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TWO-DIMENSIONAL POTENTIALS GENERATING A GIVEN ONE-PARAMETER FAMILY OF ORBITS ON A SURFACE

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Abstract. We say that a potential *generates a curve* on a surface if a unit mass traces the curve under the action of the potential. We consider the following problem: a one-parameter family of regular curves $f(u,v) = c$ on a surface $\vec{r}(u,v) = \{x(u,v),\ y(u,v),\ z(u,v)\}$ is given. We seek two-dimensional potentials of the form $V(u,v) = u^m R(\frac{v}{u})$, R being an arbitrary C^2-function, which generate this family of regular curves as trajectories on the above surface. We show that if the given family of orbits satisfies exactly two differential conditions, then such a potential exists and it is determined uniquely. Special cases are also studied and pertinent examples are given for each case. At a second step, if we consider that the "*slope function*" $\gamma(u,v) = f_v/f_u$ is homogeneous of zero degree and the components of the metric tensor are homogeneous functions of zero degree too, then a potential of the above form always exists and it is found as a solution of an ordinary second-order O.D.E. Several examples are offered and implications of this study are discussed.

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1. INTRODUCTION

The inverse problem of dynamics in a broad sense consists of the determination of forces, parameters and constraints which are required for the realization of the motion of a mechanical system with some properties given in advance [12]. Especially, the inverse problem of dynamics, as introduced by [17], seeks all the potentials $V = V(x,y)$ which can generate a one-parameter family of planar orbits $f(x,y) = c$, traced in the xy Cartesian plane by a material point of unit mass, with a preassigned dependence of its total energy $\mathcal{E} = \mathcal{E}(f(x,y))$ on the given family. There results a first-order partial differential equation, linear in the unknown function $V = V(x,y)$ and the coefficients depend on the family of orbits. This equation (written again by [5]) reads:

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(\mathcal{E} - V) = 0, \tag{1}$$

where

$$\gamma = \frac{f_y}{f_x}, \quad \Gamma = \gamma\gamma_x - \gamma_y. \tag{2}$$

Bozis [6] presented a second order linear partial differential equation giving the potential functions $V = V(x,y)$ which give rise to a preassigned family of planar curves $f(x,y)=c$. This equation is the following one:

$$-V_{xx}+\kappa V_{xy}+V_{yy}=\lambda V_x+\mu V_y, \tag{3}$$

where

$$\kappa=\frac{1-\gamma^2}{\gamma},\ \ \lambda=\frac{\Gamma_y-\gamma\Gamma_x}{\gamma\Gamma},\ \ \mu=\lambda\gamma+\frac{3\Gamma}{\gamma}. \tag{4}$$

Bozis' equation does not include the energy $\mathcal{E}$ and consequently no assumption about the energy dependence $\mathcal{E}=\mathcal{E}(f)$ needs to be made.

Mertens [15] studied a family of curves $f(u,v)=c$ on a smooth surface S in 3-D space using Szebehely's method and obtained a linear partial differential equation in the potential function $V(u,v)$. This equation is the following one:

$$(g_{22}f_u-g_{12}f_v)V_u+(g_{11}f_v-g_{12}f_u)V_v=2W(\mathcal{E}-V), \tag{5}$$

where

$$\begin{aligned} W=&\frac{1}{A}[g(f_v^2f_{uu}-2f_uf_vf_{uv}+f_u^2f_{vv})-\\ &-B_1(g_{22}f_u-g_{12}f_v)-B_2(g_{11}f_v-g_{12}f_u)],\\ A=&\,g_{11}f_v^2-2g_{12}f_uf_v+g_{22}f_u^2,\\ B_1=&\frac{1}{2}(g_{11})_uf_v^2+[(g_{12})_v-\frac{1}{2}(g_{22})_u]f_u^2-(g_{11})_vf_uf_v,\\ B_2=&[(g_{12})_u-\frac{1}{2}(g_{11})_v]f_v^2+\frac{1}{2}(g_{22})_vf_u^2-(g_{22})_uf_uf_v,\\ g=&\,g_{11}g_{22}-(g_{12})^2. \end{aligned} \tag{6}$$

The subscripts denote partial differentiation with respect to the corresponding variable.

Furthermore, Bozis and Mertens [7] derived a second order partial differential equation of hyperbolic type for the potential V in which all the coefficients are known functions of the coordinates u,v and gave some examples. This PDE reads:

$$k_1V_{uu}+k_2V_{uv}-\beta V_{vv}+k_3V_u+k_4V_v=0, \tag{7}$$

where

$$\begin{aligned} k_1&=\alpha\gamma,\ k_2=\beta\gamma-\alpha,\ k_3=\gamma+\gamma\alpha_u-\alpha_v,\ k_4=\gamma\beta_u-\beta_v-1,\\ \alpha&=\frac{1}{2W}(g_{22}f_u-g_{12}f_v),\ \ \beta=\frac{1}{2W}(-g_{12}f_u+g_{11}f_v),\ \ \gamma=\frac{f_v}{f_u}. \end{aligned} \tag{8}$$

Other works related to this topic are those by [3], [9], [13], [14].

