f(z,t)$ is a univalent subordination chain such that $Df(0,t) = e^t I_n$ for $t \geq 0$, we say that $f(z,t)$ is a normalized univalent subordination chain (Loewner chain).

The above subordination condition implies the existence of a Schwarz mapping $v = v(z,s,t)$, called the *transition mapping associated with* $f(z,t)$, such that $f(z,s) = f(v(z,s,t),t)$ for $z \in B^n$ and $0 \leq s \leq t < \infty$.

Lemma 1.4 below was essentially proved in [9, Theorems 2.1 and 2.3]. The argument that $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$ can be found in the proof of [9, Theorem 2.6] (see also the proof of [8, Theorem 2.8]). The Loewner differential equation (1.7) was deduced in [6, Theorem 1.4]. Lemma 1.4 is a generalization of [8, Theorems 2.1 and 2.3] (in the case $A(t) \equiv A$), [17, Theorem 2.1] and [6, Theorem 1.4] (in the case $A(t) \equiv I_n$). In the case of one complex variable, see [18].

LEMMA 1.4. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfy the conditions in Definition 1.1. Assume that*

$$\sup_{s\geq 0} \int_s^\infty \|e^{\int_s^t [A(\tau)-2m(A(\tau))I_n]d\tau}\| dt < \infty. \tag{1.4}$$

Also let $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping which satisfies the following conditions:

(i) *$h_t(\cdot) = h(\cdot,t) \in \mathcal{N}$ and $Dh(0,t) = A(t)$ for $t \geq 0$.*

(ii) *$h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B^n$.*

Then for each $s \geq 0$ and $z \in B^n$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \ \ a.e. \ \ t \geq s, \quad v(z,s,s) = z, \tag{1.5}$$

has a unique solution $v = v(z,s,t)$ such that $v(\cdot,s,t)$ is a univalent Schwarz mapping, $v(z,s,\cdot)$ is Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z \in B^n$, and $Dv(0,s,t) = \exp(-\int_s^t A(\tau)d\tau)$ for $t \geq s \geq 0$. In addition, there exists the limit

$$\lim_{t\to\infty} e^{\int_0^t A(\tau)d\tau} v(z,s,t) = f(z,s) \tag{1.6}$$

locally uniformly on B^n for each $s \geq 0$, and $f(z,t)$ is a univalent subordination chain such that $f(z,s) = f(v(z,s,t),t)$ for $z \in B^n$ and $0 \leq s \leq t < \infty$. Also, $\{e^{-\int_0^t A(\tau)d\tau} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n, $Df(0,t) = e^{\int_0^t A(\tau)d\tau}$, $t \geq 0$, $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$, and

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \ \ a.e. \ \ t \geq 0, \quad \forall z \in B^n. \tag{1.7}$$

DEFINITION 1.5. (i) A mapping $h(z,t)$ which satisfies the assumptions (i) and (ii) of Lemma 1.4 will be called a generating vector field (cf. [1] and [3]).

(ii) The univalent subordination chain $f(z,t)$ given by (1.6) will be called the canonical solution of the Loewner differential equation (1.7).

(iii) Let $g(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping such that $g(\cdot,t) \in H(B^n)$, $g(0,t) = 0$, $t \geq 0$, and $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$. Assume that $g(z,t)$ satisfies the Loewner differential equation (1.7). In this case, $g(z,t)$ will be called a standard solution of the Loewner differential equation (1.7).

(iv) Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable operator that satisfies the assumptions of Definition 1.1. A mapping $f \in S(B^n)$ has a *generalized parametric representation with respect to A* if there exists a generating vector field $h(z,t)$ such that

$$f(z) = \lim_{t\to\infty} e^{\int_0^t A(\tau)d\tau} v(z,t)$$

locally uniformly on B^n, where $v(z,t) = v(z,0,t)$ and $v(z,s,t)$ is the unique solution of the initial value problem (1.5) (see [9]; see also [8] in the case $A(t) \equiv A$; [7] and [20, 21] in the case $A(t) \equiv I_n$).

Remark 1.6. (i) The condition (1.4) is satisfied if there is $\eta > 0$ such that

$$k(A(t)) - 2m(A(t)) \leq -\eta, \quad t \in [0, \infty). \tag{1.8}$$

Indeed, in view of the conditions (1.1) and (1.2), we deduce that

$$\|e^{\int_s^t [A(\tau) - 2m(A(\tau))I_n] d\tau}\| \leq e^{\int_s^t [k(A(\tau)) - 2m(A(\tau))] d\tau} \leq e^{-\eta(t-s)}.$$

(ii) If $A(t) \equiv A$ is a constant operator, then the condition (1.4) reduces to

$$\int_0^\infty \|e^{(A - 2m(A)I_n)t}\| dt < \infty.$$

The above condition is also equivalent to $k_+(A) < 2m(A)$ (see [3]).

