MATHEMATICA, Tome 50 (73), N° 1, 2008, pp. 75–83

ON NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

S. LATHA and N. POORNIMA

Abstract. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0,\ n \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. In this note, we define certain subclasses $\mathcal{S}_n^*(A,B)$, $\mathcal{C}_n(A,B)$, $\mathcal{R}_n(A,B)$, $\mathcal{Q}_n(A,B)$, $\mathcal{S}_n(A,B;C,D)$ and $\mathcal{C}_n(A,B;C,D)$ of $\mathcal{A}(n)$ and some properties of neighborhoods are studied for these classes.

MSC 2000. 30C45.
Key words. Univalent functions, neighborhoods, convex functions, starlike functions.

1. INTRODUCTION

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0,\ n \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by Ω the class of analytic functions ω on $\mathcal{U}$ satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$, for $z \in \mathcal{U}$.

For any function $f \in \mathcal{A}(n)$ and $\delta \geq 0$ we define

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k,\ \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\} \tag{2}$$

which is the (n,δ)-neighborhood of f.

For $e(z) = z$ we see that

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k,\ \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{3}$$

The concept of neighborhoods was first introduced by Goodman and then generalized by Ruschweyh [7].

In this paper, we consider (n,δ)-neighborhoods for functions with negative coefficients in $\mathcal{U}$.

We designate $\mathcal{P}_n(A,B)$ as the class of functions defined on $\mathcal{U}$, which are of the form

$$\frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1 \quad \text{and} \quad \omega \in \Omega.$$

2. NEIGHBORHOODS FOR CLASSES $\mathcal{S}_N^*(A,B)$ AND $\mathcal{C}_N(A,B)$

In this section we obtain a necessary and sufficient conditions for functions to be in the classes $\mathcal{S}_n^*(A,B)$ and $\mathcal{C}_n(A,B)$. Further, neighborhoods of these classes are determined. The classes $\mathcal{S}_n^*(A,B)$ and $\mathcal{C}_n(A,B)$ are defined as follows:

$$\mathcal{S}_n^*(A,B) = \left\{ f : f \in \mathcal{A}(n) \text{ and } z \mapsto \frac{zf'(z)}{f(z)} \text{ is in } P_n(A,B) \right\}$$

and

$$\mathcal{C}_n(A,B) = \left\{ f : f \in \mathcal{A}(n) \text{ and } z \mapsto \frac{(zf'(z))'}{f'(z)} \text{ is in } P_n(A,B) \right\}.$$

Lemma 1. *A function $f \in \mathcal{S}_n^*(A,B)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k(B+1)-(A+1)}{B-A} a_k \le 1, \quad -1 \le A < B \le 1. \tag{4}$$

Proof. Suppose $f \in \mathcal{S}_n^*(A,B)$. Then

$$\frac{zf'(z)}{f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1.$$

That is,

$$\omega(z) = \frac{1 - \dfrac{zf'(z)}{f(z)}}{B\dfrac{zf'(z)}{f(z)} - A}, \qquad \omega(0) = 0,$$

and

$$|\omega(z)| = \left| \frac{zf'(z) - f(z)}{Bzf'(z) - Af(z)} \right| = \left| \frac{\displaystyle\sum_{k=n+1}^{\infty} (k-1)a_k z^k}{(B-A)z - \displaystyle\sum_{k=n+1}^{\infty} (Bk-A)a_k z^k} \right| < 1.$$

Thus

$$\Re \left\{ \frac{\displaystyle\sum_{k=n+1}^{\infty} (k-1)a_k z^k}{(B-A)z - \displaystyle\sum_{k=n+1}^{\infty} (Bk-A)a_k z^k} \right\} < 1. \tag{5}$$

Let $z=r,$ with $0<r<1.$ Then, for sufficiently small r, the denominator of (5) is positive and so it is positive for all r, with $0<r<1,$ since ω is regular for $|z|<1$. Then (5) gives

$$\sum_{k=n+1}^{\infty}(k-1)a_k r^k<(B-A)r-\sum_{k=n+1}^{\infty}(Bk-A)a_k r^k.$$

That is,

$$\sum_{k=n+1}^{\infty}[k(B+1)-(A+1)]a_k r^k<(B-A)r.$$

Letting $r\mapsto 1,$ we obtain (4).

