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STARLIKENESS CONDITIONS FOR DIFFERENTIABLE OPEN MAPPINGS IN THE PLANE

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Abstract. We find starlikeness conditions for mappings defined on general simply connected domains $D \subset \mathbb{C}$. We extend results given by W.C. Royster [12], by P.T. Mocanu [8] for the unit ball and the results from [9], [10], [11] given for the ellipse. Our results improves the preceding theorems also by working with more general mappings, as the class of open, discrete mappings $f : D \to \mathbb{C}$, differentiable near ∂D and satisfying the differential condition $(*)$ near ∂D. This class of mappings is larger than the class of C^1 mappings $f : D \to \mathbb{C}$ satisfying condition $(*)$ on D, as is required in [8], [9], [10], [11], [12].

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The class

$$F(D) = \{f : D \to \mathbb{C} \mid D \subset \mathbb{C} \text{ is a domain and } f \text{ is continuous, open, discrete}\}$$

is the topological model of the analytic functions, introduced in mathematics by S. Stoilow [13]. For such mappings, the set

$$B_f = \{x \in D \mid f \text{ is not a local homeomorphism in } x\}$$

is an isolated set in D. Several of the classical theorems from complex analysis concerning univalence and the number of zeros hold for mappings from the class $F(D)$, as the univalence on the border theorem [3], [4], Rouché's theorem [2], Hurwitz's theorem [1], the argument principle [5], [6].

If $D \subset \mathbb{C}$ is a domain, $z \in D$ and $f : D \to \mathbb{C}$ is a map, we denote by $f'_F(z)$ the Fréchet derivative of f in z. We shall need the following argument principle, which follows immediately from Theorem 3 in [7]:

THEOREM 1. *Let $D \subset \mathbb{C}$ be a Jordan domain bounded by a Jordan curve $\gamma : [0,1] \to \partial D$, $f \in C(\overline{D}, \mathbb{C}) \cap F(D)$, $z_0 \in D \setminus B_f$ such that $f^{-1}(f(z_0)) \cap \overline{D} = \{z_0\}$, and $\Gamma = f \circ \gamma$ with Γ differentiable on $(0,1)$ and*

$$\operatorname{Im} \frac{\Gamma'(t)}{\Gamma(t) - f(z_0)} > 0 \text{ on } (0,1).$$

Then f is injective on D and $f(D)$ is a Jordan domain bounded by Γ which is starlike with respect to $f(z_0)$.

From this result we immediately obtain:

THEOREM 2. *Let $D \subset \mathbb{C}$ be a domain with the property that there exists Jordan domains $D_p \subset D$ bounded by Jordan curves $\gamma_p : [0,1] \to \partial D_p$, $p \in \mathbb{N}$ such that $D_p \nearrow D$. Let $f : D \cup \bigcup_{p=1}^{\infty} \partial D_p \to \mathbb{C}$ be continuous with $f \in F(D)$, and let $z_0 \in D$ be such that $f^{-1}(f(z_0)) = \{z_0\}$ and $z_0 \notin B_f$. Let $\Gamma_p = f \circ \gamma_p$ for $p \in \mathbb{N}$ and suppose that each curve Γ_p is differentiable on $(0,1)$ and*

$$(*) \qquad \operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} > 0$$

for every $t \in (0,1)$ and every $p \in \mathbb{N}$.

Then f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$.

Proof. We see from Theorem 1 that each domain $f(D_p)$ is a Jordan domain bounded by the Jordan curve Γ_p, which is starlike with respect to $f(z_0)$ and f is injective on D_p for every $p \in \mathbb{N}$. It results that f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$. $\square$