The following result provides a sufficient condition for a standard solution to be the canonical solution of (1.7) (see [9, Theorem 2.5]). Note that Lemma 1.7 is a generalization of [8, Theorem 2.6] and [17, Theorems 2.2 and 2.3].

Lemma 1.7. *Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfy the condition (1.4) and the assumptions of Definition 1.1. Let $h = h(z,t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a generating vector field such that $Dh(0,t) = A(t)$, $t \geq 0$. Also let $f = f(z,t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a standard solution of the Loewner differential equation (1.7) such that $Df(0,t) = e^{\int_0^t A(\tau) d\tau}$ for $t \geq 0$, and $\{e^{-\int_0^t A(\tau) d\tau} f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n. Then $f(z,t)$ coincides with the canonical solution of the Loewner differential equation (1.7).*

We next present the notion of generalized spirallikeness (see [9]).

Definition 1.8. Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping such that $m(A(t)) > 0$ for $t \geq 0$. Also let Ω be a domain in $\mathbb{C}^n$ which contains the origin. We say that Ω is generalized spirallike with respect to A if $e^{-\int_s^t A(\tau) d\tau}(w) \in \Omega$ for all $w \in \Omega$ and $t \geq s \geq 0$. A mapping $f \in S(B^n)$ is called generalized spirallike with respect to A if $f(B^n)$ is a generalized spirallike domain with respect to A.

We remark that if $A(t)$ is a constant linear operator in $\mathbb{C}^n$, then Definition 1.8 reduces to the usual definition of spirallikeness (see [25]). On the other hand, if $A(t) \equiv I_n$, we obtain the usual notion of starlikeness (see [5], [10], [25]). Various results concerning spirallike mappings with respect to constant linear operators may be found in [4], [12, 13, 14, 15, 16], [22, 23, 24, 25].

The authors [9] proved the following characterization of generalized spirallikeness in terms of univalent subordination chains.

Lemma 1.9. *Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition (1.1). Assume that $m(A(t)) > 0$ for $t \geq 0$. Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping. Then f is generalized spirallike with respect to A if and only if $f(z,t) = e^{\int_0^t A(\tau) d\tau} f(z)$ is a univalent subordination chain.*

We conclude this section with the notion of generalized asymptotic spirallikeness with respect to a measurable linear operator. This notion is a generalization of asymptotic starlikeness and spirallikeness (see [7, 8]; cf. [20, 21]).

DEFINITION 1.10. Let $\Omega \subseteq \mathbb{C}^n$ be a domain which contains the origin and let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping which satisfies the conditions in Definition 1.1. We say that Ω is *generalized A-asymptotically spirallike* if there exists a mapping $Q = Q(z,t) : \Omega \times [0,\infty) \to \mathbb{C}^n$, which satisfies the following conditions:

(i) $Q(\cdot,t) \in H(\Omega)$, $Q(0,t) = 0$, $DQ(0,t) = A(t)$, $t \geq 0$, and $\{Q(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family on Ω;

(ii) $Q(z,\cdot)$ is measurable on $[0,\infty)$ for all $z \in \Omega$;

(iii) The initial value problem

$$\frac{\partial w}{\partial t} = -Q(w,t) \quad \text{a.e.} \quad t \geq s, \quad w(z,s,s) = z, \tag{1.9}$$

has a unique solution $w = w(z,s,t)$ for each $z \in \Omega$ and $s \geq 0$, such that $w(\cdot,s,t)$ is a holomorphic mapping of Ω into Ω for $t \geq s$, $w(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$ for $s \geq 0$, and there exists a sequence $\{t_\mu\}_{\mu\in\mathbb{N}}$, $0 < t_\mu \to \infty$, such that

$$\lim_{\mu\to\infty} e^{\int_0^{t_\mu} A(\tau)d\tau} w(z,0,t_\mu) = z$$

locally uniformly on Ω.

A domain $\Omega \subseteq \mathbb{C}^n$, which contains the origin, is called *generalized asymptotically spirallike* if there exists a measurable operator $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$, which satisfies the conditions in Definition 1.1, such that Ω is generalized A-asymptotically spirallike.

