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CHARACTERIZATIONS OF CLOSED SETS IN PRODUCT SPACES

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Abstract. In this paper we characterize different types of closed sets in product spaces using the notion of upper limit.

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1. INTRODUCTION

Let X be a set. A *net* in X is a map $S\colon \Lambda \to X$ of a directed set Λ into X. The net S is also denoted by $\{s_\lambda,\ \lambda \in \Lambda\}$, where $s_\lambda = S(\lambda)$ (see, for example, [3] and [10]).

Let X be a topological space, A be a subset of X, and $x \in X$. By $Cl(A)$ (respectively, $Int(A)$) we denote the closure (respectively, the interior) of A in X. It is known that:

(i) The point x of X belongs to the closure of A in X if and only if there is a net in A converging to x (see [10]).

(ii) The point x is in the *ϑ-closure* (respectively, in the *δ-closure*) of A of X, $x \in Cl_\vartheta(A)$ (respectively, $x \in Cl_\delta(A)$), if each open subset V containing x satisfies $A \cap Cl(V) \neq \emptyset$ (respectively, $A \cap Int(Cl(V)) \neq \emptyset$). A is *ϑ-closed* (respectively, *δ-closed*) if $Cl_\vartheta(A) = A$ (respectively, $Cl_\delta(A) = A$) (see, for example, [12] and [15]).

(iii) A net $\{x_\lambda : \lambda \in \Lambda\}$ in X *ϑ-converges* (respectively, δ-converges) (see, for example, [2], [5], [6], and [7]) to x if for every neighborhood U of x there is some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in Cl(U)$ (respectively, $x_\lambda \in Int(Cl(U))$).

Let Y, Z be topological spaces and f be a map of Y into Z. The map f is said to be *ϑ-continuous* (respectively, *δ-continuous*) at $y \in Y$ if for every open neighborhood V of $f(y)$ there exists an open neighborhood U of y such that $f(Cl(U)) \subseteq Cl(V)$ (respectively, $f(Int(Cl(U))) \subseteq Int(Cl(V))$). The map f is said to be *ϑ-continuous* (respectively, *δ-continuous*) on Y if it is ϑ-continuous (respectively, δ-continuous) at each point of Y (see, for example, [4], [8], and [13]).

Let Y, Z be topological spaces and f be a map of Y into Z. The map f is said to be *δ-continuous* (respectively, *quasi ϑ-continuous*) if for every δ-closed subset (respectively, ϑ-closed subset) A of Z, $f^{-1}(A)$ is δ-closed set (respectively, ϑ-closed) in Y (see, for example, [13] and [14]).

Let $\mathcal{P}(X)$ be the family of all subsets of a topological space X. A net in $\mathcal{P}(X)$ is also called a *directed set of subsets of* X. If Λ is a directed set, then by $\overline{\lim_{\Lambda}}(A_\lambda)$ (respectively, ϑ-$\overline{\lim_{\Lambda}}(A_\lambda)$) where $A_\lambda \subseteq X$, we denote the *upper limit* (respectively, ϑ-*upper limit*) of the net $\{A_\lambda, \lambda \in \Lambda\}$ in $\mathcal{P}(X)$, that is, the set of all points x of X such that for every $\lambda_0 \in \Lambda$ and for every open neighborhood U of x in X there exists an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap U \neq \emptyset$ (respectively, $A_\lambda \cap Cl(U) \neq \emptyset$) (see, for example, [1], [5], [9] and [11]).

If Λ is a directed set, then by w-ϑ-$\overline{\lim_{\Lambda}}(A_\lambda)$, where $A_\lambda \subseteq X$, we denote the *weakly ϑ-upper limit* of the net $\{A_\lambda, \lambda \in \Lambda\}$ in $\mathcal{P}(X)$, that is, the set of all points x of X such that for every $\lambda_0 \in \Lambda$ and for every open neighborhood U of x in X there exists an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap Int(Cl(U)) \neq \emptyset$ (see [6] and [7]).