In the present work we deal with the second order PDE (7) and we seek homogeneous solutions of the form $V(u,v)=u^mR(w)$ where $w=\frac{v}{u}$ and R is an arbitrary C^2-function. Similar works concerning homogeneous potentials in the planar and three-dimensional version of the problem are those by [8] and [11]. A review on basic facts of inverse problem in dynamics was presented in [10] and [1]. Recently, Betsakos and Grigoriadou [2] studied the problem of the determination of all measures supported in a compact set K whose logarithmic potentials generate each of the given curve.

In Section 2 we give a full description of this problem and we modify the eqs. (5) and (6) in such a way that we can handle them easier than previously. In Section 3 we prove there is no homogeneous potential which generates this family of orbits unless *two* differential conditions are fulfilled by the one-parameter family of orbits on a certain surface. In Section 4 we make *two* more restrictions: (a) we take into account that the "*slope function*" $\gamma = \gamma(u, v)$ is homogeneous of zero degree and (b) the components of the metric tensor are homogeneous functions of zero degree too. In this case a homogeneous potential of degree m always exists. These potentials are solutions of a second-order ODE in the unknown function $R(w)$. Pertinent examples are given. We conclude in Section 5.

2. DESCRIPTION OF THE PROBLEM

In an Euclidean 3D-space $\mathbb{E}^3$ with an orthonormal Cartesian system of reference $Oxyz$ we assign a smooth surface S:

$$P = P(u, v) \Longleftrightarrow \{x = x(u, v),\ y = y(u, v),\ z = z(u, v)\} \tag{9}$$

with $u,\ v$ as curvilinear coordinates on S. On this surface we also consider a one-parameter family of regular curves given in the solved form

$$f(u, v) = c, \quad c \in \mathbb{R}, \tag{10}$$

where c is the parameter of the family (10). By regular curve we mean that the function f is of C^3-class on a domain $D \subset \mathbb{E}^3$ and such that $\nabla f \neq 0$.

For the given family of orbits we define γ as follows: $\gamma = f_v/f_u$. The "*slope function*" γ represents the family (10) in the sense that if the family (10) is given, then γ is determined uniquely. On the other hand, if γ is given, we can obtain a unique family (10). The inverse problem of dynamics consists in finding potentials V which can give rise to this family of curves as trajectories on a given surface. In all over the paper we shall regard $V = V(u, v)$ as the potential function.

The line-element on the surface S in this system of parameters is given by:

$$\mathrm{d}s^2 = g_{11}\mathrm{d}u^2 + 2g_{12}\mathrm{d}u\mathrm{d}v + g_{22}\mathrm{d}v^2, \tag{11}$$

where $g_{11},\ g_{12},\ g_{22}$ are known functions of $u,\ v$.

Now, we consider a particle of unit mass which describes any member of the given family (10). Here we have to clear out that trajectories are bound to a given surface by constraints. The kinetic energy (T) of the test particle is given by ([15])

$$T = \frac{1}{2}(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2), \tag{12}$$

where the dot denotes differentiation with respect to time. Then the Hamiltonian governing this system reads: $H = T + V$.

Using the notations:

$$\gamma = \left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial f}{\partial u}\right)^{-1} \text{ and } \Gamma = \gamma\frac{\partial \gamma}{\partial u} - \frac{\partial \gamma}{\partial v}, \tag{13}$$

the equation (5) takes a simpler form:

$$(g_{22} - \gamma g_{12})V_u + (\gamma g_{11} - g_{12})V_v + \frac{2\Delta}{A_1}(\mathcal{E} - V) = 0, \tag{14}$$

where

$$\begin{aligned}
\Delta &= g\Gamma + B_1'(g_{22} - \gamma g_{12}) + B_2'(\gamma g_{11} - g_{12}),\\
A_1 &= g_{11}\gamma^2 - 2g_{12}\gamma + g_{22},\\
B_1' &= \frac{1}{2}(g_{11})_u\gamma^2 + [(g_{12})_v - \frac{1}{2}(g_{22})_u] - (g_{11})_v\gamma,\\
B_2' &= [(g_{12})_u - \frac{1}{2}(g_{11})_v]\gamma^2 + \frac{1}{2}(g_{22})_v - (g_{22})_u\gamma.
\end{aligned} \tag{15}$$