Note that if $A(t) \equiv A \in L(\mathbb{C}^n, \mathbb{C}^n)$ in Definition 1.10, then Ω is asymptotically spirallike (see [8]). Also, if $A(t) \equiv I_n$ in Definition 1.10, then Ω is asymptotically starlike (see [6, Definition 2.1] and [21, Definition 3] in the case of the maximum norm).

DEFINITION 1.11. Let $f \in S(B^n)$ and let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping which satisfies the conditions in Definition 1.1. We say that f is generalized A-asymptotically spirallike (generalized asymptotically spirallike) if f is biholomorphic on B^n and $f(B^n)$ is a generalized A-asymptotically spirallike (generalized asymptotically spirallike) domain. In particular, if $A(t) \equiv A \in L(\mathbb{C}^n, \mathbb{C}^n)$, we say that f is A-asymptotically spirallike, and if $A(t) \equiv I_n$, we say that f is asymptotically starlike.

REMARK 1.12. (i) The notion of asymptotic starlikeness was introduced by Poreda [21] in the case of the maximum norm in $\mathbb{C}^n$, and by Graham, Hamada, Kohr and Kohr [7] in the case of the Euclidean norm. The authors [7] proved that asymptotic starlikeness is equivalent to the notion of parametric representation: $f \in S(B^n)$ is asymptotically starlike if and only if f has parametric representation, i.e. there exists a Loewner chain $f(z,t)$ such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f = f(\cdot,0)$ (see [6], [10, 11]; cf. [20, 21]). Also, the authors [8] proved that if $A \in L(\mathbb{C}^n,\mathbb{C}^n)$ is such that $k_+(A) < 2m(A)$, then a mapping $f \in S(B^n)$ is A-asymptotically spirallike if and only if there exists a univalent subordination chain $f(z,t)$ such that $Df(0,t) = e^{At}$, $\{e^{-At}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f = f(\cdot,0)$.

(ii) If $Q(\cdot,t) = A(t)$ for $t \geq 0$, in Definition 1.10, where $A : [0,\infty) \to L(\mathbb{C}^n,\mathbb{C}^n)$ is a measurable mapping which satisfies the conditions in Definition 1.1, then Ω is generalized spirallike with respect to A. In particular, if $Q(\cdot,t) \equiv A \in L(\mathbb{C}^n,\mathbb{C}^n)$ in Definition 1.10, where $m(A) > 0$, then Ω is spirallike, and if $Q(\cdot,t) \equiv I_n$, then Ω is starlike. Indeed, taking into account (1.1), we deduce that the initial value problem (1.9) has the unique solution $w(z,s,t) = e^{-\int_s^t A(\tau)d\tau}(z)$ such that $w(z,s,t) \in \Omega$, $z \in \Omega$, $t \geq s$. Hence Ω is generalized spirallike with respect to A.

(iii) Conversely, if Ω is a generalized spirallike domain with respect to a measurable operator $A : [0,\infty) \to L(\mathbb{C}^n,\mathbb{C}^n)$ which satisfies the assumptions of Definition 1.1, then Ω is generalized A-asymptotically spirallike. Indeed, let $Q(\cdot,t) = A(t)$ for $t \geq 0$. Then $Q(\cdot,t) \in H(\Omega)$, $Q(0,t) = 0$, $DQ(0,t) = A(t)$ for $t \geq 0$, and it is not difficult to deduce that the initial value problem

$$\frac{\partial w}{\partial t}(z,s,t) = -A(t)w(z,s,t), \quad w(z,s,s) = z,$$

has the unique solution $w(z,s,t) = e^{-\int_s^t A(\tau)d\tau}(z)$, for all $z \in \Omega$ and $t \geq s \geq 0$. Then $w(\cdot,s,t)$ is a holomorphic mapping of Ω into Ω, since Ω is generalized spirallike with respect to A, $w(z,s,\cdot)$ is locally Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$, since $\|A(\cdot)\|$ is uniformly bounded on $[0,\infty)$, and $e^{\int_0^t A(\tau)d\tau}w(z,0,t) = z$ for $z \in \Omega$ and $t \geq 0$. Hence Ω is generalized A-asymptotically spirallike.