REMARK 1. Let $\gamma_p(t) = (x_p(t), y_p(t))$, $t \in [0,1]$, $p \in \mathbb{N}$, $f = P + \mathrm{i}Q$, $f(z_0) = P_0 + \mathrm{i}Q_0$. We have

$$\Gamma_p'(t) = f_F'(\gamma_p(t))(\gamma_p'(t)) = \begin{pmatrix} \frac{\partial P}{\partial x}(\gamma_p(t)) & \frac{\partial P}{\partial y}(\gamma_p(t)) \\ \frac{\partial Q}{\partial x}(\gamma_p(t)) & \frac{\partial Q}{\partial y}(\gamma_p(t)) \end{pmatrix} \begin{pmatrix} x_p'(t) \\ y_p'(t) \end{pmatrix}$$

$$= \left(\frac{\partial P}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial P}{\partial y}(\gamma_p(t))y_p'(t), \frac{\partial Q}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial Q}{\partial y}(\gamma_p(t))y_p'(t) \right)$$

for $t \in (0,1)$ and $p \in \mathbb{N}$.

Using the usual identification $(x,y) \to (x + \mathrm{i}y)$, we see that

$$\begin{aligned} \operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} &= \operatorname{Im} \frac{\left(\frac{\partial P}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial P}{\partial y}(\gamma_p(t))y_p'(t)\right)}{(P(\gamma_p(t)) - P_0) + \mathrm{i}(Q(\gamma_p(t)) - Q_0)} \\ &+ \operatorname{Im} \frac{\mathrm{i}\left(\frac{\partial Q}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial Q}{\partial y}(\gamma_p(t))y_p'(t)\right)}{(P(\gamma_p(t)) - P_0) + \mathrm{i}(Q(\gamma_p(t)) - Q_0)} \\ &= \frac{(P(\gamma_p(t)) - P_0)\left(\frac{\partial Q}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial Q}{\partial y}(\gamma_p(t)y_p'(t))\right)}{(P(\gamma_p(z)) - P_0)^2 + (Q(\gamma_p(t)) - Q_0)^2} \\ &- \frac{(Q(\gamma_p(t)) - Q_0)\left(\frac{\partial P}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial P}{\partial y}(\gamma_p(t))y_p'(t)\right)}{(P(\gamma_p(z)) - P_0)^2 + (Q(\gamma_p(t)) - Q_0)^2}. \end{aligned}$$

It results that condition $(*)$ from Theorem 2 holds if Royster's condition (8.2) in [12]

$$x'_p(t)[(P(\gamma_p(t) - P_0))\frac{\partial Q}{\partial x}(\gamma_p(t)) - (Q(\gamma_p(t)) - Q_0)\frac{\partial P}{\partial x}(\gamma_p(t))]$$
$$+ y'_p(t)[(P(\gamma_p(t)) - P_0)\frac{\partial Q}{\partial y}(\gamma_p(t)) - (Q(\gamma_p(t)) - Q_0)\frac{\partial P}{\partial y}(\gamma_p(t))] > 0$$

holds for every $t \in (0,1)$ and every $p \in \mathbb{N}$.

P.T. Mocanu [8] introduced the operator $Df = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}$, where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right).$$

He proved in [8] that if $f \in C^1(B, \mathbb{C})$ is such that $f(0) = 0$, $f(z) \neq 0$ for $z \neq 0$, $J_f(z) \neq 0$ on B and

$$(a) \qquad \operatorname{Re} \frac{Df(z)}{f(z)} > 0 \text{ on } B \setminus \{0\}$$

then f is injective on B and $f(B)$ is starlike with respect to 0.

Letting $f = P + \mathrm{i}Q$, condition (a) is equivalent to

$$x\left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) + y\left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) > 0 \text{ on } B \setminus \{0\}.$$

In the papers [9], [10], [11] there have been considered C^1 mappings $f : E \to \mathbb{C}$ defined on the ellipse

$$E = \left\{z = x + \mathrm{i}y \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}$$

and also the operator $D_{a,b}f = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}$, where

$$\frac{\partial f}{\partial z} = \frac{1}{2ab}\left(a^2\frac{\partial f}{\partial x} - \mathrm{i}b^2\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2ab}\left(a^2\frac{\partial f}{\partial x} + \mathrm{i}b^2\frac{\partial f}{\partial y}\right).$$

It has been proved that if $f \in C^1(E, \mathbb{C})$ is such that $f(0) = 0$, $f(z) \neq 0$ for $z \neq 0$, $J_f(z) \neq 0$ on E and

$$(a') \qquad \operatorname{Re} \frac{Df(z)}{f(z)} > 0 \text{ on } E \setminus \{0\},$$

then f is injective on E and $f(E)$ is starlike with respect to 0.

