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DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES AND WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. We give a characterization for the essential norm of differences of weighted composition operators acting between weighted Bergman spaces and weighted Banach spaces of analytic functions with sup-norms.

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1. INTRODUCTION

Let $\phi_1, \phi_2 : D \to D$ be analytic mappings, where D is the open unit disk in the complex plane. Through composition each such map induces a linear composition operator $C_{\phi_1}(f) = f \circ \phi_1$ resp. $C_{\phi_2}(f) = f \circ \phi_2$ acting on the space $H(D)$ of all holomorphic functions on D. Let now $\psi_1, \psi_2 : D \to \mathbb{C}$ be analytic mappings. We want to study differences of weighted composition operators $(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})(f) = \psi_1(f \circ \phi_1) - \psi_2(f \circ \phi_2)$.

Next, let us describe the setting in which these differences operate. Let v and w be strictly positive bounded continuous functions (*weights*) on D. We are interested in differences $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}$ acting between the weighted Bergman space

$$A_{v,p} = \{f \in H(D);\ \|f\|_{v,p} := \left(\int_D |f(z)|^p v(z)\, \mathrm{d}A(z)\right)^{\frac{1}{p}} < \infty\},\ 1 \leq p < \infty,$$

where $\mathrm{d}A(z)$ is the area measure on D normalized so that area of D is 1 and the weighted Banach space of holomorphic functions (weighted Bergman space of infinite order)

$$H_w^\infty = \{f \in H(D);\ \|f\|_w := \sup_{z \in D} w(z)|f(z)| < \infty\}.$$

These spaces appear in the study of growth conditions of analytic functions and have been studied in various articles, see e.g. [14], [15], [1], [10], [11], [2]. Concerning general information on Bergman spaces we refer the reader to the monographs [5] and [7].

Recently, Nieminen [12] characterized compactness of $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}$ acting on weighted Banach spaces of holomorphic functions generated by standard weights.

In [9] his results were generalized to more general weights and an expression (up to equivalence) was given for the essential norm. In this article we want to study the essential norm of differences of weighted composition operators in the setting described above.

2. PRELIMINARIES

For notation and general information on composition operators we refer the reader to the monographs [3] and [13]. By B_w^∞ we denote the closed unit ball of the space H_w^∞.

The formulation of many results on weighted spaces of analytic functions and on operators between them requires the so-called *associated weights* (see [2]). For a weight w the associated weight $\tilde{w}$ is defined as follows

$$\tilde{w}(z) := \frac{1}{\sup\{|f(z)|;\ f \in B_w^\infty\}}.$$

The associated weights have the following properties (see [2]):

(1) $\tilde{w}$ is continuous and subharmonic,
(2) $\tilde{w} \geq w > 0$,
(3) for every $z \in D$ there is $f_z \in B_w^\infty$ with $|f_z(z)| = \frac{1}{\tilde{w}(z)}$.

In order to handle differences of weighted composition operators we need some geometric data of the open unit disk as well as of the involved weights.

First, recall that the pseudohyperbolic metric $\rho(z,a)$ for $z, a \in D$ is defined by $\rho(z,a) = |\varphi_z(a)|$, where $\varphi_z(a) = \frac{z-a}{1-\overline{z}a}$. Furthermore we use the fact that

$$\varphi_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2},\ z \in D.$$

We denote by $A \sim B$ that A/B is bounded from above and below by two positive constants.