Eliminating the energy-dependence function, we obtain the PDE of [7]:

$$k_1V_{uu} + k_2V_{uv} - \beta V_{vv} + k_3V_u + k_4, V_v = 0 \tag{16}$$

where

$$k_1 = \alpha\gamma,\ k_2 = \beta\gamma - \alpha,\ k_3 = \gamma + \gamma\alpha_u - \alpha_v,\ k_4 = \gamma\beta_u - \beta_v - 1 \tag{17}$$

and the quantities α and β are given now as follows:

$$\alpha = -\frac{A_1(g_{22} - \gamma g_{12})}{2\Delta},\ \ \beta = -\frac{A_1(\gamma g_{11} - g_{12})}{2\Delta}. \tag{18}$$

In the present work we consider the PDE (16) and we shall look for solutions of the form $V(u,v)= u^m R(\frac{v}{u})$ of any degree m. We have to solve a linear, second order PDE in $V(u,v)$ and we shall try to find adequate triplets $\Big[\gamma,\ (g_{ij}),\ V(u,v)\Big]$ $(i,\ j = 1,2)$ satisfying the above equation. The choice of the potential function of this form makes the mathematical calculations simpler. To our knowledge, there are not so many results in the literature concerning this subject.

3. THE GENERIC CASE

We emphasize in homogeneous potentials

$$V(u,v) = u^m R(w),\ \ w = \frac{v}{u}, \tag{19}$$

where R is an arbitrary C^2-function. We shall offer *two* criteria which must be fulfilled by the given family of orbits (10) on a certain surface so that the problem has a solution of type (19). Since $V(u,v)$ is a homogeneous function of degree m, the following relation holds

$$uV_u + vV_v = mV. \tag{20}$$

We differentiate both members of (20) twice with respect to u, v and we get the relations

$$uV_{uu} + vV_{uv} = (m-1)V_u,\quad uV_{uv} + vV_{vv} = (m-1)V_v. \tag{21}$$

Then we solve algebraically the system of $1+2=3$ eqs. ((16) and (21)) for the second-order derivatives of the potential function V. We find

$$V_{uu} = \frac{D_1}{D_0},\quad V_{uv} = \frac{D_2}{D_0},\quad V_{vv} = \frac{D_3}{D_0}, \tag{22}$$

where:

$$\begin{gathered} D_1 = \lambda_1 V_u + \lambda_2 V_v,\ \ D_2 = \lambda_3 V_u + \lambda_4 V_v,\ \ D_3 = \lambda_5 V_u + \lambda_6 V_v, \\ \lambda_1 = -[(m-1)\beta u + v^2 k_3 + (m-1)k_2 v],\ \ \lambda_2 = -[v^2 k_4 - (m-1)\beta v], \\ \lambda_3 = (m-1)vk_1 + uvk_3,\ \ \lambda_4 = (1-m)u\beta + uvk_4, \\ \lambda_5 = -[(m-1)uk_1 + u^2 k_3],\ \ \lambda_6 = (m-1)(vk_1 - uk_2) - u^2 k_4 \end{gathered} \tag{23}$$

and

$$D_0 = k_1 v^2 - k_2 uv - \beta u^2. \tag{24}$$

We continue our work assuming $D_0 \neq 0$. Thus, we write *two* necessary and sufficient conditions $V_{(uu)v} = V_{(uv)u}$, $V_{(uv)v} = V_{(vv)u}$ for the system (22) to be compatible. In doing so, there appear again second order derivatives of V which we intend to replace by the expressions (22) themselves. Thus we end up to a system of *two* equations in V_u, V_v. This system is

$$TV_u + SV_v = 0,\ \ CV_u + BV_v = 0, \tag{25}$$

where

$$\begin{gathered} T = \mu_1 - \lambda_1 p_2 + \lambda_3 p_1,\ \ S = \mu_2 - \lambda_2 p_2 + \lambda_4 p_1, \\ C = \mu_3 - \lambda_3 p_2 + \lambda_5 p_1,\ \ B = \mu_4 - \lambda_4 p_2 + \lambda_6 p_1 \end{gathered} \tag{26}$$

and

$$\begin{aligned} \mu_1 &= (\lambda_{1v} - \lambda_{3u})D_0 + (\lambda_2\lambda_5 - \lambda_3\lambda_4), \\ \mu_2 &= (\lambda_{2v} - \lambda_{4u})D_0 + (\lambda_2\lambda_6 - \lambda_4^2 + \lambda_1\lambda_4 - \lambda_2\lambda_3), \\ \mu_3 &= (\lambda_{3v} - \lambda_{5u})D_0 + (\lambda_3^2 - \lambda_1\lambda_5 + \lambda_4\lambda_5 - \lambda_3\lambda_6), \\ \mu_4 &= (\lambda_{4v} - \lambda_{6u})D_0 + (\lambda_3\lambda_4 - \lambda_2\lambda_5), \\ p_1 &= \frac{\partial D_0}{\partial u},\ \ p_2 = \frac{\partial D_0}{\partial v}. \end{aligned} \tag{27}$$