Letting $f = P + \mathrm{i}Q$, we see that condition (a') is equivalent to

$$\frac{b}{a}x\left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) + \frac{a}{b}y\left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) > 0 \text{ on } E \setminus \{0\}.$$

We shall generalize these results for more general mappings defined on more general domains.

THEOREM 3. *Let $D \subset \mathbb{C}$ be a domain with the property that there exist Jordan domains $D_p \subset D$, bounded by Jordan curves $\gamma_p : [0,1] \to \partial D_p$ which are differentiable on $(0,1)$ such that there exists differentiable mappings $g_p : \mathbb{C} \to \mathbb{C}$ such that $\gamma_p' = g_p \circ \gamma_p$ on $(0,1)$ for every $p \in \mathbb{N}$ and $D_p \nearrow D$. Let $f : D \cup \bigcup_{p=1}^{\infty} \partial D_p \to \mathbb{C}$ be continuous with $f \in F(D)$, f differentiable on $\bigcup_{p=1}^{\infty} \partial D_p$, $z_0 \in D \setminus B_f$, and $f^{-1}(f(z_0)) = \{z_0\}$. Denote $f = P + \mathrm{i}Q$, $f(z_0) = P_0 + \mathrm{i}Q_0$ and $g_p = g_{1p} + \mathrm{i}g_{2p}$ for $p \in \mathbb{N}$. If the relation*

$$(1) \qquad (P - P_0)\left(\frac{\partial Q}{\partial x}g_{1p} + \frac{\partial Q}{\partial y}g_{2p}\right) - (Q - Q_0)\left(\frac{\partial P}{\partial x}g_{1p} + \frac{\partial P}{\partial y}g_{2p}\right) > 0$$

holds on $\partial D_p \setminus \{\gamma_p(0)\}$ for every $p \in \mathbb{N}$, then f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$.

Proof. Let $\gamma_p(t) = x_p(t) + \mathrm{i}y_p(t)$ for $t \in [0,1]$ and $p \in \mathbb{N}$, and $\Gamma_p = f \circ \gamma_p$ for $p \in \mathbb{N}$. Then Γ_p is continuous on $[0,1]$ and differentiable on $(0,1)$ for every $p \in \mathbb{N}$ and $x_p'(t) = g_{1p}(\gamma_p(t))$, $y_p'(t) = g_{2p}(\gamma_p(t))$ for every $t \in (0,1)$ and every $p \in \mathbb{N}$. We have

$$\begin{aligned}
& x_p'(t)[(P(\gamma_p(t)) - P_0)\frac{\partial Q}{\partial x}(\gamma_p(t)) - (Q(\gamma_p(t)) - Q_0)\frac{\partial P}{\partial x}(\gamma_p(t))] \\
& + y_p'(t)[(P(\gamma_p(t)) - P_0)\frac{\partial Q}{\partial y}(\gamma_p(t)) - Q(\gamma_p(t) - Q_0)\frac{\partial P}{\partial y}(\gamma_p(t))] \\
& = (P(\gamma_p(t)) - P_0)\left(\frac{\partial Q}{\partial x}(\gamma_p(t))x_p'(t) + \frac{\partial Q}{\partial y}(\gamma_p(t))y_p'(t)\right) \\
& - (Q(\gamma_p(t)) - Q_0)\left(\frac{\partial P}{\partial x}(\gamma_p(t)) \cdot x_p'(t) + \frac{\partial P}{\partial y}(\gamma_p(t)) \cdot y_p'(t)\right) \\
& = (P(\gamma_p(t)) - P_0)\left(\frac{\partial Q}{\partial x}(\gamma_p(t)) \cdot g_{1p}(\gamma_p(t)) + \frac{\partial Q}{\partial y}(\gamma_p(t))g_{2p}(\gamma_p(t))\right) \\
& - (Q(\gamma_p(t)) - Q_0)\left(\frac{\partial P}{\partial x}(\gamma_p(t))g_{1p}(\gamma_p(t)) + \frac{\partial P}{\partial y}(\gamma_p(t))g_{2p}(\gamma_p(t))\right) > 0
\end{aligned}$$

