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DIFFERENTIAL SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS DEFINED BY THE DZIOK-SRIVASTAVA LINEAR OPERATOR

VERONICA O. NECHITA

Abstract. In this paper we extend previously known results and obtain two sandwich theorems for analytic functions in the unit disk defined with the Dziok-Srivastava linear operator.

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1. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \dots\}.$$

We also consider the class

$$\mathcal{A} = \left\{f \in \mathcal{H} : f(z) = z + a_2 z^2 + \dots\right\}.$$

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

We use the terms of subordination and superordination, so we review here these definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

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Let $\psi\colon \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in \mathcal{Q}$. In [4], the authors considered the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \tag{1}$$

implies $p(z) \prec q(z)$ for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1). Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1).

Let $\varphi\colon \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. Recently, in [5], the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \tag{2}$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination. Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superodination (2).

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For $l, m \in \mathbb{N}$, $l \le m+1$, $\alpha_j \in \mathbb{C}$, $j = 1, 2, \ldots, l$, and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, \ldots, m$, the generalized hypergeometric function

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

is given by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$
$$(l \le m+1; l, m \in \mathbb{N})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)(a+2)\ldots(a+n-1) & \text{if } n \in \mathbb{N}^*. \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\,{}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

the Dziok-Srivastava operator $H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)$ is given in [3] by the Hadamard product

$$\begin{aligned}&H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z)\\&:=h(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)*f(z)\\&=z+\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{a_{n+1}z^{n+1}}{n!}.\end{aligned}$$

For brevity, we write

$$H_m^l[\alpha_1]f(z):=H_m^l(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z).$$

In this paper we will determine some properties on admissible functions defined with the Dziok-Srivastava linear operator.

2. PRELIMINARIES

In our present investigation we shall need the folllowing results.

THEOREM 2.1 ([4], Theorem 3.4h., p. 132). *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$, when $w\in q(U)$. Set $Q(z)=zq'(z)\cdot\phi[q(z)]$, $h(z)=\theta[q(z)]+Q(z)$ and suppose that either*

(i) *h is convex*

or

(ii) *Q is starlike.*

In addition, assume that

(iii) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)}>0.$

If p is analytic in U, with $p(0)=q(0)$, $p(U)\subset D$ and

$$\theta[p(z)]+zp'(z)\cdot\phi[p(z)]\prec\theta[q(z)]+zp'(z)\cdot\phi[q(z)]=h(z),$$

then $p\prec q$, and q is the best dominant.

By taking $\theta(w):=w$ and $\phi(w):=\gamma$ in Theorem 2.1, we get

COROLLARY 2.2. *Let q be univalent in U, $\gamma\in\mathbb{C}^*$ and suppose*

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\mathrm{Re}\,\frac{1}{\gamma}\right\}.$$

If p is analytic in U, with $p(0)=q(0)$ and

$$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z),$$

then $p\prec q$, and q is the best dominant.

THEOREM 2.3 ([6]). *Let θ and ϕ be analytic in a domain D and let q be univalent in U, with $q(0)=a$, $q(U)\subset D$. Set $Q(z)=zq'(z)\cdot\phi[q(z)]$, $h(z)=\theta[q(z)]+Q(z)$ and suppose that*

(i) $\operatorname{Re}\left[\frac{\theta'[q(z)]}{\phi[q(z)]}\right] > 0$

and

(ii) $Q(z)$ *is starlike.*

If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(U) \subset D$ *and* $\theta[p(z)] + zp'(z) \cdot \phi[p(z)]$ *is univalent in* U, *then*

$$\theta[q(z)] + zp'(z) \cdot \phi[q(z)] \prec \theta[p(z)] + zp'(z) \cdot \phi[p(z)] \Rightarrow q \prec p$$

and q *is the best subordinant.*

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.3, we get

COROLLARY 2.4 ([1]). *Let* q *be convex in* U, $q(0) = a$ *and* $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. *If* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ *and* $p(z) + \gamma zp'(z)$ *is univalent in* U, *then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \Rightarrow q \prec p$$

and q *is the best subordinant.*

3. MAIN RESULTS

THEOREM 3.1. *Let* q *be univalent in* U *with* $q(0) = 1$, $\gamma \in \mathbb{C}^*$ *and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ *and*