However, the calculations show that the above *two* eqs. (25) are not independent. Thus, we have left only with one of them, let's say the first one. At this point we assume that $T \neq 0$, $S \neq 0$ and we define the ratio

$$\rho = \frac{S}{T}. \tag{28}$$

Hence the first of eqs. (25) is written as

$$V_u + \rho V_v = 0. \tag{29}$$

Since $\rho = -\frac{V_u}{V_v}$, the map ρ must be a homogeneous function of zero degree, namely

$$u\rho_u + v\rho_v = 0. \tag{30}$$

We replace ρ (ρ defined in (28)) into (30) and we obtain the first differential condition:

$$(uS_u + vS_v)T - (uT_u + vT_v)S = 0. \tag{31}$$

We solve the system of eqs. (20) and (29) and we find:

$$\begin{aligned} \frac{V_u}{V} &= -\frac{m\rho}{v - \rho u}, \\ \frac{V_v}{V} &= \frac{m}{v - \rho u}. \end{aligned} \tag{32}$$

It can be proved directly that the compatibility condition for the system (32) is the differential condition (31). Since the potential function $V(u, v)$ is determined uniquely (apart from a multiplicative constant), we have to take only into consideration that it satisfies the PDE (16). Hence, we prepare the derivatives of second order from (32) and we replace them into (16). In doing so, we obtain the second differential condition for the family of orbits on a certain surface:

$$\begin{aligned} &(m-1)(k_1 S^2 - k_2 TS - \beta T^2) + k_1 v(ST_u - TS_u) + \\ &\quad + (k_2 v + \beta u)(ST_v - TS_v) + (k_3 S - k_4 T)(uS - vT) = 0. \end{aligned} \tag{33}$$

Now we can formulate the following:

Proposition 1. *If for the given family of orbits* (10) *the following conditions hold:*

(i) $D_0, \ T, \ S \neq 0$,
(ii) *there exists a real m for which the two differential conditions* (31) *and* (33) *are satisfied,*

then this family of orbits is generated by a homogeneous potential of degree m. The corresponding potential is determined uniquely, apart from a multiplicative factor, by the system (32).

Remark 1. If $\rho = 0$ (ρ is defined in (28)), then from (29) we obtain $V_u = 0$ and, hence, the potential is one-dimensional: $V = V(v)$.

Example 1. It is known ([18], p. 70, example 7) that the Liouville-type dynamical system, described in curvilinear coordinates $(u,\ v)$ by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(u^2+v^2)(\dot{u}^2+\dot{v}^2) - \frac{1}{u^2+v^2} \tag{34}$$

admits the one-parameter family of orbits

$$f(u,v) = \frac{1-u^2}{v^2} = c. \tag{35}$$

Indeed, we assign the metric

$$g_{11} = g_{22} = u^2+v^2,\ \ g_{12} = 0 \tag{36}$$

and the family of curves (35). Here we have $D_0 \neq 0$ and $\rho = -\frac{u}{v}$. We checked the two conditions (31) and (33) and we ascertained that the first one is satisfied for *any* value of m, while the second one is satisfied only for $m = -2$ and $m = -4$. Thus, for $m = -2$, the system (32) reads

$$\begin{aligned} \frac{V_u}{V} &= \frac{-2u}{u^2+v^2}, \\ \frac{V_v}{V} &= \frac{-2v}{u^2+v^2} \end{aligned} \tag{37}$$

and the potential is

$$V(u,v) = \frac{1}{u^2+v^2}. \tag{38}$$

Remark 2. We note here that the value $m = -4$ leads to the special case $T = S = 0$ (ρ is undefined in this case) and it will be studied in section 4.2.

Example 2. We assign the surface S:

$$\vec{r}(u,v) = \{u+v,\ u-v,\ u+v^2\}$$

and the one-parameter family of curves $f = u + v^2 = c$ on it. Then we have:

$$g_{11} = 3,\ \ g_{12} = 2v,\ \ g_{22} = 2+4v^2. \tag{39}$$

It is: $D_0 \neq 0$ and $\rho = \frac{(m+2)u}{(m-1)v}$. We checked the two conditions (31) and (33); the first one is satisfied for *any* value of m, while the second one is satisfied for $m \in \{2, 1, -2\}$. Thus, for $m = 2$, the system (32) reads

$$\begin{aligned} \frac{V_u}{V} &= \frac{8u}{4u^2+v^2}, \\ \frac{V_v}{V} &= \frac{2v}{4u^2+v^2} \end{aligned} \tag{40}$$

and the potential is

$$V(u,v) = 4u^2+v^2. \tag{41}$$

For $m = -2$ we take $S = 0$ and $T \neq 0$. Then $\rho = 0$ and the corresponding potential is one-dimensional: $V = V(v)$. The second of the eqs. (32) becomes:

$$\frac{V_v}{V} = -\frac{2}{v} \tag{42}$$

and the potential is: $V = V(v) = 1/v^2$.