In what follows by X, Y, Z, Y_1,…,Y_n we denote topological spaces.

2. CLOSED SETS IN PRODUCT SPACES

Definition 1. Let D be a subset of $X \times Y_1 \times \cdots \times Y_n$. For every $x \in X$ we denote by D_x the subset $D \cap (\{x\} \times Y_1 \times \cdots \times Y_n)$ of D and by $D[x]$ the subset of $Y_1 \times \cdots \times Y_n$ for which $D_x = \{x\} \times D[x]$.

Theorem 1. *A subset D of $X \times Y_1$ is closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to a point x of X we have*

$$\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Proof. Let D be a closed subset of $X \times Y_1$ and let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X converging to $x \in X$. Consider

$$y \in \overline{\lim_{\Lambda}}(D[x_\lambda]).$$

Then, for every open neighborhood V_y of y in Y_1 and for every $\lambda \in \Lambda$ there exists an element $\lambda' \geq \lambda$ such that $V_y \cap D[x_{\lambda'}] \neq \emptyset$.

Let V_x and V_y be arbitrary open neighborhoods of x and y in X and Y_1, respectively, and let $U = V_x \times V_y$. Since the net $\{x_\lambda, \lambda \in \Lambda\}$ of X converges to $x \in X$, there exists an element $\lambda \in \Lambda$ such that $\{x_{\lambda_1} : \lambda_1 \geq \lambda\} \subseteq V_x$. Let $\lambda' \geq \lambda$ and $V_y \cap D[x_{\lambda'}] \neq \emptyset$. If $y' \in V_y \cap D[x_{\lambda'}]$, then $(x_{\lambda'}, y') \in U$, that is, $U \cap D \neq \emptyset$, hence, $(x, y) \in Cl(D) = D$. This means that $(x, y) \in D_x$, that is, $y \in D[x]$.

Conversely, let D be a subset of $X \times Y_1$ such that

$$\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x]$$

for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to $x \in X$. We prove that D is closed. Indeed, suppose that $(x, y) \in Cl(D)$. We prove that $(x, y) \in D$. There exists a net $\{(x_\lambda, y_\lambda) : \lambda \in \Lambda\}$ in D converging to (x, y). This means that for every open neighborhood V_x and V_y of x and y in X and Y_1, respectively, there exists an element $\lambda \in \Lambda$ such that $x_{\lambda'} \in V_x$ and $y_{\lambda'} \in V_y$ for every $\lambda' \geq \lambda$. In particular from this it follows that the net $\{x_\lambda : \lambda \in \Lambda\}$ in X converges to x.

Also, $(x_\lambda, y_\lambda) \in D$ or $y_\lambda \in D[x_\lambda]$. So the above means that $V_y \cap D[x_{\lambda'}] \neq \emptyset$, for every $\lambda' \geq \lambda$, that is,

$$y \in \overline{\lim_{\Lambda}}(D[x_\lambda])$$

and therefore $y \in D[x]$, that is, $(x, y) \in D$. □

Corollary 1. *A subset D of $X \times Y_1 \times Y_2 \times \cdots \times Y_n$ is closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to a point x of X we have*

$$\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Definition 2. Let $F\colon X \times Y \to Z$ be a continuous map. By F_x, where $x \in X$, we denote the continuous map of Y into Z, for which $F_x(y) = F(x, y)$, for every $y \in Y$.

Theorem 2. *A map $F : X \times Y_1 \to Z$ is continuous if and only if for every closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to $x \in X$ we have*

$$\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Proof. Let us suppose that the map F is continuous and let K be a closed subset of Z. Then, the subset $D = F^{-1}(K)$ is closed in $X \times Y_1$.

We observe first that for every map $F\colon X \times Y_1 \to Z$ and for every $K \subseteq Z$ we have $(F^{-1}(K))[x] = F_x^{-1}(K)$. Indeed, $y \in (F^{-1}(K))[x]$ if and only if $(x, y) \in F^{-1}(K)$, that is, if and only if $y \in F_x^{-1}(K)$.