for every $t \in (0,1)$ and every $p \in \mathbb{N}$. We used here the fact that condition (1) holds on every set $\partial D_p \setminus \{\gamma_p(0)\}$, $p \in \mathbb{N}$. Using Remark 1, we see that

$$\operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} > 0$$

for every $t \in (0,1)$ and every $p \in \mathbb{N}$. We apply now Theorem 2. □

REMARK 2. Usually, we take the domains $D_p \subset D$ from Theorem 2 and Theorem 3 such that $\overline{D}_p \subset D$ for every $p \in \mathbb{N}$ and such that $D_p \nearrow D$. Using Riemann's theorem, if $D \subset \mathbb{C}$ is a domain such that $\operatorname{card} \partial D > 1$, then D is simply connected if and only if there exists Jordan domains D_p such that

$\overline{D}_p \subset D$ for every $p \in \mathbb{N}$ and $D_p \nearrow D$. It results that the preceding theorems hold on simply connected domains $D \subset \mathbb{C}$.

THEOREM 4. *Let $D \subset \mathbb{C}$ be a bounded domain which is starlike with respect to a point $z_0 \in D$, bounded by a Jordan curve $\gamma : [0,1] \to \partial D$ that is differentiable on $(0,1)$, and such that there exists $g \in \mathcal{L}(\mathbb{C},\mathbb{C})$ such that $\gamma' = g \circ \gamma$ on $(0,1)$. Let $f \in F(D)$ satisfy the properties that $z_0 \in D \setminus B_f$ and $f^{-1}(f(z_0)) = \{z_0\}$. Let $f = P + \mathrm{i}Q$, $f(z_0) = P_0 + \mathrm{i}Q_0$, $g = g_1 + \mathrm{i}g_2$, and suppose that there exists a neighborhood U of ∂D on which f is differentiable and the relation*

$$\begin{aligned}(2)\qquad &(P-P_0)\left(\frac{\partial Q}{\partial x}(g_1 + rg_1(z_0)) + \frac{\partial Q}{\partial y}(g_2 + rg_2(z_0))\right)\\ &-(Q-Q_0)\left(\frac{\partial P}{\partial x}(g_1 + rg_1(z_0)) + \frac{\partial P}{\partial y}(g_2 + rg_2(z_0))\right) > 0\end{aligned}$$

holds on U for every $r > 0$. Then f is injective on D and $f(D)$ is starlike with respect to z_0.

Proof. We take $D_r = r(D \setminus \{z_0\})$, $\gamma_r : [0,1] \to \partial D_r$, $\gamma_r(t) = r(\gamma(t) - z_0)$ for $t \in [0,1]$ and $0 < r \le 1$, $g_r = g + rg(z_0)$ for $0 < r \le 1$. Then

$$\begin{aligned}\gamma_r'(t) = r\gamma'(t) = rg(\gamma(t)) = g(r\gamma(t)) &= g(r\gamma(t) - rz_0 + rz_0)\\ &= g(\gamma_r(t)) + rg(z_0) = g_r(\gamma_r(t))\end{aligned}$$

for $t \in [0,1]$ and $0 < r \le 1$. Since D is a bounded domain, we can find $0 < r_0 < 1$ such that $\partial D_r \subset U$ for $0 < r_0 < r \le 1$. Then f is differentiable on $\bigcup_{r_0 < r \le 1} \partial D_r$ and relation (2) holds on ∂D_r for $r_0 < r < 1$. We apply now Theorem 3. □