Conversely, for $|z|=r,\ 0<r<1,$ we have, by (4) and since $r^k<r,$

$$\sum_{k=n+1}^{\infty}[k(B+1)-(A+1)]a_k r^k<r\sum_{k=n+1}^{\infty}[k(B+1)-(A+1)]a_k<(B-A)r.$$

So we get

$$\begin{aligned}\left|\sum_{k=n+1}^{\infty}(k-1)a_k z^k\right| &\le \sum_{k=n+1}^{\infty}(k-1)a_k r^k< \\ &<(B-A)r-\sum_{k=n+1}^{\infty}(Bk-A)a_k r^k< \\ &<\left|(B-A)z-\sum_{k=n+1}^{\infty}(Bk-A)a_k z^k\right|.\end{aligned}$$

This proves that $\dfrac{zf'(z)}{f(z)}$ is of the form

$$\frac{1+A\omega(z)}{1+B\omega(z)},\quad (\omega(z)\in\Omega).$$

Therefore $f\in\mathcal{S}_n^*(A,B)$. Thus the proof is complete. □

Lemma 2. *A function* $f\in\mathcal{C}_n(A,B)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\frac{k[k(B+1)-(A+1)]}{B-A}a_k\le 1,\quad -1\le A<B\le 1. \tag{6}$$

Proof. A function $f\in\mathcal{C}_n(A,B)$ if and only if the function $z\mapsto zf'(z)$ belongs to $\mathcal{S}_n^*(A,B)$, so the conclusion follows from Lemma 1. □

Theorem 1. $\mathcal{S}_n^*(A,B)\subset\mathcal{N}_{n,\delta}(e),$ *where*

$$\delta=\frac{(n+1)(B-A)}{(n+1)(B+1)-(A+1)}$$

and $\mathcal{S}_n^*(-1,1)\subset\mathcal{N}_{n,1}(e)$.

Proof. It follows from Lemma 1 that, if $f \in \mathcal{S}_n^*(A,B)$, then

$$\sum_{k=n+1}^{\infty} k a_k \le \frac{(n+1)(B-A)}{(n+1)(B+1)-(A+1)} = \delta.$$

This implies, $\mathcal{S}_n^*(A,B) \subset \mathcal{N}_{n,\delta}(e)$.

Further, for $A=-1$ and $B=1$, we have $f \in \mathcal{S}_n^*(-1,1)$ if and only if

$$\sum_{k=n+1}^{\infty} k a_k \le 1. \tag{7}$$

This gives that $f(z) \in \mathcal{N}_{n,1}(e)$. □

Corollary 1. *For* $n=1$, $\mathcal{S}_1^*(A,B) \subset \mathcal{N}_{1,\delta}(e)$, *where*

$$\delta = \frac{2(B-A)}{2B-A+1}$$

and $\mathcal{S}_1^*(-1,1) \subset \mathcal{N}_{1,1}(e)$.

Corollary 2. *For* $A = 2\alpha - 1$ *and* $B=1$, *we obtain Theorem* 2.1 *in* [4] *which reads as:* $\mathcal{S}_n^*(\alpha) \subset \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{(n+1)(1-\alpha)}{n+1-\alpha},$$

and $\mathcal{S}_1^*(0) \subset \mathcal{N}_{n,1}(e)$.

Theorem 2. $\mathcal{C}_n(A,B) \subset \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{B-A}{(n+1)(B+1)-(A+1)}.$$

Proof. It follows from Lemma 2 that, if $f \in \mathcal{C}_n(A,B)$ then

$$\sum_{k=n+1}^{\infty} k a_k \le \frac{B-A}{(n+1)(B+1)-(A+1)} = \delta.$$

This implies that $\mathcal{C}_n(A,B) \subset \mathcal{N}_{n,\delta}(e)$. □

Corollary 3. *For* $A = 2\alpha - 1$ *and* $B=1$, *we obtain Theorem* 2.2 *in* [4] *which reads as:* $\mathcal{C}_n(\alpha) \subset \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{1-\alpha}{n+1-\alpha}.$$