In this paper, we give an answer to the following question (cf. [8]):

Question 1.13. *Does there exist a connection between non-normalized univalent subordination chains and generalized asymptotic spirallikeness?*

2. MAIN RESULTS

In this section we consider the connection between univalent subordination chains and generalized asymptotically spirallike mappings. The following result provides examples of generalized asymptotically spirallike mappings that can be imbedded in non-normalized univalent subordination chains which satisfy the assumptions of Lemma 1.4. This result was obtained in [8] for $Dh(0,t) = A \in L(\mathbb{C}^n,\mathbb{C}^n)$, $t \geq 0$; in [7] for $Dh(0,t) = I_n$, $t \geq 0$ (in the Euclidean case), and in [21, Theorem 1] (in the case of the maximum norm).

THEOREM 2.1. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping which satisfies the assumptions of Definition* 1.1 *and the condition* (1.4). *Let $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a generating vector field such that $Dh(0,t) = A(t)$ for $t \geq 0$. Also let $f(z,t)$ be the canonical solution of the Loewner differential equation* (1.7) *and let $f = f(\cdot, 0)$. Then f is generalized A-asymptotically spirallike.*

Proof. Let $\Omega = f(B^n)$. In view of (1.6), $f(z,s) = \lim_{t\to\infty} \mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau} v(z,s,t)$ locally uniformly on B^n for $s \geq 0$, where $v = v(z,s,t)$ is the unique solution of the initial value problem (1.5) such that $v(z,s,\cdot)$ is Lipschitz continuous on $[s,\infty)$, $v(\cdot,s,t)$ is a univalent Schwarz mapping and $Dv(0,s,t) = \mathrm{e}^{-\int_s^t A(\tau)\mathrm{d}\tau}$.

Now, let $Q : \Omega \times [0,\infty) \to \mathbb{C}^n$ be given by $Q(w,t) = Df(z)h(z,t)$ for $w = f(z) \in \Omega$ and $t \geq 0$. Then $Q(\cdot,t) \in H(\Omega)$, $Q(0,t) = 0$, $DQ(0,t) = A(t)$ for $t \geq 0$. It is clear that $Q(w,\cdot)$ is measurable on $[0,\infty)$ for $w \in \Omega$ by the measurability of $h(z,\cdot)$ on $[0,\infty)$ for $z \in B^n$. Since $h(\cdot,t) \in \mathcal{N}$, $Dh(0,t) = A(t)$ and $\|A(\cdot)\|$ is uniformly bounded on $[0,\infty)$, we deduce in view of [8, Lemma 1.2] (see also [9]) that for each $r \in (0,1)$, there exists some $L = L(r) > 0$ such that $\|h(z,t)\| \leq L(r)$, $\|z\| \leq r$, $t \geq 0$. Since f is uniformly bounded on each closed ball $\overline{B_r^n}$ contained in B^n, we obtain in view of the above relation that the family $\{Q(\cdot,t)\}_{t\geq 0}$ is locally uniformly bounded on B^n.

Further, let $\nu(w,s,t) = f(v(z,s,t))$ for $w = f(z) \in \Omega$ and $t \geq s \geq 0$. Then $\nu(\cdot,s,t)$ is a holomorphic mapping of Ω into Ω, $\nu(0,s,t) = 0$ and $\nu(f(z),s,\cdot)$ is locally Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z \in B^n$. Taking into account (1.5), it is not difficult to deduce that $\nu = \nu(f(z),s,t)$ satisfies the initial value problem

$$(2.1) \qquad \frac{\partial \nu}{\partial t}(f(z),s,t) = -Q(\nu(f(z),s,t),t) \text{ a.e. } t \geq s,\ \nu(f(z),s,s) = f(z),$$

for each $z \in B^n$ and $s \geq 0$. In view of the uniqueness of solutions to the initial value problem (1.5), we deduce that the initial value problem (2.1) has the unique locally absolutely continuous solution $\nu = \nu(f(z),s,t) = f(v(z,s,t))$ on $[s,\infty)$ for $z \in B^n$ and $t \geq s \geq 0$.

Next, let $v(z,t) = v(z,0,t)$ for $z \in B^n$ and $t \geq 0$. Since f is normalized, f has a power series expansion near the origin of the form

$$f(z) = z + \frac{1}{2}D^2 f(0)(z,z) + R(z),$$

where $R(z)/\|z\|^2 \to 0$ as $\|z\| \to 0$. On the other hand, since $m(A(t)) > 0$ for $t \geq 0$, $\int_0^\infty m(A(\tau))\mathrm{d}\tau = \infty$, and since

$$(2.2) \qquad \|v(z,t)\| \leq \mathrm{e}^{-\int_0^t m(A(\tau))\mathrm{d}\tau} \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B^n, \quad t \geq 0,$$

by [9, Theorem 2.1], we deduce that $v(z,t) \to 0$ locally uniformly on B^n as $t \to \infty$. Moreover, taking into account the relation (1.4), we deduce that there exists a sequence $\{t_\mu\}_{\mu\in\mathbb{N}}$ such that $0 < t_\mu \to \infty$ and

$$\|e^{\int_0^{t_\mu}[A(\tau)-2m(A(\tau))I_n]d\tau}\| \to 0 \quad \text{as} \quad \mu \to \infty.$$