We are interested in radial weights w such that the following condition (which is due to Lusky [11]) holds

$$(L1) \quad \inf_k \frac{w(1-2^{-k-1})}{w(1-2^{-k})} > 0.$$

By Lusky [11] we know that each of the following weights has condition $(L1)$

$$\begin{aligned} v_p(z) &= (1-|z|)^p, \quad 0 < p < \infty, \text{ the standard weights,} \\ w_1(z) &= (1-\log(1-|z|))^{-\beta}, \quad \beta > 0, \\ w_2(z) &= (1-|z|)^p(1-\log(1-|z|))^{-\beta}, \quad 0 < p < \infty \text{ and } \beta > 0. \end{aligned}$$

For radial weights v satisfying $(L1)$, we have that v and $\tilde{v}$ are equivalent. For example if $v(z) = 1/\max_{|\lambda|=1} |f(\lambda z)|$ is a weight for some $f \in H(D)$, then $\tilde{v} = v$ (see [2]). From this we see that $v_p = \tilde{v}_p$ and $w_1 = \tilde{w}_1$.

Let v be a radial weight on D which is continuously differentiable with respect to $|z|$. Then it is known (see [4], [11], [6]) that Lusky's condition $(L1)$ is equivalent to each of the following three conditions:

(A) there are $0 < r < 1$ and $1 < C < \infty$ with $\frac{v(z)}{v(p)} \leq C$ for all $p, z \in D$ with $\rho(p, z) \leq r$,

(U) there is $\alpha > 0$ such that $\frac{v(z)}{(1-|z|)^\alpha}$ is increasing near the boundary of D,

(B) $\sup_{r\in[0,1[} \frac{(1-r)|v'(r)|}{v(r)} < \infty$.

3. RESULTS

In the sequel we consider the following weights. Let ν be a holomorphic function on D, non-vanishing, strictly positive on $[0, 1[$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v as follows $v(z) = \nu(|z|^2)$ for every $z \in D$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1 - z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1 - |z|^2)^\alpha$.
(ii) Select $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.
(iii) Choose $\nu(z) = \sin(1 - z)$ and the corresponding weight is given by $v(z) = \sin(1 - |z|^2)$.

For a fixed point $a \in D$ we introduce a function $v_a(z) := \nu(\overline{a}z)$ for every $z \in D$. Since ν is holomorphic on D, so is the function v_a. By [2] Corollary 1.6 we have $v = \tilde{v}$ and hence v is subharmonic.

We stated the following lemma already in [17], but for the sake of completeness we want to repeat the proof of it here.

Lemma 1. *Let v be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that*

$$\sup_{a\in D} \sup_{z\in D} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C_* < \infty.$$

Then there is $C > 0$ such that

$$|f(z)| \leq \frac{C}{(1 - |z|^2)^{\frac{2}{p}} v^{\frac{1}{p}}(z)} \|f\|_{v,p}$$

for all $z \in D$, $f \in A_{v,p}$.

Proof. Let $\alpha \in D$ be an arbitrary point. Consider the map

$$T_\alpha : A_{v,p} \to A_{v,p},\ T_\alpha(f(z)) = f(\varphi_\alpha(z))\varphi'_\alpha(z)^{\frac{2}{p}} v_\alpha(\varphi_\alpha(z))^{\frac{1}{p}}.$$

Then a change of variables yields

$$\begin{aligned}\|T_\alpha f\|_{v,p}^p &= \int_D v(z)|f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2|v_\alpha(\varphi_\alpha(z))|\,\mathrm{d}A(z)\\ &= \int_D \frac{v(z)|v_\alpha(\varphi_\alpha(z))|}{v(\varphi_\alpha(z))}|f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2 v(\varphi_\alpha(z))\,\mathrm{d}A(z)\\ &\le \sup_{z\in D}\frac{v(z)|v_\alpha(\varphi_\alpha(z))|}{v(\varphi_\alpha(z))}\int_D |f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2 v(\varphi_\alpha(z))\,\mathrm{d}A(z)\\ &\le C_*\int_D v(t)|f(t)|^p\,\mathrm{d}A(t) = C_*\|f\|_{v,p}^p.\end{aligned}$$

Now put $g(z) = T_\alpha(f(z))$. By the mean-value property we obtain

$$v(0)|g(0)|^p \le \int_D v(z)|g(z)|^p\,\mathrm{d}A(z) = \|g\|_{v,p}^p \le C_*\|f\|_{v,p}^p.$$