3. NEIGHBORHOODS FOR CLASSES $\mathcal{Q}_N(A,B)$ AND $\mathcal{R}_N(A,B)$

In this section we obtain necessary and sufficient conditions for functions to be in $\mathcal{Q}_n(A,B)$ and $\mathcal{R}_n(A,B)$, and also neighborhoods of these classes are determined. For $-1 \le A < B \le 1$ we define the classes $\mathcal{Q}_n(A,B)$ and $\mathcal{R}_n(A,B)$ as:

$$\mathcal{Q}_n(A,B) = \left\{ f : f \in \mathcal{A}(n) \text{ and } z \mapsto \frac{f(z)}{z} \text{ is in } P_n(A,B) \right\}$$

and

$$\mathcal{R}_n(A,B) = \left\{ f : f \in \mathcal{A}(n) \text{ and } f' \in P_n(A,B) \right\}.$$

LEMMA 3. *A function* $f \in \mathcal{Q}_n(A,B)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{(B+1)}{B-A} a_k \le 1, \quad -1 \le A < B \le 1. \tag{8}$$

Proof. Suppose $f \in \mathcal{Q}_n(A,B)$. Then

$$\frac{f(z)}{z} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1, \quad \omega \in \Omega, \quad z \in \mathcal{U}.$$

That is,

$$\omega(z) = \frac{1 - \dfrac{f(z)}{z}}{B\dfrac{f(z)}{z} - A}, \qquad \omega(0) = 0,$$

and

$$|\omega w(z)| = \left| \frac{z - f(z)}{Bf(z) - Az} \right| = \left| \frac{\sum_{k=n+1}^{\infty} a_k z^k}{(B-A)z - \sum_{k=n+1}^{\infty} Ba_k z^k} \right| < 1.$$

Thus

$$\Re \left\{ \frac{\sum_{k=n+1}^{\infty} k a_k z^k}{(B-A)z - \sum_{k=n+1}^{\infty} Ba_k z^k} \right\} < 1. \tag{9}$$

Let $z = r$, with $0 < r < 1$. Then, for sufficiently small r, the denominator of (9) is positive and so it is positive for all r, with $0 < r < 1$, since ω is regular for $|z| < 1$. Then (9) gives

$$\sum_{k=n+1}^{\infty} a_k r^k < (B-A)r - \sum_{k=n+1}^{\infty} Ba_k r^k.$$

That is,

$$\sum_{k=n+1}^{\infty} (B+1) a_k r^k < (B-A)r.$$

Letting $r \mapsto 1$, we obtain (8).

Conversely, for $|z| = r$, $0 < r < 1$, we have, by (8) and since $r^k < r$,

$$\sum_{k=n+1}^{\infty} (B+1) a_k r^k < r \sum_{k=n+1}^{\infty} (B+1) a_k < (B-A)r.$$

So we have

$$\left|\sum_{k=n+1}^{\infty} k a_k z^k\right| \le \sum_{k=n+1}^{\infty} a_k r^k < (B-A)r - \sum_{k=n+1}^{\infty} B a_k r^k <$$
$$< \left|(B-A)z - \sum_{k=n+1}^{\infty} B a_k z^k\right|.$$

This proves that $\frac{f(z)}{z}$ is of the form

$$\frac{1+A\omega(z)}{1+B\omega(z)} \quad (\omega(z)\in\Omega).$$

Therefore $f \in \mathcal{Q}_n(A,B)$. Thus the proof is complete. □

LEMMA 4. *A function* $f \in \mathcal{R}_n(A,B)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k(B+1)}{B-A} a_k \le 1, \quad -1 \le A < B \le 1. \tag{10}$$

Proof. A function $f \in \mathcal{R}_n(A,B)$ if and only if $zf'(z) \in \mathcal{Q}_n(A,B)$ and hence the conclusion follows from Lemma 3. □

REMARK 1. From Lemma 3 and Lemma 4 we have $\mathcal{R}_n(A,B) \subset \mathcal{Q}_n(A,B)$.