$$\begin{aligned}\gamma\alpha_1 + (1+\gamma)\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} - \\ -\gamma(\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z) \cdot H_m^l[\alpha_1]f(z)}{\left\{H_m^l[\alpha_1+1]f(z)\right\}^2} \prec q(z) + \gamma zq'(z),\end{aligned} \tag{3}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} \prec q(z)$$

and q *is the best dominant.*

Proof. We define the function p by

$$p(z) := \frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)}. \tag{4}$$

We calculate the derivative of $p(z)$ and we get

$$\begin{aligned}&p'(z) = \\ &= \frac{\left\{H_m^l[\alpha_1]f(z)\right\}' H_m^l[\alpha_1+1]f(z) - \left\{H_m^l[\alpha_1+1]f(z)\right\}' H_m^l[\alpha_1]f(z)}{\left\{H_m^l[\alpha_1+1]f(z)\right\}^2}.\end{aligned} \tag{5}$$

By using the identity

$$z\left\{H_m^l[\alpha_1]f(z)\right\}' = \alpha_1 H_m^l[\alpha_1+1]f(z) - (\alpha_1-1)H_m^l[\alpha_1]f(z), \tag{6}$$

we obtain from (5) that

$$zp'(z) = \alpha_1 + \frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} - (\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z)\cdot H_m^l[\alpha_1]f(z)}{\left\{H_m^l[\alpha_1+1]f(z)\right\}^2}$$

and

$$p(z) + \gamma zp'(z) = \gamma\alpha_1 + (1+\gamma)\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} -$$
$$-\gamma(\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z)\cdot H_m^l[\alpha_1]f(z)}{\left\{H_m^l[\alpha_1+1]f(z)\right\}^2}.$$

The subordination (3) from the hypothesis becomes

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z).$$

We obtain the conclusion of our theorem by applying now Corrolary 2.2. □

For $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$, $\beta_1 = c$, the Dziok-Srivastava linear oparator $H_m^l[\alpha_1]f(z)$ becomes the Carlson-Shaffer linear operator $L(a,c)f(z)$ introduced in [2]. By taking these values in Theorem 3.1, we obtain the following corollary.

Corollary 3.2 ([7]). *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\gamma a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - \gamma(a+1)\frac{L(a+2,c)f(z)\cdot L(a,c)f(z)}{\left\{L(a+1,c)f(z)\right\}^2} \prec$$
$$\prec q(z) + \gamma zq'(z),$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z)$$

and q is the best dominant.

By taking $l = 1$, $m = 0$ and $\alpha_1 = 1$ in Theorem 3.1, we get the following result.

Corollary 3.3 ([7]). *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ *and*

$$\gamma \left\{ 1 - \frac{f''(z) f(z)}{[f'(z)]^2} \right\} + (1-\gamma) \frac{f(z)}{z f'(z)} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{f(z)}{z f'(z)} \prec q(z)$$

and q *is the best dominant.*

We give an application of Theorem 3.1 for a particular convex function q.

Corollary 3.4. *Let* $A, B \in \mathbb{C}$, $A \neq B$, $|B| \leq 1$ *and* $\gamma \in \mathbb{C}$ *such that* $\operatorname{Re} \gamma > 0$. *If* $f \in \mathcal{A}$ *and*

$$\gamma \alpha_1 + (1+\gamma) \frac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)} - \gamma (\alpha_1 + 1) \frac{H_m^l [\alpha_1 + 2] f(z) \cdot H_m^l [\alpha_1] f(z)}{\{H_m^l [\alpha_1 + 1] f(z)\}^2} \prec$$

$$\prec \frac{1 + Az}{1 + Bz} + \gamma \frac{(A - B) z}{(1 + Bz)^2},$$

then

$$\frac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)} \prec \frac{1 + Az}{1 + Bz}$$

and $q(z) = \dfrac{1 + Az}{1 + Bz}$ *is the best dominant.*

We next give a result concerning superordinations.