Hence

$$v(0)|g(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2 v(\alpha) \le C_*\|f\|_{v,p}^p.$$

Thus $|f(\alpha)| \le C_*^{\frac{1}{p}}\frac{\|f\|_{v,p}}{v^{\frac{1}{p}}(0)(1-|\alpha|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(\alpha)} \le C\frac{\|f\|_{v,p}}{(1-|\alpha|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(\alpha)}$. Since α was arbitrary, the claim follows. □

Lemma 2. *Let v be a weight on D as defined in the previous section (i.e. $v(z) = \nu(|z|^2)$ for every $z \in D$) such that*

$$\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \le C_* < \infty.$$

Moreover we assume that v satisfies the Lusky condition $(L1)$. There is a constant $M < \infty$ such that if $f \in A_{v,p}$, then

$$|(1-|a|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(a)f(a) - (1-|q|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(q)f(q)| \le M||f||_{v,p}\rho(a,q)$$

for all $a, q \in D$.

Proof. Since v and the standard weights have condition (A), there exist $0 < r < 1$ and $1 < M_1 < \infty$ such that $\frac{v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}}}{v^{\frac{1}{p}}(\xi)(1-|\xi|^2)^{\frac{2}{p}}} \le M_1$ for all $z, \xi \in D$ with $\rho(z,\xi) \le r$. An application of Cauchy's formula and of Lemma 1 yields for $z \in D$,

$$\begin{aligned}|f'(z)| &= \frac{1}{2\pi}\left|\int_{|\xi-z|=(1-|z|)r}\frac{f(\xi)}{(\xi-z)^2}\,d\xi\right|\\ &\le \frac{C\|f\|_{v,p}}{2\pi r^2(1-|z|)^2}\int_{|\xi-z|=(1-|z|)r}\frac{|d\xi|}{v^{\frac{1}{p}}(\xi)(1-|\xi|^2)^{\frac{2}{p}}}.\end{aligned}$$

Since $\rho(\xi,z)\le\frac{|\xi-z|}{1-|z|}=r$, we get

$$|f'(z)| \;\le\; \frac{CM_1}{2\pi r^2}\frac{2\pi(1-|z|)r}{(1-|z|)^2}\frac{\|f\|_{v,p}}{v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}}}=\frac{CM_1\|f\|_{v,p}}{r(1-|z|)v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}}}.$$

Let $h(z):=v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}}f(z)$, where $v(z)=v(z\cdot\overline{z})$. The total differential of h is given by $dh=\frac{\partial h}{\partial z}dz+\frac{\partial h}{\partial\overline{z}}d\overline{z}$. Then

$$\begin{aligned}\frac{\partial h}{\partial z}(z) &= \left(v'(z\cdot\overline{z})\overline{z}\frac{1}{p}v^{\frac{1}{p}-1}(z)(1-z\cdot\overline{z})^{\frac{2}{p}}-\overline{z}v^{\frac{1}{p}}(z\cdot\overline{z})\frac{2}{p}(1-z\cdot\overline{z})^{\frac{2}{p}-1}\right)f(z)\\ &+ v^{\frac{1}{p}}(z\cdot\overline{z})(1-z\cdot\overline{z})^{\frac{2}{p}}f'(z)\text{ and}\end{aligned}$$

$$\frac{\partial h}{\partial\overline{z}}(z)=\left(v'(z\cdot\overline{z})z\frac{1}{p}v^{\frac{1}{p}-1}(z\cdot\overline{z})(1-z\cdot\overline{z})^{\frac{2}{p}}-zv^{\frac{1}{p}}(z\cdot\overline{z})\frac{2}{p}(1-z\cdot\overline{z})^{\frac{2}{p}-1}\right)f(z)$$

yield using Lemma 1

$$\begin{aligned}|dh(z)| &\le (\frac{2}{p}|v'(|z|^2)|v^{\frac{1}{p}-1}(z)(1-|z|^2)^{\frac{2}{p}}|f(z)|+\frac{4}{p}v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}-1}|f(z)|\\ &+ v^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}}|f'(z)|)|dz|\\ &\le \left(\frac{2C}{p}\frac{|v'(|z|^2)|}{v(z)}\|f\|_{v,p}+\frac{4C}{p}\frac{1}{(1-|z|^2)}\|f\|_{v,p}+\frac{CM_1\|f\|_{v,p}}{r(1-|z|)}\right)|dz|\end{aligned}$$

