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ON THE MULTIPLICITY MODULE OF A POINTED GROUP

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Abstract. In this paper we give a module-theoretic approach to the notion of multiplicity module of a pointed group using techniques from the theory of group-graded algebras.

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1. INTRODUCTION

1.1. We fix a finite group G and a complete local commutative noetherian ring $\mathcal{O}$ with residue field k of nonzero characteristic p (not necessary algebraically closed). All algebras and modules will be assumed to be free over $\mathcal{O}$ and of finite rank. Let A be a G-algebra over $\mathcal{O}$. Recall that a pointed group H_α on A is a pair (H, α) where H is a subgroup of G and $\alpha \in \mathcal{P}(A^H)$ is a point of A^H.

1.2. With any pointed group H_α on A are associated several mathematical objects that we will now describe, following Thévenaz [6, Section 13] (see also Puig [5]). First, there is a unique maximal ideal m_α of A^H such that $\alpha \subsetneq m_\alpha$ and the simple k-algebra $S(\alpha) = A^H/m_\alpha$ is called the multiplicity algebra of the pointed group H_α. We let $N_G(H_\alpha)$ denote the stabilizer of α in $N_G(H)$ and denote

$$\bar{N}_G(H_\alpha) = N_G(H_\alpha)/H.$$

The group $N_G(H_\alpha)$ acts on the quotient A^H/m_α and, therefore, $S(\alpha)$ is an $N_G(H_\alpha)$-algebra. Since H acts trivially on A^H, it is convenient to view $S(\alpha)$ as an $\bar{N}_G(H_\alpha)$-algebra.

If k is algebraically closed, then $S(\alpha) = \mathrm{End}_k(V(\alpha))$ for some simple $S(\alpha)$-module $V(\alpha)$, and this simple module is called the multiplicity module of H_α. Actually the multiplicity module $V(\alpha)$ can be canonically endowed with a module structure over a twisted group algebra $k_\sharp\hat{\bar{N}}_G(H_\alpha)$ which is associated with $S(\alpha)$. By the multiplicity module $V(\alpha)$ of a pointed group H_α, we shall always mean the k-vector space $V(\alpha)$ endowed with its $k_\sharp\hat{\bar{N}}_G(H_\alpha)$-module structure.

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1.3. By using the bijection established in [1, Proposition 2.4] we can interpret a pointed group H_α on A as an isomorphism class of indecomposable R_H-direct summand of A, where $R = A * G$ is the G-graded skew group algebra of A and G. More precisely, the indecomposable module corresponding to H_α is the $A * H$-module Ai, where $i \in \alpha$. This correspondence allows us to give module-theoretic proofs some results from G-algebras theory, concerning pointed groups (see [1] and [2]).

1.4. In this article we discuss the notions of the multiplicity module and the multiplicity algebra of a pointed group from a module theoretic point of view. We start with a strongly G-graded algebra R, an R-module $\tilde{M}$ and U an indecomposable direct summand of $\mathrm{Res}_H^G(\tilde{M})$. Note that we consider the general case, when k is not necessary algebraically closed. We will define the notion of multiplicity module of U. This construction applied to the G-graded algebra $A * G$ will imply in the case k algebraically closed, the usual notion of multiplicity module of a pointed group.

Our approach will require techniques from the theory of group-graded algebras, so we shall recall the needed facts in Section 2. We will focus mainly on the categorial approach to the Clifford theory for indecomposable modules over strongly graded algebras. We refer to the reader to [3] and [4] for general facts on graded rings and on [3] for results on Clifford theory. Our main result is the construction of the multiplicity module of U in Section 3, which generalizes the corresponding notion for pointed groups.

2. BACKGROUND AND NOTATIONS

Let $\mathcal{O}$, k and G be as in the introduction. Let $R = \bigoplus_{g \in G} R_g$ be a strongly G-graded $\mathcal{O}$-algebra (that is, R_g is an $\mathcal{O}$-summand of R, and $R_g R_h = R_{gh}$, for all $g, h \in G$). For a subset X of G we denote $R_X = \bigoplus_{x \in G} R_x$. If M is a (left) R-module, then $(R|M)$-mod is the full subcategory of R-mod consisting of direct summands of finite direct copies of M, that is, $(R|M)$-mod is the smallest additive subcategory of R-mod containing M. The objects of this category are called R-modules lying over M.