Hence, by Theorem 1, we have

$$\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x],$$

that is,

$$\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Conversely, suppose that for every closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to $x \in X$ we have

$$\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

We prove that F is continuous. Indeed, let K be any closed subset of Z and let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X converging to $x \in X$. Then we have

$$\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K)$$

or

$$\overline{\lim_{\Lambda}}(F^{-1}(K)[x_\lambda]) \subseteq F^{-1}(K)[x].$$

Hence, by Theorem 1, the set $F^{-1}(K)$ is closed in $X \times Y_1$ and therefore the map F is continuous. □

Corollary 2. *A map $F : X \times Y_1 \times \cdots \times Y_n \to Z$ is continuous if and only if for every closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X converging to $x \in X$ we have*

$$\overline{\lim_{\Lambda}}(F^{-1}_{x_\lambda}(K)) \subseteq F^{-1}_x(K).$$

3. δ-CLOSED SETS IN PRODUCT SPACES

Theorem 3. *Let $x \in X$. The point x belongs to the δ-closure of a subset A of X if and only if there is a net in A which δ-converges to x.*

Proof. Let $\{x_\lambda, \lambda \in \Lambda\}$ be a net in A which δ-converges to x in X. We consider an open set V of x in X. Since the net $\{x_\lambda, \lambda \in \Lambda\}$ δ-converges to x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in Int(Cl(V))$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Hence $Int(Cl(V)) \cap A \neq \emptyset$ and $x \in Cl_\delta(A)$.

Conversely, let $x \in Cl_\delta(A)$. Then for every open neighborhood U of x it follows that $Int(Cl(U)) \cap A \neq \emptyset$. Let $\mathcal{N}(x)$ be the set of all open neighborhoods of x in X. The set $\mathcal{N}(x)$ with the relation of inverse inclusion is a directed set. For every $U \in \mathcal{N}(x)$, let $x_U \in Int(Cl(U)) \cap A$. Clearly, the net $\{x_U, U \in \mathcal{N}(x)\}$ δ-converges to x. □

Theorem 4. *A subset D of $X \times Y_1$ is δ-closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which δ-converges to a point x of X we have*

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Proof. Let D be a δ-closed subset of $X \times Y_1$ and let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X which δ-converges to $x \in X$. Consider

$$y \in w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]).$$

We prove that $y \in D[x]$. Clearly, for every open neighborhood V_y of y in Y_1 and for every $\lambda \in \Lambda$ there exists an element $\lambda' \geq \lambda$ such that

$$Int(Cl(V_y)) \cap D[x_{\lambda'}] \neq \emptyset.$$

Let V_x and V_y be arbitrary open neighborhoods of x and y in X and Y, respectively, and let $U = V_x \times V_y$.

There exists an element $\lambda \in \Lambda$ such that $\{x_{\lambda_1} : \lambda_1 \geq \lambda\} \subseteq Int(Cl(V_x))$. Let $\lambda' \geq \lambda$ and $Int(Cl(V_y)) \cap D[x_{\lambda'}] \neq \emptyset$. If $y' \in Int(Cl(V_y)) \cap D[x_{\lambda'}]$, then $(x_{\lambda'}, y') \in Int(Cl(U))$, that is, $Int(Cl(U)) \cap D \neq \emptyset$ and, hence, $(x, y) \in Cl_\delta(D) = D$. This means that $(x, y) \in D_x$, that is, $y \in D[x]$.