By condition (B) there is $C_1>0$ with

$$\frac{|v'(|z|^2)|}{v(z)}\le\frac{C_1}{1-|z|}\text{ for all }z\in D.$$

Moreover, $\frac{1}{1-|z|^2}=\frac{1}{(1-|z|)(1+|z|)}\le\frac{1}{1-|z|}$ for every $z\in D$. Therefore

$$|dh(z)|\le\left(\frac{2C_1C}{p}+\frac{4C}{p}+\frac{CM_1}{r}\right)\frac{\|f\|_{v,p}}{(1-|z|)}|dz|.$$

If $\rho(a,q)\le r$, then by using $1-\rho(a,q)^2=\frac{(1-|q|^2)(1-|a|^2)}{|1-\overline{a}q|^2}$ we have that $1-|z|\sim 1-|a|$ for all z on the line between a and q and that $\frac{|q-a|}{1-|a|}\sim\rho(q,a)$. Here the constants only depend on r. By integration on both sides of the above inequality we can find constants $C_2,C_3>0$ with

$$\begin{aligned}|h(q)-h(a)| &\le C_2\|f\|_{v,p}\frac{1}{1-|a|}|q-a|\\ &\le C_3\|f\|_{v,p}\rho(q,a)\end{aligned}$$

for all $\rho(q,a)\le r$. If $\rho(a,q)>r$, then

$$\begin{aligned}&|v^{\frac{1}{p}}(q)(1-|q|^2)^{\frac{2}{p}}f(q)-v^{\frac{1}{p}}(a)(1-|a|^2)^{\frac{2}{p}}f(a)|\\ \le\;& 2\max\{v^{\frac{1}{p}}(q)(1-|q|^2)^{\frac{2}{p}}|f(q)|,v^{\frac{1}{p}}(a)(1-|a|^2)^{\frac{2}{p}}|f(a)|\}\\ \le\;& 2C\|f\|_{v,p}\le\frac{2C}{r}\|f\|_{v,p}\rho(a,q),\end{aligned}$$

and the claim follows. □

For weights of another form the following theorem was stated in [16]. There we obtained the same conditions.

THEOREM 1. *Let w be a radial weight and v be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \le C_* < \infty$. In addition we assume that v satisfies condition $(L1)$. Moreover let $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be analytic mappings. Then the weighted composition operator $\psi C_\phi : A_{v,p} \to H_w^\infty$ is bounded if and only if*

$$\sup_{z\in D}\frac{w(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(\phi(z))} < \infty.$$

Proof. By [2] we know that under the given assumptions v and $\tilde{v}$ are equivalent. First suppose that

$$M_1 = \sup_{z\in D}\frac{w(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}} < \infty.$$

By Lemma 1 we know

$$|f(z)| \le \frac{C\|f\|_{v,p}}{(1-|z|^2)^{\frac{2}{p}}v^{\frac{1}{p}}(z)}$$

for every $z \in D$ and every $f \in A_{v,p}$. Thus, for $z \in D$, we get

$$\begin{aligned}\|\psi C_\phi f\|_w &= \sup_{z\in D} w(z)|\psi(z)||f(\phi(z))|\\ &\le \sup_{z\in D}\frac{Cw(z)|\psi(z)|}{v^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2}{p}}}\|f\|_{v,p}.\end{aligned}$$