For $m = 1$ we take $S \neq 0$ and $T = 0$. Now we define $\rho = T/S = 0$ and the corresponding potential is again one-dimensional: $V = V(u)$. The first of the eqs. (32) is:

$$\frac{V_u}{V} = \frac{1}{u} \tag{43}$$

and the potential is: $V = V(u) = u$.

Example 3. We assign the surface S:

$$\vec{r}(u,v) = \left\{u - \frac{u^3}{3} + uv^2,\ -v + \frac{v^3}{3} - vu^2,\ u^2 - v^2\right\} \quad \text{(“Enneper's” surface)}$$

and the one-parameter family of curves $f = u + v = c$ on it. Then we have:

$$g_{11} \;=\; g_{22} = (1 + u^2 + v^2)^2,\ \ g_{12} = 0. \tag{44}$$

It is: $D_0 \neq 0$ and $\rho = -\frac{u}{v}$. We checked the two conditions (31) and (33); the first one is satisfied for *any* value of m, while the second one is satisfied only for $m = 2$. Thus, for $m = 2$, the system (32) becomes

$$\begin{aligned} \frac{V_u}{V} &= \frac{2(u+3v)}{u^2 + 6uv + v^2}, \\ \frac{V_v}{V} &= \frac{2(3u+v)}{u^2 + 6uv + v^2} \end{aligned} \tag{45}$$

and the potential is

$$V(u,v) = u^2 + 6uv + v^2. \tag{46}$$

Counterexample. We assign the previous surface and the family of curves $f = u^2 - v^2 = c$ now. We found that none of the conditions (31) and (33) is satisfied. So, this family of curves is not generated by a potential of the form (19).

4. SPECIAL CASES

Up to now we supposed that $D_0 \neq 0$ and T, $S \neq 0$. In this paragraph we shall examine the cases $D_0 = 0$ and $T = S = 0$.

4.1. The case $D_0 = 0$. The linear system (22) is meaningful only if $D_0 \neq 0$. Otherwise, i.e., if $D_0 = 0$, then each of the three quantities in the rhs of (22) should be zero. This leads to a (linear, homogeneous) algebraic system in $V_u,\ V_v$ of "three" equations. It can be checked, however, that, for the given family satisfying the relation $D_0 = 0$, the rank of the matrix

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \\ \lambda_5 & \lambda_6 \end{pmatrix}$$

is equal to *one*. This means that the aforementioned system consists of just one equation, say the equation

$$\lambda_1 V_u + \lambda_2 V_v = 0 \tag{47}$$

which ensures the compatibility of equations (16) and (20). We define $\rho = \frac{\lambda_2}{\lambda_1}$ and then we combine (47) with (20). So, we obtain the system (32). Then we have to check two differential conditions analogous with (31) and (33). In this case it is: $S \to \lambda_2$ and $T \to \lambda_1$.

EXAMPLE 4. We assign the sphere:

$$\vec{r}(u,v) = \left\{ \frac{u^2+v^2-1}{u^2+v^2+1},\ \frac{2u}{u^2+v^2+1},\ \frac{2v}{u^2+v^2+1} \right\}$$

and we consider the one-parameter family of curves $f = u^2 + v^2 = c$ on it. Then we have:

$$g_{11} = g_{22} = \frac{4}{(1+u^2+v^2)^2},\quad g_{12} = 0. \tag{48}$$

In this case we have $D_0 = 0$. We checked the *two* conditions (31) and (33) and we found that these conditions are both verified for *any* value of m. Thus, the system (32) becomes:

$$\begin{aligned} \frac{V_u}{V} &= \frac{mu}{u^2+v^2}, \\ \frac{V_v}{V} &= \frac{mv}{u^2+v^2}. \end{aligned} \tag{49}$$

Hence the potential function $V(u,v)$ is

$$V(u,v) = (u^2+v^2)^{m/2}. \tag{50}$$

4.2. The case $T = S = 0$. If for the given one-parameter family of orbits we have:

$$T = 0 \ \text{ and } \ S = 0, \tag{51}$$

but $D_0 \neq 0$, then the ratio ρ cannot be determined in (28). Thus, for the pertinent family (10), none of the eqs. (25) provides fruitful information for the determination of the potential function V. In other words, no additional condition is imposed for the compatibility of the system (22). So, we deal only with the second-order PDE (16) and solve it when it is possible.