THEOREM 5. *Denote by E be the ellipse*

$$\left\{ z = x + \mathrm{i}y \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}.$$

Let $f \in F(D)$ be such that $f(0) = 0$, $0 \in E \setminus B_f$, and $f(z) \neq 0$ on $E \setminus \{0\}$. Suppose that there exists a neighborhood U of ∂E with the property that f is differentiable on U and the relation

$$(3)\qquad \frac{b}{a}x\left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) + \frac{a}{b}y\left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) > 0$$

holds on U. Then f is injective on E and $f(E)$ is starlike with respect to 0.

Proof. Let $\gamma : [0, 2\pi] \to \partial E$, $\gamma(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$. Then $\operatorname{Im}\gamma = \partial E$ and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$g(x,y) = \left(-\frac{a}{b}y, \frac{b}{a}x\right) \text{ for } (x,y) \in \mathbb{R}^2.$$

Then $g \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, $\gamma' = g \circ \gamma$ on $(0, 2\pi)$ and we have

$$\begin{aligned} &P\left(\frac{\partial Q}{\partial x}g_1 + \frac{\partial Q}{\partial y}g_2\right) - Q\left(\frac{\partial P}{\partial x}g_1 + \frac{\partial P}{\partial y}g_2\right) \\ &= P\left(\frac{-a}{b}y\frac{\partial Q}{\partial x} + \frac{b}{a}x\frac{\partial Q}{\partial y}\right) - Q\left(\frac{\partial P}{\partial x}\left(\frac{-a}{b}y\right) + \frac{\partial P}{\partial y}\frac{b}{a}x\right) \\ &= \frac{b}{a}x\left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) + \frac{a}{b}y\left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) > 0 \end{aligned}$$

on U. We apply now Theorem 4. □

We study now the case when D is an unbounded domain which is bounded by a simple curve $\gamma : (0,1) \to \partial D$ such that $\lim_{t\to 0} \gamma(t) = \infty$ and $\lim_{t\to 1} \gamma(t) = \infty$.

THEOREM 6. *Let $D \subset \mathbb{C}$ be an unbounded domain with the properties that* $\operatorname{card} \partial D > 1$ *and that there exist the domains $D_p \subset D$, bounded by simple curves $\gamma_p : (0,1) \to \partial D_p$ such that $\lim_{t\to 0} \gamma_p(t) = \infty$, $\lim_{t\to 1} \gamma_p(t) = \infty$, and $D_p \nearrow D$. Let $f \in F(D)$ be such that $\lim_{z\to\infty} f(z) = l \in \mathbb{C}$, $z_0 \in D \setminus B_f$, $f^{-1}(f(z_0)) = \{z_0\}$, and $\Gamma_p = f \circ \gamma_p$ for $p \in \mathbb{N}$. Furthermore, suppose that Γ_p is differentiable and*

$$\operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} > 0 \text{ on } (0,1)$$

for every $p \in \mathbb{N}$. Then f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$.

Proof. Using Riemann's theorem, we can find $\phi : B \to D$ a conformal bijection. Let $Q_p = \phi^{-1}(D_p)$ and $w_p = \phi^{-1} \circ \gamma_p$ for $p \in \mathbb{N}$. Then $w_p : (0,1) \to \partial Q_p$ extends on $[0,1]$ to a Jordan curve which bounds the Jordan domain Q_p for every $p \in \mathbb{N}$ and $Q_p \subset B$ for $p \in \mathbb{N}$, $Q_p \nearrow B$. Let $F : B \to \mathbb{C}$, $F = f \circ \phi$ and let $W_p = F \circ w_p$ and $\Gamma_p = f \circ \gamma_p$ for $p \in \mathbb{N}$. Then