For the converse let $a \in D$ be arbitrary. There exists $f_a^p \in B_v^\infty$ such that $|f_a(a)|^p = \frac{1}{\tilde{v}(a)}$. Now put $g_a(z) := f_a(z)\varphi_a'(z)^{\frac{2}{p}}$. Then a change of variables yields

$$\begin{aligned}\|g_a\|_{v,p}^p &= \int_D |g_a(z)|^p v(z)\,\mathrm{d}A(z) = \int_D |f_a(z)|^p|\varphi_a'(z)|^2 v(z)\,\mathrm{d}A(z)\\ &\le \sup_{z\in D} v(z)|f_a(z)|^p\int_D|\varphi_a'(z)|^2\,\mathrm{d}A(z) \le \int_D|\varphi_a'(z)|^2\,\mathrm{d}A(z) = \int_D \mathrm{d}A(t) = 1.\end{aligned}$$

Next, we assume that there is a sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that $|\phi(z_n)| \to 1$ and

$$\frac{|\psi(z_n)|w(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi(z_n))(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \ge n$$

for every $n \in \mathbb{N}$.

Thus consider now $g_n(z) := g_{\phi(z_n)}(z)$ for every $n \in \mathbb{N}$ as defined above. Then we obtain that $(g_n)_n$ lies in the closed unit ball of $A_{v,p}$ and

$$c \geq w(z_n)|\psi(z_n)||g_n(\phi(z_n))| = \frac{w(z_n)|\psi(z_n)|}{\tilde{v}^{\frac{1}{p}}(\phi(z_n))(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq n$$

for every $n \in \mathbb{N}$, which is a contradiction. □

THEOREM 2. *Let w be a radial weight and v be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that*

$$\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C_* < \infty.$$

Moreover, we assume that v satisfies the Lusky condition $(L1)$. Let $\psi_1, \psi_2 \in H_w^\infty$.

If $\phi_1, \phi_2 : D \to D$ are analytic maps such that $\max\{||\phi_1||_\infty, ||\phi_2||_\infty\} = 1$ *and $\psi_1 C_{\phi_1}, \psi_2 C_{\phi_2} : A_{v,p} \to H_w^\infty$ are bounded, then the essential norm*

$$||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}||_e$$

is equivalent to the maximum of the following expressions:

(i) $\limsup_{|\phi_1(z)|\to 1} \frac{w(z)|\psi_1(z)|}{v^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}}\rho(\phi_1(z), \phi_2(z))$,

(ii) $\limsup_{|\phi_2(z)|\to 1} \frac{w(z)|\psi_2(z)|}{v^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}}\rho(\phi_1(z), \phi_2(z))$,

(iii) $\limsup_{\min\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} w(z)\left|\frac{\psi_1(z)}{v^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z)}{v^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}}\right|$.

Proof. We first prove the lower estimate of the essential norm.

(i) We can find a sequence $(z_n)_n \in D$ with $|\phi_1(z_n)| \to 1$ such that

$$\begin{aligned} &\lim_{n\to\infty} \frac{|\psi_1(z_n)|w(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}}\rho(\phi_1(z_n), \phi_2(z_n)) \\ = &\limsup_{|\phi_1(z)|\to 1} \frac{w(z)|\psi_1(z)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}}\rho(\phi_1(z), \phi_2(z)). \end{aligned}$$