2.1. Let M be an indecomposable R_1-module and let

$$G_M = \{g \in G \mid R_g \otimes_{R_1} M \simeq M \text{ in } R_1\text{-mod}\}$$

be the stabilizer of M. Let further $E = \mathrm{End}_{R_1}(R \otimes_{R_1} M)^{\mathrm{op}}$. Then it is well-known that E is a G-graded $\mathcal{O}$-algebra (not necessarily strongly graded) with

$$E_g \simeq \{f \in E \mid f(R_x \otimes_{R_1} M) \subseteq R_{xg} \otimes_{R_1} M \text{ for all } x \in G\}.$$

In particular, $E_1 \simeq \mathrm{End}_{R_1}(M)^{\mathrm{op}}$, and moreover, for any subgroup H of G,

$$E_H \simeq \mathrm{End}_{R_H}(R_H \otimes_{R_1} M)^{\mathrm{op}}.$$

2.2. Let $J_{\rm gr}(E)$ be the graded Jacobson radical of E, that is, $J_{\rm gr}(E)$ is the intersection of the maximal graded left ideals of E. Then $D = E/J_{\rm gr}(E)$ is a strongly graded k-algebra with $D_1 \simeq E_1/J(E_1)$. By the graded version of the Fitting's Lemma, E_1 is a local ring and D is a crossed product of the skew-field D_1 with the group G_M. If k is algebraically closed, $D_1 = k$ and D is a twisted group algebra of G_M and k, so there is $\alpha \in Z^2(G_M, k^*)$ such that $D = k^\alpha G_M$.

2.3. By [3, Theorem 2.3.10], we have that the additive functor

$$D \otimes_E \mathrm{Hom}_R(R \otimes_{R_1} M, -) : (R|R \otimes_{R_1} M)\text{-mod} \to D\text{-proj}$$

induces an isomorphism between the Grothendieck groups associated to these categories. Moreover, this functor commutes with induction from subgroups, restriction and truncation.

Recall also that, the categories $(R|R\otimes_{R_1} M)$-mod and E-proj are equivalent and, if U is an indecomposable E-module, the corresponding indecomposable D-module is isomorphic to $U/J_{\rm gr}(E)U$.

2.4. Note that passing from E to D relies on the following general remark concerning the endomorphism ring of projective modules. Assume that P is a projective G-graded R-module. Then $P/J_{\rm gr}(P)$ is a gr-simple R-module and the stabilizers of P and $P/J_{\rm gr}(P)$ are equal. Moreover

$$\mathrm{End}_R(P)^{\rm op}/J_{\rm gr}(\mathrm{End}_R(P)^{\rm op}) \simeq \mathrm{End}_R(P/J_{\rm gr}(P))^{\rm op}$$

as G_P-graded rings.

2.5. Finally, we recall the connection between G-invariant ideals of R_1 and graded ideals of R. Recall that an ideal I of R_1 is called G-invariant if $R_g I R_{g^{-1}} = I$, for all $g \in G$. The lattice of graded ideals of R is isomorphic to the lattice of G-invariant ideals of R_1. If I is a G-invariant ideal of R, then the corresponding graded ideal of R is $RI = IR$, and conversely, if $J = \bigoplus_{g\in G} J_g$ is a graded ideal of R, then the corresponding G-invariant ideal of R_1 is J_1.