Conversely, let D be a subset of $X \times Y_1$ such that

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x]$$

for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X δ-converging to $x \in X$. We prove that D is δ-closed. Indeed, suppose that $(x, y) \in Cl_\delta(D)$. Then, there exists a net $\{(x_\lambda, y_\lambda) : \lambda \in \Lambda\}$ in D δ-converging to (x, y). This means that for every open neighborhood V_x and V_y of x and y in X and Y_1, respectively, there exists an element $\lambda \in \Lambda$ such that $x_{\lambda'} \in Int(Cl(V_x))$ and $y_{\lambda'} \in Int(Cl(V_y))$ for every $\lambda' \geq \lambda$. In particular from this it follows that the net $\{x_\lambda : \lambda \in \Lambda\}$ δ-converges to x in X.

Also, since $(x_\lambda, y_\lambda) \in D$ or $y_\lambda \in D[x_\lambda]$, the above means that

$$Int(Cl(V_y)) \cap D[x_{\lambda'}] \neq \emptyset,$$

for every $\lambda' \geq \lambda$, that is,

$$y \in w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Hence, $y \in D[x]$ and $(x, y) \in D$. □

Corollary 3. *A subset D of $X \times Y_1 \times Y_2 \times \cdots \times Y_n$ is δ-closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which δ-converges to a point x of X we have*

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Definition 3. Let $F\colon X \times Y \to Z$ be a δ-continuous map. By F_x, where $x \in X$, we denote the δ-continuous map of Y into Z, for which $F_x(y) = F(x, y)$, for every $y \in Y$.

Theorem 5. *A map $F\colon X \times Y_1 \to Z$ is δ-continuous if and only if for every δ-closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which δ-converges to $x \in X$ we have*

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Proof. Let $F : X \times Y_1 \to Z$ be a δ-continuous map and K be a δ-closed subset of Z. Then, the subset $D = F^{-1}(K)$ is δ-closed in $X \times Y_1$. First we observe that for the δ-closed subset K of Z we have $(F^{-1}(K))[x] = F_x^{-1}(K)$. Hence, by Theorem 4, we have

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x],$$

that is,

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Conversely, let K be any δ-closed subset of Z. We prove that $F^{-1}(K)$ is δ-closed. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X which $\delta-$converges to $x \in X$. Then we have

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K)$$

or

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(F^{-1}(K)[x_\lambda]) \subseteq F^{-1}(K)[x].$$

Hence, by Theorem 4, the set $F^{-1}(K)$ is δ-closed in $X \times Y_1$ and therefore the map $F\colon X \times Y_1 \to Z$ is δ-continuous. □

Corollary 4. *A map $F\colon X \times Y_1 \times \cdots \times Y_n \to Z$ is δ-continuous if and only if for every δ-closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which δ-converges to $x \in X$ we have*

$$w\text{-}\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

4. ϑ-CLOSED SETS IN PRODUCT SPACES

Theorem 6. *Let $x \in X$. The point x belongs to the ϑ-closure of a subset A of X if and only if there is a net in A which ϑ-converges to x.*

Proof. Let $\{x_\lambda, \lambda \in \Lambda\}$ be a net in A which ϑ-converges to x in X. We consider an open set V of x in X. Since the net $\{x_\lambda, \lambda \in \Lambda\}$ is ϑ-converging to x there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in Cl(V)$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Thus $Cl(V) \cap A \neq \emptyset$ and therefore $x \in Cl_\vartheta(A)$.

Conversely, let $x \in Cl_\vartheta(A)$. Then for every open neighborhood U of x it follows that $Cl(U) \cap A \neq \emptyset$. Let $\mathcal{N}(x)$ be the set of all open neighborhoods of x in X. The set $\mathcal{N}(x)$ with the relation of inverse inclusion is a directed set. For every $U \in \mathcal{N}(x)$, let $x_U \in Cl(U) \cap A$. Clearly, the net $\{x_U, U \in \mathcal{N}(x)\}$ ϑ-converges to x. □

Theorem 7. *A subset D of $X \times Y_1$ is ϑ-closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which ϑ-converges to a point x of X we have*

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Proof. Let D be a ϑ-closed subset of $X \times Y_1$ and let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X which ϑ-converges to $x \in X$. Consider

$$y \in \vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]).$$

Clearly, for every open neighborhood V_y of y in Y_1 and for every $\lambda \in \Lambda$ there exists an element $\lambda' \geq \lambda$ such that $Cl(V_y) \cap D[x_{\lambda'}] \neq \emptyset$. Let V_x and V_y be arbitrary open neighborhoods of x and y in X and Y_1, respectively, and let $U = V_x \times V_y$.