Theorem 3.5. *Let* q *be convex in* U, $q(0) = 1$ *and* $\gamma \in \mathbb{C}$, $\operatorname{Re} \gamma > 0$. *If* $f \in \mathcal{A}$, $\dfrac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$,

$$\gamma \alpha_1 + (1+\gamma) \frac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)} - \gamma (\alpha_1 + 1) \frac{H_m^l [\alpha_1 + 2] f(z) \cdot H_m^l [\alpha_1] f(z)}{\{H_m^l [\alpha_1 + 1] f(z)\}^2}$$

is univalent in U *and*

$$q(z) + \gamma z q'(z) \prec \gamma \alpha_1 + (1+\gamma) \frac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)} -$$

$$- \gamma (\alpha_1 + 1) \frac{H_m^l [\alpha_1 + 2] f(z) \cdot H_m^l [\alpha_1] f(z)}{\{H_m^l [\alpha_1 + 1] f(z)\}^2},$$

then

$$q(z) \prec \frac{H_m^l [\alpha_1] f(z)}{H_m^l [\alpha_1 + 1] f(z)}$$

and q *is the best subordinant.*

Proof. The conclusion follows immediately by applying Corollary 2.4 to the function p defined in (4). $\square$

We can combine the results of Theorem 3.1 and Theorem 3.5 to obtain the following “sandwich theorem”.

Corollary 3.6. *Let* q_1, q_2 *be convex in* U, $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. *If* $f \in \mathcal{A}$, $\dfrac{H_m^l[\alpha_1] f(z)}{H_m^l[\alpha_1+1] f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$,

$$\gamma\alpha_1 + (1+\gamma)\frac{H_m^l[\alpha_1] f(z)}{H_m^l[\alpha_1+1] f(z)} - \gamma(\alpha_1+1)\frac{H_m^l[\alpha_1+2] f(z) \cdot H_m^l[\alpha_1] f(z)}{\left\{H_m^l[\alpha_1+1] f(z)\right\}^2}$$

is univalent in U *and*

$$q_1(z) + \gamma z q_1'(z) \prec \gamma\alpha_1 + (1+\gamma)\frac{H_m^l[\alpha_1] f(z)}{H_m^l[\alpha_1+1] f(z)} -$$

$$-\gamma(\alpha_1+1)\frac{H_m^l[\alpha_1+2] f(z) \cdot H_m^l[\alpha_1] f(z)}{\left\{H_m^l[\alpha_1+1] f(z)\right\}^2} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{H_m^l[\alpha_1] f(z)}{H_m^l[\alpha_1+1] f(z)} \prec q_2(z)$$

and the functions q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

Theorem 3.7. *Let* q *be univalent in* U *with* $q(0) = 1$, $\gamma \in \mathbb{C}^*$ *and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ *and*

$$\begin{aligned} &[1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1] f(z)}{\left\{H_m^l[\alpha_1] f(z)\right\}^2} + \gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2] f(z)}{\left\{H_m^l[\alpha_1] f(z)\right\}^2} - \\ &- 2\alpha_1\gamma\frac{z\left\{H_m^l[\alpha_1+1] f(z)\right\}^2}{\left\{H_m^l[\alpha_1] f(z)\right\}^3} \prec q(z) + \gamma z q'(z), \end{aligned} \tag{7}$$

then

$$z\frac{H_m^l[\alpha_1+1] f(z)}{\left\{H_m^l[\alpha_1] f(z)\right\}^2} \prec q(z)$$

and q *is the best dominant.*

Proof. Let

$$p(z) := z\frac{H_m^l[\alpha_1+1] f(z)}{\left\{H_m^l[\alpha_1] f(z)\right\}^2}. \tag{8}$$

A simple computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z\left\{H_m^l[\alpha_1+1] f(z)\right\}'}{H_m^l[\alpha_1+1] f(z)} - 2\frac{z\left\{H_m^l[\alpha_1] f(z)\right\}'}{H_m^l[\alpha_1] f(z)}. \tag{9}$$

By using the identity (6), we obtain from (9) that

$$\frac{zp'(z)}{p(z)} = (\alpha_1 - 1) + (\alpha_1 + 1)\frac{H_m^l[\alpha_1 + 2] f(z)}{H_m^l[\alpha_1 + 1] f(z)} - 2\alpha_1 \frac{H_m^l[\alpha_1 + 1] f(z)}{H_m^l[\alpha_1] f(z)}$$

and

$$p(z) + \gamma z p'(z) = [1 + \gamma(\alpha_1 - 1)] \frac{z H_m^l[\alpha_1 + 1] f(z)}{\{H_m^l[\alpha_1] f(z)\}^2} +$$

$$+\gamma(\alpha_1 + 1)\frac{z H_m^l[\alpha_1 + 2] f(z)}{\{H_m^l[\alpha_1] f(z)\}^2} - 2\alpha_1\gamma \frac{z\{H_m^l[\alpha_1 + 1] f(z)\}^2}{\{H_m^l[\alpha_1] f(z)\}^3}.$$

Hence the hypothesis (7) yields the subordination.

