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A MAXIMUM MODULUS PRINCIPLE FOR A CLASS OF NON-ANALYTIC FUNCTIONS DEFINED IN THE UNIT DISK

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Dedicated to the loving memory of our father, Professor Nicolae N. Pascu (22.VII.1942 – 22.IX.2004)

Abstract. We obtain a maximum modulus principle for a large class of non-analytic functions defined in the unit disk. A corollary and an application in the case of real valued functions of two variables are also given.

MSC 2000. 30C80.

Key words. Maximum modulus principle, non-analytic functions.

1. INTRODUCTION

Maximum modulus principle (for analytic functions) states that an analytic function defined on a simply connected domain attains its maximum modulus at an interior point of the domain, then the function is constant. Maximum modulus principle does not apply without the additional assumption that the function is analytic, as it can be easily be seen by considering the function $f(z,\bar{z}) = \frac{1}{2} - z\bar{z}$ defined for $z \in U = \{z \in \mathbb{C} : |z| < 1\}$ (this function attains its maximum modulus at $z = 0$ without being constant in U). Therefore, by removing the hypothesis on the analyticity of the function, one needs to add supplementary hypotheses in order to insure that the maximum modulus principle holds.

Maximum principles are important tools in various areas of mathematics (they give uniqueness of solutions in Differential equations, Potential theory and they are the key ingredients for proving important inequalities, for example Schwarz lemma in Complex analysis).

In the present paper we give sufficient conditions for a non-analytic function defined in the unit disk to satisfy a maximum modulus principle. We consider the class of non-analytic functions having a power series expansion of the form

$$f(z,\bar{z}) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} a_{kn} z^{n-k}\bar{z}^{k}, \quad z \in U. \tag{1}$$

In Theorem 2 we show that under certain conditions on the coefficients a_{kn} the maximum modulus principle holds for $f(z,\bar{z})$; moreover, we show that $|f(z,\bar{z})|$ is an increasing function of $|z|$. The proof uses a result in [1] on the maximum principle for non-analytic functions.

Our choice of the class of functions in 1 is motivated by the fact that it is a large enough class of functions which includes some important classes of functions. In particular, it includes the class of real-valued functions of two variables having a Taylor series expansion in the whole unit disk (see Theorem 3).

2. PRELIMINARIES

For a complex number $z = x + \mathrm{i}y$ we denote by $\bar{z} = x - \mathrm{i}y$ its complex conjugate, and by $|z| = \sqrt{x^2 + y^2}$ its modulus. We denote by $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk in $\mathbb{C}$.

In order to prove our main result, we need the following result from [1]:

THEOREM 1. *Let $f(z, \bar{z})$ defined for have a series expansion of the form*

$$f(z, \bar{z}) = \sum_{n=1}^{\infty} f_n(z, \bar{z}), \quad z \in U, \tag{2}$$

where $f_n(z, \bar{z})$ are functions of $z \in \bar{U}$ satisfying

$$f_n(rz, r\bar{z}) = r^n f_n(z, \bar{z}) \tag{3}$$

for all $z \in \bar{U}$ and real numbers $r > 0$ for which $rz \in \bar{U}$, $n = 1, 2, \ldots$.

If for some $\theta \in [0, 2\pi)$ we have

$$\sum_{n=2}^{\infty} n \left| f_n(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) \right| \leq \left| f_1(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) \right| \neq 0, \tag{4}$$

then $|f(z, \bar{z})|$ is an increasing function of $|z|$ on $\arg z = \theta$, that is

$$|f(z_1, \bar{z}_1)| < |f(z_2, \bar{z}_2)|,$$

for any $z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta}$, $z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta} \in U$ with $0 < r_1 < r_2 < 1$.

In particular, if (4) holds for all $\theta \in [0, 2\pi)$, then $|f(z, \bar{z})|$ is radially increasing in the whole disk U, and it cannot therefore attain its maximum at an interior point of U.

3. MAIN RESULTS

The main result is contained in the the following:

THEOREM 2. *If the function $f(z, \bar{z})$ defined for $z \in U$ has a series expansion of the form:*

$$f(z, \bar{z}) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} a_{kn} z^{n-k} \bar{z}^k, \quad z \in U, \tag{5}$$

where the coefficients $a_{kn} \in \mathbb{C}$ satisfy the inequality

$$\sum_{n=2}^{\infty} n \sum_{k=0}^{n} |a_{kn}| \leq ||a_{01}| - |a_{11}|| \neq 0, \tag{6}$$

then $|f(z, \bar{z})|$ is a radially increasing function in the unit disk, that is

$$|f(z_1, \bar{z}_1)| < |f(z_2, \bar{z}_2)|,$$

for any $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta} \in U$ *with* $0 < r_1 < r_2 < 1$.