Example 5. We consider again the metric

$$g_{11} = g_{22} = u^2 + v^2, \ \ g_{12} = 0 \tag{52}$$

and the one-parameter family of curves

$$f(u, v) = \frac{1 - u^2}{v^2} = c. \tag{53}$$

We look for potentials of degree $m = -4$. As we explained in Example 1, we have to face the case $T = S = 0$. We replace the expression for the potential

$$V(u, v) = \frac{1}{u^4} R(w), \ \ w = \frac{v}{u} \tag{54}$$

into (16). Then the PDE (16) is transformed into the following second-order ODE

$$(w + w^3)R''(w) + (7w^2 + 3)R'(w) + 8wR(w) = 0, \tag{55}$$

where "$'$" denotes differentiation of R with respect to its argument w. The general solution of (55) is

$$R(w) = \Big[c_2 + \frac{c_1}{8}\int \frac{\mathrm{e}^{h(w)}}{w}\mathrm{d}w\Big]\mathrm{e}^{-h(w)} \quad (c_1, \ c_2 \text{ const.}) \tag{56}$$

with

$$h(w) = \frac{7}{16}w^2 - \frac{5}{8}\log w. \tag{57}$$

5. THE RESTRICTIVE CASE

We shall make here the following assumptions:

(i) We suppose that such potentials can give rise, among others, to a family of curves (10), where $f(u, v)$ is homogeneous function in $(u, \ v)$ of any degree. Then the "*slope function*" $\gamma(u, v)$ is homogeneous of zero degree, i.e.,

$$\gamma = \gamma(w), \ \ w = \frac{v}{u}. \tag{58}$$

(ii) We consider that the components of the metric tensor are homogeneous functions of zero degree, namely

$$g_{11} = g_{11}(w), \ \ g_{12} = g_{12}(w), \ \ g_{22} = g_{22}(w). \tag{59}$$

After some straightforward calculations, the PDE (16) is transformed into a *new* ODE of second order for the function $R(w)$:

$$C_2R''(w) + C_1R'(w) + C_0R(w) = 0, \tag{60}$$

where the primes denote total differentiation of the function R with respect to its argument w. The coefficients C_2, C_1, C_0 are given as follows

$$C_2 = w^2F_1 - wF_2 + F_3,$$
$$C_1 = -2(m-1)wF_1 + (m-1)F_2 - wF_4 + F_5, \tag{61}$$
$$C_0 = m[(m-1)F_1 + F_4],$$

and the expressions of F_j $(j = 1, \ldots, 5)$ are given in the *Appendix*.

REMARK 3. If we search for potentials of zero degree ($m = 0$), then we have $C_0 = 0$ and the ODE (60) is solvable (e.g. Example 7).

5.1. Pertinent Examples. In this paragraph we shall present some results for potentials possessing one-parameter families of regular orbits (10) on a smooth surface (9) under the previous restrictions. Let us begin with

EXAMPLE 6. We consider the metric: $\mathrm{d}s^2 = w(\mathrm{d}u^2 + \mathrm{d}v^2)$ $(u,\ v > 0)$ and the one-parameter family of curves $f = v^2 - u^2 = c$. It is: $\gamma = -\frac{v}{u} = -w$. Here all the coefficients C_2, C_1 and C_0 in (8) depend on the variable w. We shall search for potentials of fourth degree ($m = 4$). From (8) we obtain

$$C_2 = \frac{1}{2}w(w^4 - 1),$$
$$C_1 = -\frac{1}{2}(w^2+1)(3w^2+1), \tag{62}$$
$$C_0 = 2w(1+w^2).$$

Thus, using the method of *undetermined coefficients*, we found a particular solution of (7) for the unknown function $R(w)$. This is:

$$r(w) = w^2 + 1. \tag{63}$$

Now, applying the transformation $R(w) = r(w)\int z(w)\mathrm{d}w$, we computed the general solution of (31). We replaced the expression $R(w) = r(w)\int z(w)\mathrm{d}w$ into (7) and we found that the unknown function $z(w)$ has to satisfy the following ODE:

$$C_2 r\frac{\mathrm{d}z}{\mathrm{d}w} + \left[2C_2\frac{\mathrm{d}r}{\mathrm{d}w} + C_1 r\right]z = 0. \tag{64}$$