Since $|\phi_1(z_n)| \to 1$, by going to a subsequence if necessary, we can use the proof of Theorem 3.1 in [8] to find functions $(g_n)_n \in H^\infty$ such that

$$\sum_{n=1}^{\infty}|g_n(z)| \leq 1 \quad \text{for all} \quad z \in D,$$

and $|g_n(\phi_1(z_n))| > 1-(\frac{1}{2})^n$ for every n. Hence $\lim_n |g_n(\phi_1(z_n))| = 1$. For every n there is f_n^p in B_v^∞ such that $f_n^p(\phi_1(z_n)) = \frac{1}{\tilde{v}(\phi_1(z_n))}$. Put

$$h_n(z) := g_n(z)\ \varphi_{\phi_2(z_n)}(z)\varphi'_{\phi_1(z_n)}(z)^{\frac{2}{p}}\ f_n(z)$$

for every $z \in D$ and every $n \in \mathbb{N}$. Then a change of variables yields

$$\begin{aligned}\|h_n\|_{v,p}^p &= \int_D |h_n(z)|^p v(z)\ \mathrm{d}A(z)\\ &= \int_D |f_n(z)|^p |\varphi'_{\phi_1(z_n)}(z)|^2 |\varphi_{\phi_2(z_n)}(z)|^p |g_n(z)|^p v(z)\ \mathrm{d}A(z)\\ &\le \sup_{z\in D} v(z)|f_n(z)|^p \sup_{z\in D} |\varphi_{\phi_2(z_n)}(z)|^p \sup_{z\in D} |g_n(z)|^p \int_D |\varphi'_{\phi_1(z_n)}(z)|^2\ \mathrm{d}A(z)\\ &= \int_D \mathrm{d}A(t) = 1.\end{aligned}$$

Thus, $h_n \in A_{v,p}$ with $||h_n||_{v,p} \le 1$, and the map $(\xi_k)_k \mapsto \sum_k \xi_k h_k$ is a well-defined, bounded operator from c_0 into $A_{v,p}$. Since the standard basis $(e_n)_n$ for c_0 tends weakly to zero, this implies that so does $(h_n)_n$.

Now let $K : A_{v,p} \to H_w^\infty$ be a compact operator. Then

$$\lim_{n\to\infty} ||Kh_n||_w = 0.$$

For each n,

$$||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K|| \ge ||(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})h_n||_w - ||Kh_n||_w,$$

and thus we conclude that

$$\begin{aligned}&||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K|| \ge \limsup_n ||\psi_1(h_n\circ\phi_1) - \psi_2(h_n\circ\phi_2)||_w\\ &\ge \limsup_n w(z_n)|\psi_1(z_n)h_n(\phi_1(z_n)) - \psi_2(z_n)h_n(\phi_2(z_n))|\\ &= \limsup_n w(z_n)|\psi_1(z_n)|\cdot|g_n(\phi_1(z_n))|\cdot|\varphi_{\phi_2(z_n)}(\phi_1(z_n))\ f_n(\phi_1(z_n))|\\ &\quad\cdot|\varphi'_{\phi_1(z_n)}(\phi_1(z_n))|^{\frac{2}{p}}\\ &= \limsup_n \frac{w(z_n)|\psi_1(z_n)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}}\rho(\phi_2(z_n),\phi_1(z_n)),\end{aligned}$$

and we obtain the claim.

(ii) follows analogously.

(iii) Let $(z_n)_n$ be a sequence with $|\phi_1(z_n)| \to 1$ and $|\phi_2(z_n)| \to 1$ such that

$$\lim_{n\to\infty} w(z_n)\left|\frac{\psi_1(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n)|^2)^{\frac{2}{p}}}\right| =$$

$$\limsup_{\min\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} w(z)\left|\frac{\psi_1(z)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}}\right|.$$

If $\rho(\phi_1(z_n),\phi_2(z_n)) \to \sigma \ne 0$ when $|\phi_1(z_n)| \to 1$ and $|\phi_2(z_n)| \to 1$, then (iii) follows from (i) and (ii). Therefore we can assume that $\rho(\phi_1(z_n),\phi_2(z_n)) \to 0$

if $|\phi_1(z_n)| \to 1$ and $|\phi_2(z_n)| \to 1$. Proceeding by choosing $(f_n)_n$ and $(g_n)_n$ as above we set

$$h_n(z) := g_n(z) f_n(z) \varphi'_{\phi_1(z_n)}(z).$$

Then $h_n \in A_{v,p}$, $\|h_n\|_{v,p} \le 1$, and $h_n \to 0$ weakly in $A_{v,p}$. Take a compact operator $K : A_{v,p} \to H_w^\infty$. Hence $\lim_n \|Kh_n\|_w = 0$. By assumption $\psi_2 C_{\phi_2}$ is bounded, hence by Theorem 1 we can find a constant $M_1 > 0$ such that

$$\sup_{z\in D} \frac{w(z)|\psi_2(z)|}{v^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}} \le M_1.$$