Then from (2.2) and the above relation, we obtain that

$$\|v(z,t_\mu)\|^2\|e^{\int_0^{t_\mu}A(\tau)d\tau}\| \leq \|e^{\int_0^{t_\mu}[A(\tau)-2m(A(\tau))I_n]d\tau}\|\frac{\|z\|^2}{(1-\|z\|)^4} \to 0$$

locally uniformly on B^n as $\mu \to \infty$. Therefore, we deduce that

$$\lim_{\mu\to\infty}\|v(z,t_\mu)\|^2 e^{\int_0^{t_\mu}A(\tau)d\tau} = 0$$

locally uniformly on B^n. Hence, by the inequality

$$\left\|e^{\int_0^{t_\mu}A(\tau)d\tau}D^2f(0)(v(z,t_\mu),v(z,t_\mu))\right\| \leq \|D^2f(0)\|\cdot\|e^{\int_0^{t_\mu}A(\tau)d\tau}\|\cdot\|v(z,t_\mu)\|^2,$$

we deduce that $\lim_{\mu\to\infty}e^{\int_0^{t_\mu}A(\tau)d\tau}D^2f(0)(v(z,t_\mu),v(z,t_\mu)) = 0$ locally uniformly on B^n. Taking into account the above relations, we deduce that

$$\begin{aligned}&\lim_{\mu\to\infty}e^{\int_0^{t_\mu}A(\tau)d\tau}f(v(z,t_\mu))\\ &= \lim_{\mu\to\infty}e^{\int_0^{t_\mu}A(\tau)d\tau}v(z,t_\mu) + \lim_{\mu\to\infty}\Big\{e^{\int_0^{t_\mu}A(\tau)d\tau}\frac{1}{2}D^2f(0)(v(z,t_\mu),v(z,t_\mu))\\ &+ e^{\int_0^{t_\mu}A(\tau)d\tau}\|v(z,t_\mu)\|^2\frac{R(v(z,t_\mu))}{\|v(z,t_\mu)\|^2}\Big\} = f(z)\end{aligned}$$

locally uniformly on B^n. Hence $e^{\int_0^{t_\mu}A(\tau)d\tau}\nu(w,0,t_\mu) \to w$ locally uniformly on Ω as $\mu \to \infty$. Consequently, $f(B^n)$ is a generalized A-asymptotically spirallike domain, and since $f \in S(B^n)$, we conclude that f is generalized A-asymptotically spirallike, as desired. This completes the proof. □

Taking into account Theorem 2.1 and Lemma 1.7, we obtain the following consequence, which provides examples of generalized A-asymptotically spirallike mappings that are generated by the solutions of (1.7).

Corollary 2.2. *Let $A : [0,\infty) \to L(\mathbb{C}^n,\mathbb{C}^n)$ satisfy the condition* (1.4) *and the assumptions of Definition* 1.1. *Also let $f(z,t)$ be the standard solution of the Loewner differential equation* (1.7) *such that $Df(0,t) = e^{\int_0^t A(\tau)d\tau}$ for $t \geq 0$, and $\{e^{-\int_0^t A(\tau)d\tau}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n. Then $f(\cdot,0)$ is generalized A-asymptotically spirallike.*

We next consider the converse of Theorem 2.1. The following result was obtained in [8] in the case $A(t) \equiv A \in L(\mathbb{C}^n,\mathbb{C}^n)$, and in [7] in the case $A(t) \equiv I_n$ (cf. [21, Theorem 2]).

Theorem 2.3. *Let $A : [0,\infty) \to L(\mathbb{C}^n,\mathbb{C}^n)$ be a measurable mapping which satisfies the assumptions of Definition* 1.1 *and the condition* (1.4). *Also let $f : B^n \to \mathbb{C}^n$ be a generalized A-asymptotically spirallike mapping and let*

$\Omega = f(B^n)$. *Then there exists a univalent subordination chain* $f(z,t)$ *such that* $f = f(\cdot,0)$, $Df(0,t) = \mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau}$ *for* $t \geq 0$, *and* $\{\mathrm{e}^{-\int_0^t A(\tau)\mathrm{d}\tau} f(\cdot,t)\}_{t\geq 0}$ *is a normal family on* B^n.