The general solution of (64) is:

$$z(w) = \frac{(w^2-1)^2}{w(w^2+1)^2}. \tag{65}$$

Then the function $R(w)$ is determined uniquely:

$$R(w) = 2 + (1+w^2)\log(w). \tag{66}$$

Hence the potential function $V(u,v)$ is given by:

$$V(u,v) = 2u^4 + (u^4 + u^2v^2)\log\left(\frac{v}{u}\right). \tag{67}$$

Remark 4. Working in a similar way for $m = 6$, we found the following result:

$$V(u,v) = u^6 + 6u^4v^2 + u^2v^4. \tag{68}$$

Example 7. In this example we shall consider a surface whose metric has the property (6) and we shall seek for potentials of zero degree ($m = 0$). We consider the surface: $\vec{r} = \{u,\ v,\ \frac{uv}{u+v}\}$ and the one-parameter family of helical lines $f(u,v) = u = c$ on it. It is:

$$\begin{gathered} g_{11} = \frac{1 + 4w + 6w^2 + 4w^3 + 2w^4}{(1+w)^4},\ \ g_{12} = \frac{w^2}{(1+w)^4}, \\ g_{22} = \frac{2 + 4w + 6w^2 + 4w^3 + w^4}{(1+w)^4},\ g = \frac{2(1+w+w^2)^2}{(1+w)^4}. \end{gathered} \tag{69}$$

From (8) we obtain:

$$\begin{aligned} C_2 &= \frac{4 + 14w + 30w^2 + 40w^3 + 36w^4 + 21w^5 + 7w^6 + w^7}{4w(1+w)^2}, \\ C_1 &= \frac{P_{11}}{4w^2(1+w)^3}, \\ P_{11} &= -4 - 12w - 2w^2 + 38w^3 + 96w^4 + 116w^5 + 77w^6 + 27w^7 + 4w^8, \\ C_0 &= 0. \end{aligned} \tag{70}$$

Solving (7) for $R(w)$, we found

$$R(w) = \frac{2 + 4w + 6w^2 + 4w^3 + w^4}{4(2 + 3w + 3w^2 + w^3)^{4/3}}, \tag{71}$$

consequently the potential function $V(u,v)$ is

$$V(u,v) = \frac{u^4 + (u+v)^4}{4[u^3 + (u+v)^3]^{4/3}}. \tag{72}$$

6. CONCLUDING COMMENTS

In the present work we dealt with potentials of the form $V(u,v) = u^m R(\frac{v}{u})$ generating a one-parameter family of regular orbits $f(u,v) = c$ on a certain surface S. We used the property of homogeneity for the potentials which is very common in physical problems. We studied the following *two* cases: (a) homogeneous potentials producing families of orbits on a given smooth surface and (b) homogeneous potentials producing homogeneous families of orbits on surfaces with homogeneous components of the metric tensor. Generally speaking, this problem has no solution. It is not expected to find such a solution for *any* one-parameter family of orbits (10) on a given surface (9) unless two necessary and sufficient conditions for the family of orbits are satisfied. With the aid of these conditions we can check whether a given family of orbits does indeed fulfill them or not and then find uniquely the corresponding potential.

All the potentials found are real. The mathematical treatment of the problem led us to study certain special cases. Several *new* results concerning this interesting version of the inverse problem in Dynamics were given. All the computations were aided by the symbolic algebra program *MATHEMATICA 5.2*.

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Appendix (The restrictive case of Section 4)