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

Now the conclusion of our theorem follows by simply applying Corollary 2.2. □

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$, $\beta_1 = c$ Theorem 3.7 becomes

COROLLARY 3.8 ([7]). *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that*

$$\mathrm{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\,\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$[1 + \gamma(a - 1)]\frac{zL(a + 1, c) f(z)}{\{L(a, c) f(z)\}^2} + \gamma(a + 1)\frac{zL(a + 2, c) f(z)}{\{L(a, c) f(z)\}^2} -$$

$$-2a\gamma \frac{z\{L(a + 1, c) f(z)\}^2}{\{L(a, c) f(z)\}^3} \prec q(z) + \gamma z q'(z),$$

then

$$z\frac{L(a + 1, c) f(z)}{\{L(a, c) f(z)\}^2} \prec q(z)$$

and q is the best dominant.

By taking $l = 1$, $m = 0$ and $\alpha_1 = 1$ in Theorem 3.7, we obtain:

COROLLARY 3.9 ([7]). *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that*

$$\mathrm{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\,\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)'' \prec q(z) + \gamma z q'(z),$$

then

$$\frac{z^2 f'(z)}{\{f(z)\}^2} \prec q(z)$$

and q is the best dominant.

We consider $q(z)=\dfrac{1+Az}{1+Bz}$ and give the following application of Theorem 3.7.

Corollary 3.10. *Let $A,B\in\mathbb{C}$, $A\neq B$, $|B|\leq 1$ and $\gamma\in\mathbb{C}$ such that* $\operatorname{Re}\gamma>0$*. If $f\in\mathcal{A}$ and*

$$[1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}+\gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}-$$

$$-2\alpha_1\gamma\frac{z\{H_m^l[\alpha_1+1]f(z)\}^2}{\{H_m^l[\alpha_1]f(z)\}^3}\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$z\frac{H_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ is the best dominant.

We apply Corollary 2.4 to the function p given by (8) in the proof of Theorem 3.7 to obtain the following result.

Theorem 3.11. *Let q be convex in U, $q(0)=1$ and $\gamma\in\mathbb{C}$,* $\operatorname{Re}\gamma>0$*. If $f\in\mathcal{A}$, $z\dfrac{H_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}\in\mathcal{H}[1,1]\cap\mathcal{Q}$,*

$$[1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}+$$

$$+\gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}-2\alpha_1\gamma\frac{z\{H_m^l[\alpha_1+1]f(z)\}^2}{\{H_m^l[\alpha_1]f(z)\}^3}$$

is univalent in U and

$$q(z)+\gamma zq'(z)\prec[1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}+$$

$$+\gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}-2\alpha_1\gamma\frac{z\{H_m^l[\alpha_1+1]f(z)\}^2}{\{H_m^l[\alpha_1]f(z)\}^3},$$

then

$$q(z)\prec z\frac{H_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2}$$

and q is the best subordinant.

By combining the results of Theorem 3.7 and Theorem 3.11 we finally get the following “sandwich theorem”.

COROLLARY 3.12. *Let q_1, q_2 be convex in U, $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$, $z\frac{H_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$,*

$$[1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} + \gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} - 2\alpha_1\gamma\frac{z\{H_m^l[\alpha_1+1]f(z)\}^2}{\{H_m^l[\alpha_1]f(z)\}^3}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec [1+\gamma(\alpha_1-1)]\frac{zH_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} + \gamma(\alpha_1+1)\frac{zH_m^l[\alpha_1+2]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} - 2\alpha_1\gamma\frac{z\{H_m^l[\alpha_1+1]f(z)\}^2}{\{H_m^l[\alpha_1]f(z)\}^3} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec z\frac{H_m^l[\alpha_1+1]f(z)}{\{H_m^l[\alpha_1]f(z)\}^2} \prec q_2(z)$$

and the functions q_1 and q_2 are respectively the best subordinant and the best dominant.

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"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: nechita@math.ubbcluj.ro