Proof. Let $Q : \Omega \times [0,\infty) \to \mathbb{C}^n$ be a mapping which satisfies the assumptions of Definition 1.10 such that $DQ(0,t) = A(t)$ for $t \geq 0$. Also let $\nu = \nu(f(z),s,t)$ be the unique solution of the initial value problem

$$\frac{\partial \nu}{\partial t} = -Q(\nu,t) \quad \text{a.e.} \quad t \geq s, \quad \nu(f(z),s,s) = f(z), \tag{2.3}$$

for all $z \in B^n$ and $s \geq 0$, such that $\nu(\cdot,s,t)$ is a holomorphic mapping of $f(B^n)$ into $f(B^n)$, $\nu(0,s,t) = 0$ for $t \geq s$, and $\nu(f(z),s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $f(z) \in f(B^n)$. Moreover, we know that there exists a sequence $\{t_\mu\}_{\mu\in\mathbb{N}}$, $0 < t_\mu \to \infty$, such that

$$\lim_{\mu\to\infty} \mathrm{e}^{\int_0^{t_\mu} A(\tau)\mathrm{d}\tau} \nu(w,0,t_\mu) = w \tag{2.4}$$

locally uniformly on $\Omega = f(B^n)$.

Let $v = v(z,s,t)$ be defined by $v(z,s,t) = f^{-1}(\nu(f(z),s,t))$, $z \in B^n$, $t \geq s$. Then $v(\cdot,s,t)$ is a holomorphic mapping of B^n into B^n such that $v(0,s,t) = 0$ for $t \geq s \geq 0$, and thus $v(\cdot,s,t)$ is a Schwarz mapping. Moreover, $v(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $z \in B^n$ and $v(z,s,s) = z$ for $z \in B^n$. In view of (2.4) and the fact that f is normalized, we deduce as in the proof of Theorem 2.1 that the limit

$$\lim_{\mu\to\infty} \mathrm{e}^{\int_0^{t_\mu} A(\tau)\mathrm{d}\tau} v(z,0,t_\mu) = f(z) \tag{2.5}$$

exists locally uniformly on B^n. Next, let $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be given by $h(z,t) = [Df(z)]^{-1}Q(f(z),t)$, $z \in B^n$, $t \geq 0$. Then $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B^n$, and since $DQ(0,t) = A(t)$ and f is normalized, it follows that $Dh(0,t) = A(t)$ for $t \geq 0$. Taking into account (2.3), we deduce that

$$\frac{\partial v}{\partial t} = -[Df(v(z,s,t))]^{-1}Q(f(v(z,s,t)),t) = -h(v(z,s,t),t),$$

for almost all $t \geq s$ and for all $z \in B^n$. Hence $v(z,s,t)$ is a solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \quad \text{a.e.} \quad t \geq s, \quad v(z,s,s) = z, \tag{2.6}$$

for all $s \geq 0$ and $z \in B^n$. By the uniqueness of solutions to (2.3), we deduce that $v(z,s,t) = f^{-1}(\nu(f(z),s,t))$ is the unique locally absolutely continuous solution on $[s,\infty)$ of (2.6).

Since the family $\{Q(\cdot,t)\}_{t\geq 0}$ is locally uniformly bounded on $f(B^n)$, we deduce by using arguments similar to those in the proof of [8, Theorem 3.5] that $\Re\langle h_t(z), z\rangle \geq 0$, a.e. $t \geq 0$, $\forall z \in B^n$. Now, since $h_t(0) = 0$, $Dh_t(0) = A(t)$ and $m(A(t)) > 0$ for $t \geq 0$, we obtain by the minimum principle for harmonic functions that $\Re\langle h_t(z), z\rangle > 0$, a.e. $t \geq 0$, $\forall z \in B^n \setminus \{0\}$. Thus $h_t(\cdot) = h(\cdot, t) \in \mathcal{N}$ for a.e. $t \geq 0$. Next, taking into account Lemma 1.4, we deduce that

$$\lim_{t\to\infty} \mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau} v(z, s, t) = f(z, s) \tag{2.7}$$

exists locally uniformly on B^n for each $s \geq 0$, and $f(z, s)$ is a univalent subordination chain such that $Df(0, s) = \mathrm{e}^{\int_0^s A(\tau)\mathrm{d}\tau}$ and $\{\mathrm{e}^{-\int_0^s A(\tau)\mathrm{d}\tau} f(\cdot, s)\}_{s\geq 0}$ is a normal family on B^n. From (2.5) and (2.7) we deduce that $f = f(\cdot, 0)$. This completes the proof. □

We next present the following result which provides an answer to the Question 1.13 and also a geometric characterization of univalent subordination chains. In particular, if $A(t) \equiv A \in L(\mathbb{C}^n, \mathbb{C}^n)$, Corollary 2.4 below was obtained in [8], and if $A(t) \equiv I_n$, see [7].