3. THE MULTIPLICITY MODULE OF A MODULE

3.1. Let R be a strongly G-graded $\mathcal{O}$-algebra and H a subgroup of G. Let $\tilde{M}$ be an R-module and denote

$$A = \mathrm{End}_{R_1}(\tilde{M})^{\rm op}.$$

It is known that A is a G-algebra and

$$A^H = \mathrm{End}_{R_H}(\tilde{M})^{\rm op}.$$

Moreover, A^H becomes an $N_G(H)$-algebra on which H acts trivially, and we will regard A^H as an $\bar{N}_G(H)$-algebra, where

$$\bar{N}_G(H) := N_G(H)/H.$$

3.2. Let U be an indecomposable direct summand of $\operatorname{Res}^G_H \tilde{M}$. We regard $R_{N_G(H)}$ as an $\bar{N}_G(H)$-graded algebra (with the 1-component R_H). Then

$$M := R_{N_G(H)} \otimes_{R_H} U$$

is an $\bar{N}_G(H)$-graded $R_{N_G(H)}$-module. We denote by $\bar{N}_G(H)_M$ the stabilizer of M, that is

$$\bar{N}_G(H)_M = \bar{N}_G(H)_U = \{gH \in \bar{N}_G(H) \mid R_{gH} \otimes_{R_H} U \simeq U \text{ in } R_H\text{-mod}\}.$$

3.3. Since $\tilde{M}$ is an R module, so in particular, an $R_{\bar{N}_G(H)}$-module, there is a well known isomorphism of $\bar{N}_G(H)$-graded $\mathcal{O}$-algebras

$$\operatorname{End}_{R_{\bar{N}_G(H)}}(R_{\bar{N}_G(H)} \otimes_{R_H} \tilde{M})^{\mathrm{op}} \simeq A^H * \bar{N}_G(H),$$

where $A^H * \bar{N}_G(H)$ is the skew group algebra of $A^H = \operatorname{End}_{R_H}(\tilde{M})^{\mathrm{op}}$ and $\bar{N}_G(H)$. Later we shall also apply Clifford theory to the $\bar{N}_G(H)$-graded algebra $A^H * \bar{N}_G(H)$ and its 1-component A^H.

3.4. The R_H-module U determines a unique projective indecomposable A^H-module X, and X in turn, corresponds to the A^H-module

$$\bar{X} := X/J(A^H)X,$$

which is simple as A^H (and also as $A^H/J(A^H)$)-module. Moreover, since all these correspondences preserve the group-actions, as in 2.4, we have the equality

$$\bar{N}_G(H)_U = \bar{N}_G(H)_X = \bar{N}_G(H)_{\bar{X}}$$

between the stabilizers.

3.5. Now consider the $\bar{N}_G(H)$-graded algebras

$$E := \operatorname{End}_{R_{N_G(H)}}(R_{N_G(H)} \otimes_{R_H} U)^{\mathrm{op}},$$

which is a crossed product of $E_1 \simeq \operatorname{End}_{R_H}(U)^{\mathrm{op}}$ and $\bar{N}_G(H)_U$, and

$$D := E/J_{\mathrm{gr}}(E),$$

which is a crossed product of $D_1 \simeq E_1/J(E_1)$ and $\bar{N}_G(H)_U$. Since the correspondence

$$R_{N_G(H)} \otimes_{R_H} U \mapsto (A^H * \bar{N}_G(H)) \otimes_{A^H} X$$

comes from an equivalence of categories, we have the isomorphism

$$E \simeq \operatorname{End}_{A^H * \bar{N}_G(H)}((A^H * \bar{N}_G(H)) \otimes_{A^H} X)^{\mathrm{op}}$$

of $\bar{N}_G(H)$-graded algebras. Moreover, since X is a projective A^H-module, by 2.4, we have the isomorphism

$$D \simeq \operatorname{End}_{A^H * \bar{N}_G(H)_X}((A^H * \bar{N}_G(H)) \otimes_{A^H} \bar{X})^{\mathrm{op}}$$

of $\bar{N}_G(H)$-graded k-algebras. The k-algebra on the right hand side is isomorphic to a crossed product of

$$D_1 \simeq \operatorname{End}_{A^H}(\bar{X})^{\mathrm{op}} \simeq \operatorname{End}_{A^H/J(A^H)}(X)^{\mathrm{op}}$$

with $\bar{N}_G(H)_X$.