THEOREM 3. $\mathcal{Q}_n(A,B) = \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{B-A}{B+1}.$$

Proof. The equality follows from Lemma 4. □

COROLLARY 4. *For* $A = 2\alpha - 1$ *and* $B = 1$, *we obtain Theorem* 3.1 *in* [4] *which reads as:* $\mathcal{Q}_n(\alpha) = \mathcal{N}_{n,\delta}(e)$, *where* $\delta = 1-\alpha$.

THEOREM 4. $\mathcal{N}_{n,\delta}(e) \subset \mathcal{Q}_n(A,B)$, *where*

$$A = \frac{n+1-2\delta}{n+1} \quad \text{and} \quad B = 1.$$

Proof. If $f \in \mathcal{N}_{n,\delta}(e)$, we have

$$\sum_{k=n+1}^{\infty} k a_k \le \delta, \tag{11}$$

which gives that

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\delta}{n+1} = 1 - \frac{n+1-\delta}{n+1}. \tag{12}$$

Thus we see that $f \in \mathcal{Q}_n(A,B)$. □

Corollary 5. *For* $A = 2\alpha - 1$ *and* $B = 1$ *we obtain Theorem* 3.2 *in* [4] *which reads as:* $\mathcal{N}_{n,\delta}(e) \subset \mathcal{Q}_n(\alpha)$, *where* $\alpha = \dfrac{n+1-\delta}{n+1}$.

4. NEIGHBORHOODS FOR CLASSES $\mathcal{C}_{\mathrm{N}}(A,B;C,D)$ AND $\mathcal{S}_N(A,B;C,D)$

Let $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k$, with a_k, $b_k \geq 0$. If a function $f \in \mathcal{A}(n)$ satisfies the conditions

$$\frac{f'(z)}{g'(z)} \in \mathcal{P}_n(A,B), \quad \text{for } -1 \leq A < B \leq 1, \quad z \in \mathcal{U}, \tag{13}$$

and $g \in \mathcal{S}_n^*(C,D)$, for $-1 \leq C < D \leq 1$, then we say that

$$f \in \mathcal{C}_n(A,B;C,D).$$

If we take $g(z) = z$, then $\mathcal{C}_n(A,B;C,D)$ becomes $\mathcal{R}_n(A,B)$. Further, a function $f \in \mathcal{A}(n)$ is said to be in the class $\mathcal{S}_n(A,B;C,D)$ if it satisfies

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{B-A}{1+B}, \quad \text{for } -1 \leq A < B \leq 1, \quad z \in \mathcal{U}, \tag{14}$$

and $g \in \mathcal{S}_n^*(C,D)$, for $-1 \leq C < D \leq 1$. If we put $g(z) = z$, then $\mathcal{S}_n(A,B;C,D)$ becomes $\mathcal{Q}_n(A,B)$.

We prove the following results for the classes $\mathcal{C}_n(A,B;C,D)$ and $\mathcal{S}_n(A,B;C,D)$.

Theorem 5. $\mathcal{C}_n(A,B;C,D) \subset \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{n(B-A)(D+1) + (D-C)(B+1)}{(B+1)[(n+1)(D+1) - (C+1)]}.$$

Proof. If $f \in \mathcal{C}_n(A,B;C,D)$, then we have

$$\Re\left\{\frac{1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} k b_k z^{k-1}}\right\} > \frac{1 - \sum_{k=n+1}^{\infty} k a_k}{1 - \sum_{k=n+1}^{\infty} k b_k} \geq \frac{1+A}{1+B}. \tag{15}$$

It follows from (15) that

$$\begin{aligned}\sum_{k=n+1}^{\infty} k a_k &\leq \left(\frac{B-A}{1+B}\right) + \left(\frac{1+A}{1+B}\right)\left(\sum_{k=n+1}^{\infty} k b_k\right) \leq \\ &\leq \left(\frac{B-A}{1+B}\right) + \left(\frac{1+A}{1+B}\right)\left[\frac{D-C}{(n+1)(1+D) - (1+C)}\right] \leq \\ &\leq \frac{n(B-A)(D+1) + (D-C)(B+1)}{[(n+1)(D+1) - (C+1)](B+1)} = \delta.\end{aligned}$$