In particular, $f(z,\bar{z})$ *cannot attain its maximum modulus at an interior point of* U.

Proof. Consider the functions $f_n(z,\bar{z}) = \sum_{k=0}^{n} a_{kn} z^{n-k} \bar{z}^k$ defined for $z \in \bar{U}$, where $n = 1, 2, \ldots$.

We will show that with this choice the hypotheses of Theorem 1 are satisfied, and therefore the claim of the theorem will follow.

Let us note first that from the definition of the functions $f_n(z,\bar{z})$, they are (positive real) homogeneous of degree n, that is:

$$\begin{aligned} f_n(rz, r\bar{z}) &= \sum_{k=0}^{n} a_{kn} (rz)^{n-k} \overline{(rz)}^k \\ &= r^n \sum_{k=0}^{n} a_{kn} z^{n-k} \bar{z}^k \\ &= r^n f_n(z,\bar{z}), \end{aligned}$$

for all $z \in \bar{U}$ and $r > 0$ for which $rz \in \bar{U}$, and all $n = 1, 2, \ldots$, and therefore the hypothesis (3) of Theorem 1 is satisfied.

To verify condition (4), let us note that from the hypothesis (6), we have

$$\begin{aligned} \left| f_n(e^{i\theta}, e^{-i\theta}) \right| &= \left| \sum_{k=0}^{n} a_{kn} e^{i(n-k)\theta} e^{-ik\theta} \right| \\ &\le \sum_{k=0}^{n} \left| a_{kn} e^{i(n-k)\theta} e^{-ik\theta} \right| \\ &= \sum_{k=0}^{n} |a_{kn}|, \end{aligned}$$

for all $n = 1, 2, \ldots$ and $\theta \in [0, 2\pi)$, and also

$$\begin{aligned} \left| f_1(e^{i\theta}, e^{-i\theta}) \right| &= \left| a_{01} e^{i\theta} + a_{11} e^{-i\theta} \right| \\ &\ge \left| \left| a_{01} e^{i\theta} \right| - \left| a_{11} e^{-i\theta} \right| \right| \\ &= \left| |a_{01}| - |a_{11}| \right|, \end{aligned}$$

for all $\theta \in [0, 2\pi)$.

We obtain therefore

$$\begin{aligned} \sum_{n=2}^{\infty} n \left| f_n(e^{i\theta}, e^{-i\theta}) \right| &\le \sum_{n=2}^{\infty} \sum_{k=0}^{n} |a_{kn}| \\ &\le \left| |a_{01}| - |a_{11}| \right| \\ &\le \left| f_1(e^{i\theta}, e^{-i\theta}) \right|, \end{aligned}$$

for all $\theta \in [0, 2\pi)$, which shows that the hypothesis 4 of Theorem 1 is also verified.

The claim of the theorem follows now by Theorem 1. □

REMARK 1. Let us note that the maximum modulus principle in the previous theorem also holds in the case $|a_{01}| = |a_{11}|$, provided $f(z, \bar{z})$ is not identically zero in U.

To see this, note that if $|a_{01}| = |a_{11}|$, by using the hypotheses (5) and (6) it follows that $a_{kn} = 0$ for all $n = 2, 3, \ldots$ and $k \in \{0, 1, \ldots, n\}$, and therefore we have

$$f(z, \bar{z}) = a_{01}z + a_{11}\bar{z}, \quad z \in U.$$

Also, since $f(z, \bar{z})$ is not identically zero in U, we have $|a_{01}| = |a_{11}| \neq 0$.