$$F_1 = -\frac{\gamma A_1 K_1}{2H}, \quad F_2 = \frac{\gamma A_1 (g_{22} - \gamma^2 g_{11})}{2H}, \quad F_3 = \frac{\gamma A_1 K_2}{2H},$$
$$F_4 = \gamma + \gamma da_1 - da_2, F_5 = \gamma db_1 - db_2 - 1,$$
$$da_1 = -\frac{par1}{2H^2},$$
$$par1 = (P_1 K_1 + w A_1 Q_1) H - A_1 K_1 T_1,$$
$$P_1 = -w(g_{11})_w \gamma^2 - 2\gamma w g_{11} \gamma_w + 2w\gamma_w g_{12} + 2\gamma w (g_{12})_w - w(g_{22})_w,$$
$$K_1 = g_{22} - \gamma g_{12}, \quad K_2 = \gamma g_{11} - g_{12},$$
$$Z_1 = -\tfrac{1}{2}[\gamma^2 w (g_{11})_w] - (g_{11})_w \gamma + (g_{12})_w + \tfrac{1}{2} w (g_{22})_w,$$
$$Z_2 = -[w(g_{12})_w + \tfrac{1}{2}(g_{11})_w]\gamma^2 + w(g_{22})_w \gamma + \tfrac{1}{2}(g_{22})_w,$$
$$H = -g(w\gamma\gamma_w + \gamma_w) + Z_1 K_1 + Z_2 K_2,$$
$$Q_1 = -(g_{22})_w + \gamma_w g_{12} + \gamma (g_{12})_w,$$
$$\epsilon_1 = -\tfrac{1}{2} w (g_{11})_w, \quad \epsilon_2 = -(g_{11})_w,$$
$$\eta_1 = \tfrac{1}{2} w[(g_{11})_w + w(g_{11})_{ww}], \quad \eta_2 = w(g_{11})_{ww},$$
$$\eta_3 = -w[(g_{12})_{ww} + \tfrac{1}{2}((g_{22})_w + w(g_{22})_w)], \quad \eta_4 = -w(2\epsilon_1\gamma + \epsilon_2)\gamma_w,$$
$$Z_3 = \eta_1 \gamma^2 + \eta_2 \gamma + \eta_3 + \eta_4,$$
$$\theta_1 = -[w(g_{12})_w + \tfrac{1}{2}(g_{11})_w], \quad \theta_2 = w(g_{22})_w,$$
$$\lambda_1 = [(g_{12})_w + w(g_{12})_w + \tfrac{1}{2}(g_{11})_{ww}]w,$$
$$\lambda_2 = -w[(g_{22})_w + w(g_{22})_w],$$
$$\lambda_3 = -\tfrac{1}{2} w (g_{22})_{ww}, \quad \lambda_4 = -w(2\theta_1\gamma + \theta_2)\gamma_w,$$
$$Z_4 = \lambda_1 \gamma^2 + \lambda_2 \gamma + \lambda_3 + \lambda_4,$$
$$L_1 = w[-(g_{22})_w + g_{12}\gamma_w + \gamma (g_{12})_w],$$
$$L_2 = -w[g_{11}\gamma_w + \gamma (g_{11})_w - (g_{12})_w],$$
$$N_1 = (w\gamma_w)^2 + \gamma\gamma_{ww} w^2 + 2\gamma w \gamma_w + w\gamma_{ww} + \gamma_w,$$
$$L_0 = (g_{11})_w g_{22} + g_{11}(g_{22})_w - 2g_{12}(g_{12})_w,$$
$$M_1 = L_0 w \gamma_w (\gamma w + 1),$$
$$T_1 = M_1 + g N_1 + Z_1 L_1 + Z_2 L_2 + (Z_3 - Z_1) K_1 + (Z_4 - Z_2) K_2,$$

$$da_2 = -\tfrac{par2}{2H^2},$$
$$par2 = (P_2K_1 + A_1Q_2)H - A_1K_1T_2,$$
$$P_2 = (g_{11})_w\gamma^2 + 2g_{11}\gamma\gamma_w - 2\gamma_w g_{12} - 2\gamma(g_{12})_w + (g_{22})_w,$$
$$Q_2 = (g_{22})_w - \gamma_w g_{12} - \gamma(g_{12})_w,$$
$$\mu_1 = -\tfrac{1}{2}[(g_{11})_w + w(g_{11})_{ww}],\ \mu_2 = -(g_{11})_{ww},$$
$$\mu_3 = [(g_{12})_w + \tfrac{1}{2}((g_{22})_w + w(g_{22})_{ww})],\ \mu_4 = [2\epsilon_1\gamma + \epsilon_2]\gamma_w,$$
$$Z_5 = \mu_1\gamma^2 + \mu_2\gamma + \mu_3 + \mu_4,$$
$$\rho_1 = -[(g_{12})_w + w(g_{12})_{ww} + \tfrac{1}{2}(g_{11})_{ww}]w,\ \ \rho_2 = (g_{22})_w + w(g_{22})_{ww},$$
$$\rho_3 = \tfrac{1}{2}(g_{22})_w,\ \ \rho_4 = (2\theta_1\gamma + \theta_2)\gamma_w,$$
$$Z_6 = \rho_1\gamma^2 + \rho_2\gamma + \rho_3 + \rho_4,$$
$$L_3 = (g_{22})_w - \gamma_1 g_{12} - \gamma(g_{12})_w, L_4 = \gamma_w g_{12} + \gamma(g_{11})_w - (g_{12})_w,$$
$$N_2 = -[w\gamma_w^2 + w\gamma\gamma_{ww} + \gamma\gamma_w + \gamma_{ww}],$$
$$M_2 = -L_0\gamma_w(1 + w\gamma_w),$$
$$T_2 = M_2 + gN_2 + Z_1L_3 + Z_2L_4 + Z_5K_1 + Z_6K_2,$$
$$db_1 = -\tfrac{par3}{2H^2},$$
$$par3 = (P_1K_2 + wA_1Q_3)H - A_1K_2T_1,$$
$$db_2 = -\tfrac{par4}{2H^2},$$
$$par4 = (P_2K_2 + A_1Q_4)H - A_1K_2T_2.$$

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