$$W_p = F \circ w_p = f \circ \phi \circ \phi^{-1} \circ \gamma_p = f \circ \gamma_p = \Gamma_p$$

for every $p \in \mathbb{N}$, F is continuous on every set $\overline{Q}_p$, $p \in \mathbb{N}$ and let $a_0 = \phi^{-1}(z_0)$. Then $a_0 \in Q_p$ for every $p \in \mathbb{N}$, $F \in F(B)$ and we see that

$$\operatorname{Im} \frac{W_p'(t)}{W_p(t) - F(a_0)} = \operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} > 0$$

on $(0,1)$ for every $p \in \mathbb{N}$. Then $f = F \circ \phi^{-1}$, $F(a_0) = f(z_0)$ and $F(B) = f(D)$ and using Theorem 2 we see that F is injective on B and $F(B)$ is starlike with respect to $F(a_0)$. This implies that f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$. □

THEOREM 7. *Let $D \subset \mathbb{C}$ be an unbounded domain with the properties that* $\operatorname{card} \partial D > 1$ *and that there exist domains D_p, bounded by simple differentiable curves $\gamma_p : (0,1) \to \partial D_p$ such that $\lim_{t\to 0} \gamma_p(t) = \infty$, $\lim_{t\to 1} \gamma_p(t) = \infty$, and such* *that there exist differentiable mappings* $g_p : \mathbb{R}^2 \to \mathbb{R}^2$ *with* $\gamma_p' = g_p \circ \gamma_p$ *on* $(0,1)$, *for every* $p \in \mathbb{N}$, *and* $D_p \nearrow D$. *Let* $f \in F(D)$ *and* $z_0 \in D \setminus B_f$ *satisfy* $f^{-1}(f(z_0)) = \{z_0\}$. *Also, assume that* f *is differentiable on* $\bigcup_{p=1}^{\infty} \partial D_p$ *and* $\lim_{z\to\infty} f(z) = l \in \mathbb{C}$. *Denote* $f = P + \mathrm{i}Q$, $f(z_0) = P_0 + \mathrm{i}Q_0$, *and* $g_p = g_{1p} + \mathrm{i}g_{2p}$ *for* $p \in \mathbb{N}$. *If the relation*

$$(P - P_0)\left(\frac{\partial Q}{\partial x} g_{1p} + \frac{\partial Q}{\partial y} g_{2p}\right) - (Q - Q_0)\left(\frac{\partial P}{\partial x} g_{1p} + \frac{\partial P}{\partial y} g_{2p}\right) > 0 \tag{4}$$

holds on $\bigcup_{p=1}^{\infty} \partial D_p$, *then* f *is injective on* D *and* $f(D)$ *is starlike with respect to* $f(z_0)$.

Proof. Let $\Gamma_p = f \circ \gamma_p$ for $p \in \mathbb{N}$. Then Γ_p is differentiable on $(0,1)$ and, as in Theorem 3, we use relation (4) to show that

$$\operatorname{Im} \frac{\Gamma_p'(t)}{\Gamma_p(t) - f(z_0)} > 0$$

on $(0,1)$ for every $p \in \mathbb{N}$. We apply now Theorem 6. □

Theorem 8. *Let* $D \subset \mathbb{C}$ *be an unbounded domain bounded by a simple differentiable curve* $\gamma : (0,1) \to \partial D$ *such that* $\lim_{t\to 0} \gamma(t) = \infty$, $\lim_{t\to 1} \gamma(t) = \infty$, *and such that there exists* $g \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ *with* $\gamma' = g \circ \gamma$ *on* $(0,1)$. *Suppose that there exists* $u \in \mathbb{R}^2$ *with* $\partial D + ru \subset D$ *for every* $0 < r < 1$. *Let* $f \in F(D)$ *satisfy* $\lim_{z\to\infty} f(z) = l \in \mathbb{C}$, *and* $z_0 \in D$ *be such that* $z_0 \in D \setminus B_f$, $f^{-1}(f(z_0)) = \{z_0\}$. *Denote* $f = P + \mathrm{i}Q$, $f(z_0) = P_0 + \mathrm{i}Q_0$ *and* $g = g_1 + \mathrm{i}g_2$. *Suppose that there exists* $\varepsilon > 0$ *with the properties that* f *is differentiable on* $B(\partial D, \varepsilon)$ *and the relation*