We have that $\bar{X}$ is an $(A^H, \mathrm{End}_{A^H}(\bar{X})^{\mathrm{op}})$-bimodule and since $\bar{X}$ is an $\bar{N}_G(H)_X$-invariant A^H-module, we have that

$$(A^H * \bar{N}_G(H)_X) \otimes_{A^H} \bar{X}$$
$$\simeq \bar{X} \otimes_{\mathrm{End}_{A^H}(\bar{X})^{\mathrm{op}}} \mathrm{End}_{A^H * \bar{N}_G(H)_X}((A^H * \bar{N}_G(H)) \otimes_{A^H} \bar{X})^{\mathrm{op}}$$

is an $(A^H * \bar{N}_G(H)_X, \mathrm{End}_{A^H * \bar{N}_G(H)_X}((A^H * \bar{N}_G(H)) \otimes_{A^H} \bar{X}))$-bimodule graded by $\bar{N}_G(H)_X$.

3.6. Since $\bar{X}$ is a simple A^H-module, we have that $J(A^H) \subseteq \mathrm{Ann}_{A^H}(\bar{X})$ and, moreover, $\mathrm{Ann}_{A^H}(\bar{X})$ is an $\bar{N}_G(H)_X$-invariant ideal of $A^H * \bar{N}_G(H)_X$. Using the isomorphism mentioned in 2.5, this ideal determines a unique graded ideal of $A^H * \bar{N}_G(H)_X$. Then $(A^H * \bar{N}_G(H)_X) \otimes_{A^H} \bar{X}$ is gr-simple as a $A^H * \bar{N}_G(H)_X$-module, and we will consider the $\bar{N}_G(H)_X$-graded k-algebra

$$\hat{R} := A^H * \bar{N}_G(H)_X / \mathrm{Ann}_{A^H}(\bar{X})(A^H * \bar{N}_G(H)_X).$$

Observe that

$$\hat{R} \simeq (A^H / \mathrm{Ann}_{A^H}(\bar{X})) * \bar{N}_G(H)_U$$

is a strongly $\bar{N}_G(H)_U$-graded algebra with the 1-component

$$\hat{R}_1 \simeq A^H / \mathrm{Ann}_{A^H}(\bar{X}),$$

a simple k-algebra. Denote

$$\hat{E} := \mathrm{End}_{\hat{R}}(\hat{R} \otimes_{\hat{R}_1} \bar{X})^{\mathrm{op}}$$

(hence $\hat{E}$ is isomorphic to D as $\bar{N}_G(H)_U$-graded algebras). We have the isomorphism

$$\hat{R} \otimes_{\hat{R}_1} \bar{X} \simeq \bar{X} \otimes_{\hat{E}_1} \hat{E}$$

of $\bar{N}_G(H)_U$-graded $(\hat{R}, \hat{E})$-bimodules. Therefore $\bar{X}$ is a $\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}})$-module, where $\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}})$ is the diagonal subalgebra of $\hat{R} \otimes_k \hat{E}^{\mathrm{op}}$, and

$$\hat{R} \otimes_{\hat{R}_1} \bar{X} \simeq (\hat{R} \otimes_k \hat{E}^{\mathrm{op}}) \otimes_{\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}})} \bar{X}$$

as $(\hat{R}, \hat{E})$-bimodules.

We can now define the notion of multiplicity module of the indecomposable R_H-module U.

Definition 1. *The $\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}})$-module $\bar{X}$ is called the multiplicity module of the R_H-summand U of $\tilde{M}$.*

We study now a particular case of this construction, namely the case when $\hat{E}_1 \simeq k$. This happens for example when k is algebraically closed, since $\bar{X}$ is a simple A^H-algebra and $\hat{E}_1 \simeq \mathrm{End}_{A^H}(\bar{X})$ is then isomorphic to k.

Proposition 1. *Assume that* $\hat{E}_1 \simeq k$, *so* $\hat{R}_1$ *is a central simple* k*-algebra. Then*

$$\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}}) \simeq \hat{R}_1 \otimes_k k_\gamma \bar{N}_G(H)_U,$$

where $\gamma \in Z^2(\bar{N}_G(H)_U, k^*)$.