Thus, an application of Lemma 2 and the assumption $\tilde{v} = v$ yield

$$\begin{aligned}
&\|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K\| \\
&\ge \limsup_n w(z_n)|\psi_1(z_n) h_n(\phi_1(z_n)) - \psi_2(z_n) h_n(\phi_2(z_n))| \\
&= \limsup_n w(z_n)|\psi_1(z_n) f_n(\phi_1(z_n)) g_n(\phi_1(z_n)) \varphi'_{\phi_1(z_n))}(\phi_1(z_n)) \\
&-\psi_2(z_n) f_n(\phi_2(z_n)) g_n(\phi_2(z_n)) \varphi'_{\phi_1(z_n))}(\phi_2(z_n))| \\
&\ge \limsup_n w(z_n) \left| \frac{\psi_1(z_n) g_n(\phi_1(z_n))}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z_n) g_n(\phi_1(z_n))}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n)|^2)^{\frac{2}{p}}} \right| \\
&- \limsup_n w(z_n)| \frac{\psi_2(z_n) g_n(\phi_1(z_n))}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n)|^2)^{\frac{2}{p}}} \\
&-\psi_2(z_n) f_n(\phi_2(z_n)) g_n(\phi_2(z_n)) \varphi'_{\phi_1(z_n)}(\phi_2(z_n))| \\
&= \limsup_n | \frac{w(z_n)\psi_1(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}} - \frac{w(z_n)\psi_2(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n)|^2)^{\frac{2}{p}}} | \\
&\cdot |g_n(\phi_1(z_n))| \\
&- \limsup_n \frac{w(z_n)|\psi_2(z_n)|}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n))|^2)^{\frac{2}{p}}} \cdot \\
&\cdot |\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}} h_n(\phi_1(z_n)) \\
&-\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n))|^2)^{\frac{2}{p}} h_n(\phi_2(z_n))| \\
&= \limsup_n w(z_n) \left| \frac{\psi_1(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z_n))(1-|\phi_1(z_n)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z_n)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z_n))(1-|\phi_2(z_n)|^2)^{\frac{2}{p}}} \right|,
\end{aligned}$$

which proves the claim.

We now prove the upper estimate. Take the sequence of linear operators $C_k : H(D) \to H(D)$, $k \in \mathbb{N}$, defined by $C_k f(z) = f(\frac{k}{k+1} z)$, that are continuous for the compact open topology and $C_k f \to f$ uniformly on every compact subset of D and the operators $C_k : A_{v,p} \to A_{v,p}$ are well-defined and compact with $||C_k|| \le 1$.

For fixed $k \in \mathbb{N}$ we have, $||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}||_e \leq ||(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})(Id - C_k)||$. Fix $f \in A_{v,p}$ with $||f||_{v,p} \leq 1$ and $r \in (0,1)$. Put $g_k := (Id - C_k)f$, so $g_k \in A_{v,p}$ and $||g_k||_{v,p} \leq 2$. Then