There exists an element $\lambda \in \Lambda$ such that $\{x_{\lambda_1} : \lambda_1 \geq \lambda\} \subseteq Cl(V_x)$. Let $\lambda' \geq \lambda$ and $Cl(V_y) \cap D[x_{\lambda'}] \neq \emptyset$. If $y' \in Cl(V_y) \cap D[x_{\lambda'}]$, then $(x_{\lambda'}, y') \in Cl(U)$, that is, $Cl(U) \cap D \neq \emptyset$ and, hence, $(x, y) \in Cl_\vartheta(D) = D$. This means that $(x, y) \in D_x$ and therefore $y \in D[x]$.

Conversely, let D be a subset of $X \times Y_1$ such that

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x]$$

for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which ϑ-converges to $x \in X$. We prove that D is ϑ-closed. Indeed, suppose that $(x, y) \in Cl_\vartheta(D)$. We prove that $(x, y) \in D$. There exists a net $\{(x_\lambda, y_\lambda) : \lambda \in \Lambda\}$ in D which ϑ-converges to (x, y). This means that for every open neighborhood V_x and V_y of x and y in X and Y_1, respectively, there exists an element $\lambda \in \Lambda$ such that $x_{\lambda'} \in Cl(V_x)$

and $y_{\lambda'} \in Cl(V_y)$ for every $\lambda' \geq \lambda$. In particular from this it follows that the net $\{x_\lambda : \lambda \in \Lambda\}$ in X ϑ-converges to x.

Also, since $(x_\lambda, y_\lambda) \in D$ or $y_\lambda \in D[x_\lambda]$, the above means that $Cl(V_y) \cap D[x_{\lambda'}] \neq \emptyset$, for every $\lambda' \geq \lambda$, that is,

$$y \in \vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Thus $y \in D[x]$, that is, $(x, y) \in D$. □

Corollary 5. *A subset D of $X \times Y_1 \times Y_2 \times \cdots \times Y_n$ is ϑ-closed if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which ϑ-converges to a point x of X we have*

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x].$$

Theorem 8. *A map $F\colon X \times Y_1 \to Z$ is quasi ϑ-continuous if and only if for every ϑ-closed subset K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which ϑ-converges to $x \in X$ we have*

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Proof. Suppose that $F\colon X \times Y_1 \to Z$ is quasi ϑ-continuous and let K be a ϑ-closed subset of Z. Since the map F is quasi ϑ-continuous, the set $D = F^{-1}(K)$ is ϑ-closed in $X \times Y$. Hence, by Theorem 7, we have

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(D[x_\lambda]) \subseteq D[x],$$

that is,

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

Conversely, let K be any ϑ-closed subset of Z. We prove that the subset $F^{-1}(K)$ is ϑ-closed. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X which ϑ-converges to $x \in X$. Then, we have

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K)$$

or

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(F^{-1}(K)[x_\lambda] \subseteq F^{-1}(K)[x].$$

Hence, by Theorem 7, the set $F^{-1}(K)$ is ϑ-closed in $X \times Y_1$ and therefore the map $F\colon X \times Y_1 \to Z$ is quasi ϑ-continuous. □

Corollary 6. *A map $F\colon X \times Y_1 \times Y_2 \times \cdots \times Y_n \to Z$ is quasi ϑ-continuous if and only if for every ϑ-closed K of Z and for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which ϑ-converges to $x \in X$ we have*

$$\vartheta\text{-}\overline{\lim_{\Lambda}}(F_{x_\lambda}^{-1}(K)) \subseteq F_x^{-1}(K).$$

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