Proof. Denote $\Delta := \Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}})$. The strongly $\bar{N}_G(H)_U$-graded algebra Δ is a crossed product of $\Delta_1 \simeq \hat{R}_1$ and $\bar{N}_G(H)_U$. We fix the invertible elements $u_g \in \Delta_g \cap U(\Delta)$, for all $g \in \bar{N}_G(H)_U$, and consider the map

$$\sigma\colon \bar{N}_G(H)_U \to \mathrm{Aut}(\Delta_1), \quad \sigma(g)(a) = u_g a u_g^{-1},$$

for all $g \in \bar{N}_G(H)_U$ and $a \in \Delta_1$. By the Skolem-Noether theorem ([6, Theorem 7.2]), the action $\sigma(g)$ of an element $g \in \bar{N}_G(H)_U$ on Δ_1 is an inner automorphism, thus of the form $\mathrm{Inn}(a_g)$, where $a_g \in U(\Delta_1)$. But this implies that the element $a_g^{-1}u_g$ belongs to the centralizer of Δ_1 in Δ, for all $g \in \bar{N}_G(H)_U$. Hence we can replace u_g by $a_g^{-1}u_g$, for all $g \in \bar{N}_G(H)_U$, and since $Z(\Delta_1) \simeq k$ it follows that

$$\Delta \simeq \Delta_1 \otimes_k k_\gamma \bar{N}_G(H)_U$$

for some 2-cocycle $\gamma \in Z^2(\bar{N}_G(H)_U, k^*)$. □

The next result shows that the usual notion of multiplicity module of a pointed group is a particular case of our construction above.

Proposition 2. *Assume* k *is algebraically closed and let* A *be a* G*-algebra over* $\mathcal{O}$. *Denote by* $R = A * G$ *the skew group algebra of* A *and* G *and regard* A *as an* R*-module. Let* H_α *be a pointed group on* G, *and let* Ai, *where* $i \in \alpha$, *be an indecomposable* R_H*-direct summand of* A *corresponding to* H_α.

Then the multiplicity module of Ai (*in the sense of Definition* 1) *is the multiplicity module of the pointed group* H_α.

Proof. Apply the previous construction for $A * G$ instead of R, A instead of $\tilde{M}$ and Ai instead of U. We have that

$$A \simeq \mathrm{End}_{R_1}(A)^{\mathrm{op}}$$

as G-algebras and

$$A^H \simeq \mathrm{End}_{R_H}(A)^{\mathrm{op}}.$$

Then $\bar{N}_G(H)_U$ is $N_G(H_\alpha)$, the stabilizer of H in α, the k-simple algebra $\hat{R}_1$ is $S(\alpha)$, the multiplicity algebra of H_α and $\bar{X}$ is $V(\alpha)$, the multiplicity module of H_α. Moreover, since we are in situation of Proposition 1, we have the isomorphism

$$\Delta(\hat{R} \otimes_k \hat{E}^{\mathrm{op}}) \simeq S(\alpha) \otimes_k k_\gamma \bar{N}_G(H_\gamma),$$

where $\gamma \in Z^2(\bar{N}_G(H_\alpha), k^*)$, and the proof is done. □

REFERENCES

[1] Dicu, C. and Marcus, A., *Group-graded algebras and the relative projectivity of pointed groups*, The Quarterly Journal of Mathematics, **57 (3)** (2006), 309–318.

[2] Dicu, C., *Group graded algebras and the relative freeness of pointed groups*, Mathematica, Tome **47 (70)**, nr. 2 (2005), 151–155.

[3] Marcus, A., *Representation Theory of Group Graded Algebras*, Nova Science Publishers, Commack, NY, 1999.

[4] Năstasescu, C. and Van Ostaeyen, F., *Graded ring theory*, North Holland, Amsterdam, 1982.

[5] Puig, L., *Pointed groups and constructions of modules*, J. Algebra, **116** (1988), 7–129.

[6] Thévenaz, J., *G-algebras and Modular Representation Theory*, Oxford Science Publication, The Clarendon Press, Oxford University Press, New-York, 1995.

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