This yields $f \in \mathcal{N}_{n,\delta}(e)$. □

COROLLARY 6. *For* $n = 1$ *we have* $\mathcal{C}_1(A, B; C, D) \subset \mathcal{N}_{1,\delta}(e)$, *where*

$$\delta = \frac{(2B - A + 1)(D + 1) - (C + 1)(B + 1)}{(B + 1)(2D - C + 1)}.$$

COROLLARY 7. *For* $A = 2\alpha - 1\, C = 2\beta - 1$ *and* $B = D = 1$, *we obtain Theorem* 4.1 *in* [4] *which reads as:* $\mathcal{C}_n(\alpha, \beta) \subset \mathcal{N}_{n,\delta}(e)$, *where*

$$\delta = \frac{n(1 - \alpha) + (1 - \beta)}{n + 1 - \beta}.$$

THEOREM 6. $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}_n(A, B; C, D)$, *where* $g \in \mathcal{S}_n^*(C, D)$, *for* $-1 \leq C < D \leq 1$,

$$A = 1 - \frac{2[(n + 1)(D + 1) - (C + 1)]\delta}{n(n + 1)(D + 1)} \quad \textit{and} \quad B = 1.$$

Proof. Let $f \in \mathcal{N}_{n,\delta}(g)$, for $g(z) \in \mathcal{S}_n^*(C, D)$. Then we know that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

and

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{D - C}{(n + 1)(D + 1) - (C + 1)}.$$

Thus we have

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \left(\frac{\delta}{n + 1} \right) \left(\frac{(n + 1)(D + 1) - (C + 1)}{n(D + 1)} \right) \\ &= \frac{[(n + 1)(D + 1) - (C + 1)]\delta}{n(n + 1)(D + 1)} = \frac{B - A}{B + 1}. \end{aligned}$$

This implies that $f \in \mathcal{S}_n(A, B; C, D)$. □

COROLLARY 8. *For* $n = 1$ *we obtain* $\mathcal{N}_{1,\delta}(g) \subset \mathcal{S}_1(A, B; C, D)$, *where* $g \in \mathcal{S}_1^*(C, D)$, $A = 1 - \left(\frac{2D - C + 1}{D + 1} \right) \delta$ *and* $B = 1$.

COROLLARY 9. *For* $A = 2\alpha - 1$, $C = 2\beta - 1$ *and* $B = D = 1$, *we obtain Theorem* 4.2 *in* [4] *which reads as:* $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}_n(\alpha, \beta)$, *where* $g \in \mathcal{S}_n^*(\beta)$ *and*

$$\alpha = 1 - \left(\frac{n + 1 - \beta}{n(n + 1)} \right) \delta.$$

REFERENCES

[1] Chatterjea, S.K., *On starlike functions,* J. Pure Math., **1** (1981), 23–26.
[2] Duren, P.L., *Univalent functions,* Springer-Verlag, 1983.
[3] Lakshma Reddy, G. and Padmanabhan, K.S., *p-valent regular functions with negative coefficients,* Commentarii Mathematici Universitatis Sancti Pauli, Volume **32**, No. 1, 1983.
[4] Osman Altintas and Shigeyoshi Owa, *Neighborhoods of certain analytic functions with negative coefficients,* Internat. J. Math. and Math. Sci., Volume **19**, No. 4 (1996), 797–800.
[5] Sarabgi, S.M. and Uralegaddi, B.A., *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients,* Atti. Accad. Nazion. Lincer., **65** (1978), 38–42.
[6] Srivastava, H.M., Owa, S. and Chatterjea, S.K., *A note on certain classes of starlike functions,* Rend. Semin. Mat. Univ. Pandova, **77** (1987), 115–124.
[7] Ruscheweyh, S., *Neighbohoods of univalent functions,* Proc. Amer. Math. Soc., **81** (1981), 521–527.

Received January 21, 2007

Department of Mathematics and Computer Science
Maharaja's College
University of Mysore
Mysore - 570 005, India
E-mail: drlatha@gmail.com

Department of Mathematics
Yuvaraja's College
University of Mysore
Mysore - 570 005, India
E-mail: poornimn@gmail.com