Let $a_{01} = \rho e^{i\alpha}$ and $a_{11} = \rho e^{i\beta}$, where $\rho \in (0, 1)$ and $\alpha, \beta \in [0, 2\pi)$. We have:

$$\begin{aligned}
|f(z, \bar{z})|^2 &= \left|\rho e^{i\alpha} z + \rho e^{i\beta} \bar{z}\right|^2 \\
&= \rho r^2 \left|e^{i(\alpha+\theta)} + e^{i(\beta-\theta)}\right|^2 \\
&= \rho^2 r^2 (2 + 2\cos(\alpha+\theta)\cos(\beta-\theta) + 2\sin(\alpha+\theta)\sin(\beta-\theta)) \\
&= \rho^2 r^2 (2 + 2\cos(\alpha - \beta + 2\theta)) \\
&= 4\rho^2 r^2 \cos^2\left(\theta + \frac{\alpha-\beta}{2}\right),
\end{aligned}$$

for all $z = re^{i\theta} \in U$, which shows that $|f(z, \bar{z})|$ is an increasing function of $|z| = r \in (0, 1)$, for all $\theta \in [0, 2\pi)$ for which $\cos\left(\theta + \frac{\alpha-\beta}{2}\right) \neq 0$.

However, for the values of θ for which $\cos\left(\theta + \frac{\alpha-\beta}{2}\right) = 0$, we have

$$\left|f(re^{i\theta}, e^{-i\theta})\right| = 0,$$

and since $f(z, \bar{z})$ is not identically constant, it follows that the maximum modulus principle still holds for $f(z, \bar{z})$ in this case.

Using the fact that the series

$$\zeta(a) = \sum_{n=1}^{\infty} \frac{1}{n^a}$$

converges for $a > 1$, we can obtain a maximum principle for functions having a series expansion of the form (5) for which the coefficients satisfy a simple inequality, as follows:

COROLLARY 1. *If the function $f(z, \bar{z})$ defined for $z \in U$ has a series expansion of the form:*

$$f(z, \bar{z}) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} a_{kn} z^{n-k} \bar{z}^k, \quad z \in U, \tag{7}$$

where for some real number $a > 0$ the coefficients $a_{kn} \in \mathbb{C}$ satisfy the inequality

$$\max_{0\le k\le n} |a_k| \le \frac{1}{(n+1)n^{2+a}} \frac{||a_{01}| - |a_{11}||}{\zeta(1+a) - 1} \neq 0, \quad n = 2, 3, \dots \tag{8}$$

then $|f(z, \bar{z})|$ is a radially increasing function in U, that is

$$|f(z_1, \bar{z}_1)| < |f(z_2, \bar{z}_2)|, \tag{9}$$

for any $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta} \in U$ with $0 < r_1 < r_2 < 1$.

In particular $f(z, \bar{z})$, cannot attain its maximum modulus at an interior point of U.

Remark 2. The inequalities (8) show essentially that the coefficients a_{kn} converge in absolute value to zero faster than $\frac{1}{n^3}$ as $n \to \infty$, and it provides therefore a large class of functions $f(z, \bar{z})$ for which the maximum principle holds (note that by Remark 1, the maximum principle still holds in the case $|a_{01}| = |a_{11}|$, provided $f(z, \bar{z})$ is not identically zero in U). For the convenience of the reader, in the following table we listed some approximate values of the Riemann zeta function $\zeta(a)$.

a	1.1	1.2	1.3	1.4	1.5	2.0
$\zeta(a)$	10.5844	5.9158	3.93195	3.10555	2.61238	$\frac{\pi^2}{6} \approx 1.64493$

We conclude with a version of the maximum modulus principle for real-valued functions of two real variables having a Taylor series expansion in the whole unit disk, as follows:

Theorem 3. *If the real-valued function of two real variables $f = f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ can be represented by a Taylor series*

$$f(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} C_n^k \frac{\partial^n f}{\partial x^{n-k} \partial y^k}(0,0) x^{n-k} y^k, \quad (x, y) \in U, \tag{10}$$

where for some positive real number $a > 0$ we have:

$$\max_{0\le k\le n} C_n^k \left| \frac{\partial^n f}{\partial x^{n-k} \partial y^k}(0,0) \right| \le \frac{n!}{(n+1)n^{2+a}} \frac{\left| \left| \frac{\partial f}{\partial x}(0,0) \right| - \left| \frac{\partial f}{\partial y}(0,0) \right| \right|}{\zeta(1+a) - 1} \neq 0,$$

for all $n = 2, 3, \dots$, then $|f(x, y)|$ is a radially increasing function in U, that is

$$|f(r_1 \cos\theta, r_1 \sin\theta)| < |f(r_2 \cos\theta, r_2 \sin\theta)|, \tag{11}$$

for any $0 < r_1 < r_2 < 1$ and $\theta \in [0, 2\pi)$.

In particular, $f(x, y)$ cannot attain its maximum modulus at an interior point of U.

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