$$\begin{aligned} &(P - P_0)\left[\frac{\partial Q}{\partial x}(g_1 - rg_1(u)) + \frac{\partial Q}{\partial y}(g_2 - rg_2(u))\right] \\ &\quad - (Q - Q_0)\left[\frac{\partial P}{\partial x}(g_1 - rg_1(u)) + \frac{\partial P}{\partial y}(g_2 - rg_2(u))\right] > 0 \end{aligned} \tag{5}$$

holds on $B(\partial D, \varepsilon)$. *Then* f *is injective on* D *and* $f(D)$ *is starlike with respect to* $f(z_0)$.

Proof. We take $D_t = D + ru$, $\gamma_r : (0,1) \to \partial D_r$, $\gamma_r = \gamma + ru$, $g_r : \mathbb{R}^2 \to \mathbb{R}^2$, $g_r = g - rg(u)$ for $0 < r < 1$. Then

$$g_{1r} = g_1 - rg_1(u), \quad g_{2r} = g_2 - rg_2(u)$$

for $0 < r < 1$ and

$$\gamma_r'(t) = \gamma'(t) = g(\gamma(t)) = g(\gamma_r(t) - ru) = g(\gamma_r(t)) - rg(u) = g_r(\gamma_r(t))$$

for $t \in (0,1)$ and $0 < r < 1$. Also, $\partial D_r \subset B(\partial D, \varepsilon)$ for $0 < r < \varepsilon$, hence f is differentiable on ∂D_r for $0 < r < \varepsilon$. We apply now Theorem 7. □

THEOREM 9. *Let $H = \{z = x + \mathrm{i}y \mid y > 0\}$, $z_0 \in H$, $f \in F(H)$ be such that $\lim_{z\to\infty} f(z) = l \in \mathbb{C}$, $z_0 \in H \setminus B_f$, and $f^{-1}(f(z_0)) = \{z_0\}$. Let $f = P + \mathrm{i}Q$ and $f(z_0) = P_0 + \mathrm{i}Q_0$. Suppose that there exists $\varepsilon > 0$ such that f is differentiable on $B(\partial H, \varepsilon)$ and the relation*

$$(6) \qquad (P - P_0)\frac{\partial Q}{\partial x} - (Q - Q_0)\frac{\partial P}{\partial x} > 0$$

holds on $B(\partial H, \varepsilon)$. Then f is injective on D and $f(D)$ is starlike with respect to $f(z_0)$.

Proof. We take $\gamma_r : (-\infty, \infty) \to \mathbb{R}^2$, $\gamma_r(t) = (t, r)$ for $t \in \mathbb{R}$ and $0 < r < 1$, and

$$D_r = \{(x, y) \mid y > r\} \text{ for } 0 < r < 1.$$

Then $\operatorname{Im} \gamma_r = \partial D_r$ and if $\Gamma_r = f \circ \gamma_r$ for $0 < r < 1$, we see that Γ_r is differentiable for $0 < r < \varepsilon$ and

$$\operatorname{Im} \frac{\Gamma_r'(t)}{\Gamma_r(t) - f(z_0)} = \operatorname{Im} \frac{f_F'(t,r)(1,0)}{f(t,r) - f(z_0)}$$

$$= \operatorname{Im} \frac{\begin{pmatrix} \frac{\partial P}{\partial x}(t,r) & \frac{\partial P}{\partial y}(t,r) \\ \frac{\partial Q}{\partial x}(t,r) & \frac{\partial Q}{\partial y}(t,r) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(P(t,r) - P_0) + \mathrm{i}(Q(t,r) - Q_0)}$$