$$\begin{aligned}
&\|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}\|_e \leq \sup_{||f||_{v,p}\leq 1} \|(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}) g_k\|_w \\
&\leq \sup_{||f||_{v,p}\leq 1} \sup_{\{z;|\phi_1(z)|>r\}} w(z)|\psi_1(z) g_k(\phi_1(z)) - \psi_2(z) g_k(\phi_2(z))| \\
&+ \sup_{||f||_{v,p}\leq 1} \sup_{\{z;|\phi_2(z)|>r\}} w(z)|\psi_1(z) g_k(\phi_1(z)) - \psi_2(z) g_k(\phi_2(z))| \\
&+ \sup_{||f||_{v,p}\leq 1} \sup_{\{z;|\phi_1(z)|\leq r, |\phi_2(z)|\leq r\}} w(z)|\psi_1(z) g_k(\phi_1(z)) - \psi_2(z) g_k(\phi_2(z))| \\
&=: I_{k,r} + J_{k,r} + L_{k,r}.
\end{aligned}$$

To estimate the first term $I_{k,r}$, for $z \in D$ with $|\phi_1(z)| > r$ we use Lemma 2 to get

$$\begin{aligned}
& w(z)|\psi_1(z) g_k(\phi_1(z)) - \psi_2(z) g_k(\phi_2(z))| \\
&\leq \left| \frac{w(z)\psi_1(z)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} - \frac{w(z)\psi_2(z)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}} \right| \\
&\cdot \tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}} |g_k(\phi_2(z))| \\
&+ \frac{w(z)|\psi_1(z)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} \\
&\cdot \left| \tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}} g_k(\phi_1(z)) - \tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}} g_k(\phi_2(z)) \right| \\
&\leq \left| \frac{w(z)\psi_1(z)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} - \frac{w(z)\psi_2(z)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}} \right| \\
&\cdot \tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}} |g_k(\phi_2(z))| \\
&+ 2M \frac{w(z)|\psi_1(z)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} \rho(\phi_1(z), \phi_2(z)).
\end{aligned}$$

Analogously we can estimate the term $J_{k,r}$.

The sequence of operators $(Id - C_k)_k$ satisfies $\lim_k (Id - C_k)g = 0$ for each g in $H(D)$, and the space $H(D)$ endowed with the compact open topology co is a Fréchet space. By the Banach-Steinhaus theorem, $(Id - C_k)_k$ converges to zero uniformly on the compact subsets of $(H(D), co)$. Since the closed unit ball of $A_{v,p}$ is a compact subset of $(H(D), co)$ we conclude that

$$\lim_k \sup_{||f||_{v,p}\leq 1} \sup_{|\xi|\leq r} |((Id - C_k)f)(\xi)| = 0.$$

If $|\phi_2(z)| \leq r$ in the term $I_{k,r}$, then by boundedness of $\psi_1 C_{\phi_1}$ and $\psi_2 C_{\phi_2}$, we conclude that

$$\lim_{r\to 1} \limsup_{k} I_{k,r} \leq 2M \limsup_{|\phi_1(z)|\to 1} \frac{w(z)|\psi_1(z)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} \rho(\phi_1(z),\phi_2(z)).$$

In the case $|\phi_2(z)| > r$, we have that

$$\begin{aligned}
&\lim_{r\to 1} \limsup_{k} I_{k,r} \\
\leq& 2 \limsup_{\min\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} w(z) \\
&\cdot \left| \frac{\psi_1(z)}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} - \frac{\psi_2(z)}{\tilde{v}^{\frac{1}{p}}(\phi_2(z))(1-|\phi_2(z)|^2)^{\frac{2}{p}}} \right| \\
+& 2M \limsup_{|\phi_1(z)|\to 1} \frac{w(z)|\psi_1(z)|}{\tilde{v}^{\frac{1}{p}}(\phi_1(z))(1-|\phi_1(z)|^2)^{\frac{2}{p}}} \rho(\phi_1(z),\phi_2(z)).
\end{aligned}$$

Analogously we consider the cases $|\phi_1(z)| \leq r$ and $|\phi_1(z)| > r$ in the term $J_{k,r}$.

Since $\psi_1, \psi_2 \in H^\infty_w$, we have that $\lim_{r\to 1} \limsup_k L_{k,r} = 0$, and the statement follows. $\square$

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