Corollary 2.4. *Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping which satisfies the assumptions of Definition* 1.1 *and the condition* (1.4)*. Also let $f \in S(B^n)$. Then the following statements hold:*

(i) *The mapping f is generalized A-asymptotically spirallike if and only if there exists a univalent subordination chain $f(z, t)$ such that $f = f(\cdot, 0)$, $Df(0, t) = \mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau}$ for $t \geq 0$, and $\{\mathrm{e}^{-\int_0^t A(\tau)\mathrm{d}\tau} f(\cdot, t)\}_{t\geq 0}$ is a normal family on B^n.*

(ii) *The mapping f has a generalized parametric representation with respect to A if and only if f is generalized A-asymptotically spirallike.*

Proof. First, assume that f is generalized A-asymptotically spirallike. Then the conclusion follows by Theorem 2.3. Conversely, assume that there exists a univalent subordination chain $f(z, t)$ such that $f = f(\cdot, 0)$, $Df(0, t) = \mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau}$ for $t \geq 0$, and $\{\mathrm{e}^{-\int_0^t A(\tau)\mathrm{d}\tau} f(\cdot, t)\}_{t\geq 0}$ is a normal family on B^n. In view of the proof of [9, Theorem 2.6] and by Lemma 1.7, we deduce that $f(z, t)$ is the canonical solution of the Loewner differential equation (1.7). Then $f = f(\cdot, 0)$ is generalized A-asymptotically spirallike by Theorem 2.1.

To prove the second statement, it suffices to combine Lemma 1.4, [9, Theorem 2.6] and the first statement (see also [9, Corollary 2.7]). □

Finally, we consider the connection between univalence and generalized a-asymptotic spirallikeness on the unit disc. We have (cf. [8])

Proposition 2.5. *Let $f : U \to \mathbb{C}$ be a normalized holomorphic function. Then $f \in S$ if and only if f is generalized a-asymptotically spirallike whenever $a : [0, \infty) \to \mathbb{C}$ is a measurable function such that $|a(\cdot)|$ is uniformly bounded on $[0, \infty)$ and $\Re a(t) = 1$ for almost all $t \geq 0$.*

Proof. Assume that $a : [0, \infty) \to \mathbb{C}$ is a measurable function which satisfies the assumptions in the above statement. It suffices to show that if $f \in S$, then f is generalized a-asymptotically spirallike. It is obvious that the condition (1.4) holds. Since $f \in S$, it follows that $f(z_1) = \lim_{t\to\infty} \mathrm{e}^t v(z_1, t)$ locally uniformly on U, where $v(z_1, t)$ is the unique Lipschitz continuous solution on $[0, \infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -vp(v, t) \quad \text{a.e.} \quad t \geq 0, \quad v(z_1, 0) = z_1,$$

for all $z_1 \in U$. Here $p(z_1, t)$ is a holomorphic function of $z_1 \in U$, $p(0, t) = 1$, $\Re p(z_1, t) > 0$ for $t \geq 0$, and $p(z_1, t)$ is a measurable function of $t \in [0, \infty)$ (see [18, 19] and [10]). Now, let $u(z_1, t) = \mathrm{e}^{t - \int_0^t a(\tau)\mathrm{d}\tau} v(z_1, t)$. Then $u(z_1, 0) = z_1$ and it is not difficult to deduce that $u(z_1, t)$ is the unique locally Lipschitz continuous solution on $[0, \infty)$ of the initial value problem

$$\frac{\partial u}{\partial t} = -uq(u, t) \quad \text{a.e.} \quad t \geq 0, \quad u(z_1, 0) = z_1,$$

where $q(z_1, t) = p(\mathrm{e}^{\int_0^t a(\tau)\mathrm{d}\tau - t} z_1, t) - 1 + a(t)$ for $z_1 \in U$ and $t \geq 0$. Since $\Re a(t) = 1$ for a.e. $t \geq 0$, it follows that $\Re q(z_1, t) > 0$ for a.e. $t \geq 0$ and for all $z_1 \in U$. Also $q(0, t) = a(t)$ for $t \geq 0$. Moreover, since $f(z_1) = \lim_{t \to \infty} \mathrm{e}^{\int_0^t a(\tau)\mathrm{d}\tau} u(z_1, t)$ locally uniformly on U, we deduce that f has a generalized a-parametric representation, and thus f is generalized a-asymptotically spirallike by Corollary 2.4 (ii). This completes the proof. □