$$= \operatorname{Im} \frac{\frac{\partial P}{\partial x}(t,r) + \mathrm{i}\frac{\partial Q}{\partial x}(t,r)}{(P(t,r) - P_0) + \mathrm{i}(Q(t,r) - Q_0)}$$

$$= \frac{(P(t,r) - P_0)\frac{\partial Q}{\partial x}(t,r) - (Q(t,r) - Q_0)\frac{\partial P}{\partial x}(t,r)}{(P(t,r) - P_0)^2 + (Q(t,r) - Q_0)^2} > 0$$

for every $t \in \mathbb{R}$. We apply now Theorem 6. □

THEOREM 10. *Let $D \subset \mathbb{C}$ be a domain bounded by a branch of the hyperbola*

$$G = \left\{ z = x + \mathrm{i}y \mid \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$

such that $0 \notin D$*. Let* $z_0 \in D$ *and* $f \in F(D)$ *be such that* $\lim_{z \to \infty} f(z) = l \in \mathbb{C}$, $z_0 \in D \setminus B_f$ *and* $f^{-1}(f(z_0)) = \{z_0\}$*. Denote* $f = P + \mathrm{i}Q$ *and* $f(z_0) = P_0 + \mathrm{i}Q_0$*. Suppose that there exists* $\varepsilon > 0$ *with the properties that* f *is differentiable on* $B(\partial D, \varepsilon)$ *and the relation*

$$(7)\qquad \begin{aligned} &(P-P_0)(z)\left(\frac{\partial Q}{\partial x}(z)\frac{a}{b}y + \frac{\partial Q}{\partial y}(z)\frac{b}{a}x\right) \\ &- (Q-Q_0)(z)\left(\frac{\partial P}{\partial x}(z)\frac{a}{b}y + \frac{\partial P}{\partial y}(z)\frac{b}{a}x\right) > 0 \end{aligned}$$

holds on $B(\partial D, \varepsilon)$*. Then* f *is injective on* D *and* $f(D)$ *is starlike with respect to* z_0*.*

Proof. We take $\gamma : (0, 2\pi) \to \mathbb{R}^2$, $\gamma(t) = (a\mathrm{ch}t, b\mathrm{sh}t)$ for $t \in (0, 2\pi)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$,

$$g(x,y) = \left(\frac{a}{b}y, \frac{b}{a}x\right) \text{ for } (x,y) \in \mathbb{R}^2.$$

We see that $\mathrm{Im}\,\gamma = \partial D$, that γ is differentiable on $(0, 2\pi)$ and $\gamma' = g \circ \gamma$ on $(0, 2\pi)$. Let $\gamma_r : (0, 2\pi) \to D$, $\gamma_r(t) = r\gamma(t)$ for $r > 1$, $t \in (0, 2\pi)$. Then $\mathrm{Im}\,\gamma_r \subset D$ and if $D_r \subset D$ is the domain bounded by $\mathrm{Im}\,\gamma_r$ for $r > 1$, we see that $D_r \nearrow D$ for $r \searrow 1$. Also,

$$\gamma_r'(t) = r\gamma'(t) = rg(\gamma(t)) = g(r\gamma(t)) = g(\gamma_r(t))$$

for $0 < t < 2\pi$ and $r > 1$. Taking $g_r = g$ for $r > 1$, we see that $\gamma_r' = g_r \circ \gamma_r$ on $(0, 2\pi)$, $r > 1$, that

$$g_{1r}(z) = g_1(z) = \frac{a}{b}y, \quad g_{2r}(z) = g_2(z) = \frac{b}{a}x$$

for $r > 1$ and $z \in \mathbb{R}^2$. We apply now Theorem 7. □

REMARK 3. If we assume in the Theorems 4, 5, 8, 9, and 10 that $f \in C(\overline{D}, \mathbb{C}) \cap F(D)$, then we can ask f to be differentiable only on ∂D and to satisfy condition $(*)$ only on ∂D.

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