REFERENCES

[1] Bracci, F., Contreras, M.D. and Madrigal, S.D., *Evolution families and the Loewner equation II: complex hyperbolic manifolds*, Math. Ann., **344** (2009), 947–962.

[2] Daleckii, Yu. and Krein, M.G., Stability of Solutions of Differential Equations in Banach Space, Translations of Mathematical Monographs, **43**, Amer. Math. Soc., 1974.

[3] Duren, P., Graham, I., Hamada, H. and Kohr, G., *Solutions for the generalized Loewner differential equation in several complex variables*, Math. Ann., **347** (2010), 411–435.

[4] Elin, M., Reich, S. and Shoikhet, D., *Complex dynamical systems and the geometry of domains in Banach spaces*, Dissertationes Math., **427** (2004), 1–62.

[5] Gong, S., *Convex and Starlike Mappings in Several Complex Variables*, Kluwer Acad. Publ., Dordrecht, 1998.

[6] Graham, I., Hamada, H. and Kohr, G., *Parametric representation of univalent mappings in several complex variables*, Canad. J. Math., **54** (2002), 324–351.

[7] Graham, I., Hamada, H., Kohr, G. and Kohr, M., *Parametric representation and asymptotic starlikeness in* $\mathbb{C}^n$, Proc. Amer. Math. Soc., **136** (2008), 3963–3973.

[8] Graham, I., Hamada, H., Kohr, G. and Kohr, M., *Asymptotically spirallike mappings in several complex variables*, J. Anal. Math., **105** (2008), 267–302.

[9] Graham, I., Hamada, H., Kohr, G. and Kohr, M., *Spirallike mappings and univalent subordination chains in* $\mathbb{C}^n$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **7** (2008), 717–740.

[10] Graham, I. and Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker, New York, 2003.

[11] Graham, I., Kohr, G. and Kohr, M., *Loewner chains and parametric representation in several complex variables*, J. Math. Anal. Appl., **281** (2003), 425–438.

[12] Gurganus, K., *Φ-like holomorphic functions in* $\mathbb{C}^n$ *and Banach spaces*, Trans. Amer. Math. Soc., **205** (1975), 389–406.

[13] Hamada, H. and Kohr, G., *Subordination chains and the growth theorem of spirallike mappings*, Mathematica, **42** (**65**) (2000), 155–163.

[14] Hamada, H. and Kohr, G., *An estimate of the growth of spirallike mappings relative to a diagonal matrix*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **55** (2001), 53–59.

[15] Hamada, H. and Kohr, G., *The growth theorem and quasiconformal extension of strongly spirallike mappings of type* α, Complex Var. Elliptic Equ., **44** (2001), 281–297.

[16] Hamada, H. and Kohr, G., Φ*-like and convex mappings in infinite dimensional spaces*, Rev. Roumaine Math. Pures Appl., **47** (2002), 315–328.

[17] Pfaltzgraff, J.A., *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$, Math. Ann., **210** (1974), 55–68.

[18] Pommerenke, C., *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math., **218** (1965), 159–173.

[19] Pommerenke, C., Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[20] Poreda, T., *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation, I - the geometrical properties*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **41** (1987), 105–113.

[21] Poreda, T., *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation, II - the necessary conditions and the sufficient conditions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **41** (1987), 115–121.

[22] Poreda, T., *On generalized differential equations in Banach spaces*, Dissertationes Math., **310** (1991), 1–50.

[23] Poreda, T. and Szadkowska, A., *On the holomorphic solutions of certain differential equations of first order for the mappings of the unit ball in* $\mathbb{C}^n$ *into* $\mathbb{C}^n$, Demonstratio Math., **22** (1989), 983–996.

[24] Reich, S. and Shoikhet, D., Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces, Imperial College Press, London, 2005.

[25] Suffridge, T.J., *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Math., **599** (1977), 146–159, Springer-Verlag.

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