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QUASICONFORMAL EXTENSIONS AND q-SUBORDINATION CHAINS IN $\mathbb{C}^n$

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Abstract. Let B be the unit ball with respect to Euclidean norm on $\mathbb{C}^n$. In this note we introduce the notion of a q-subordination chain defined on $B \times [0, \infty)$ and we deduce conditions for the first element of a q-subordination chain to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$. The origin $(0, 0, \ldots, 0)$ is denoted by 0 and by $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I denote the identity in $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.

We consider $\mathbb{C}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\| \cdot \|$. By $\mathcal{H}(B)$ we denote the set of function

$$f(z) = (f_1(z), \ldots, f_n(z)), \quad z = (z_1, \ldots, z_n),$$

that are holomorphic in $B = \{z \in \mathbb{C}^n : \ \|z\| < 1\}$ with values in $\mathbb{C}^n$. If $f \in \mathcal{H}(B)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Here $Df(z)$ means the first Fréchet derivative of f at $z \in B$.

We say that $f \in \mathcal{H}(B)$ is locally biholomorphic on B if f has a local holomorphic inverse at each point in B.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g if there is a Schwarz mapping v such that $f(z) = g(v(z))$, $z \in B$. We shall write $f \prec g$ to mean that f is subordinate to g.

Definition 1.1. The mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a normalized Loewner chain (normalized subordination chain) if

(i) $L(\cdot, t)$ is holomorphic and univalent on B, $t \geq 0$;
(ii) $L(0, t) = 0$, $DL(0, t) = \mathrm{e}^t I$, $t \geq 0$;
(iii) $L(\cdot, s) \prec L(\cdot, s)$ for $0 \leq s < t < \infty$;

The subordination condition (iii) is equivalent to the fact that

$$L(z, s) = L(v(z, s, t), t), \quad z \in B, \ 0 \leq s < t < \infty$$

where $v = v(z, s, t)$ is a univalent Schwarz mapping, normalized by $v(0, s, t) = 0$ and $Dv(0, s, t) = \mathrm{e}^{s-t} I$.

The mapping v is called the transition mapping associated to the Loewner chain L.

An important role in our discussion is played by the n-dimensional version of the Carathéodory set

$$\mathcal{M} = \{h \in \mathcal{H}(B) : \ h(0) = 0, \ Dh(0) = I, \ \text{Re}\ \langle h(z), z\rangle \geq 0, \ z \in B\}.$$

Recently in [4] (see also [2] and [5]), the authors proved the following result, which will be used in the next.

THEOREM 1.2. *Let $L : B \times [0,\infty) \to \mathbb{C}^n$ be a normalized Loewner chain. Then $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$, and there exists a set $E \subset (0,\infty)$ of Lebesgue measure zero such that for all $t \in [0,\infty) \setminus E$, there exists $h = h(z,t)$ such that $h(\cdot,t) \in \mathcal{M}$, $h(z,\cdot)$ is Lebesgue measurable on $[0,\infty)$ for each $z \in B$, and*

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t), \quad t \in [0,\infty) \setminus E, \ \forall\ z \in B. \tag{1}$$

DEFINITION 1.3. Let G, G' be domains in $\mathbb{R}^m$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^m$. A homeomorphism $f : \Omega \to \Omega'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(x)\|^m \leq K|\det Df(x)| \text{ a.e. in } \Omega,$$

where $Df(x)$ denotes the (real) Jacobian matrix of f, K is constant and

$$\|Df(x)\| = \sup\{\|Df(x)(a)\| : \ \|a\| = 1\}.$$

In this note we deduce conditions for the first element of a q-subordination chain to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. Other results related to quasiconformal extension of the first element of a Loewner chain were recently obtained by Hamada and Kohr ([7], [8]) and Curt and Kohr [3].

2. MAIN RESULTS

DEFINITION 2.1. Let $L : B \times [0,\infty) \to \mathbb{C}^n$ be a normalized subordination chain and let $q \in [0,1)$.

We say that L is a q-normalized subordination chain if the mapping h defined by Theorem 1.2 satisfies the following conditions:

(i) The following inequalities hold

$$\|z\|^2\frac{1-q\|z\|}{1+q\|z\|} \leq \text{Re}\ \langle h(z,t), z\rangle \leq \|z\|^2\frac{1+q\|z\|}{1-q\|z\|}, \ z \in B, \ \text{a.e. } t \in [0,\infty). \tag{2}$$

(ii) There is $q_1 > 0$ such that

$$\|h(z,t)\| \leq q_1, \ z \in B, \ \text{a.e. } t \in [0,\infty). \tag{3}$$

Next, we shall present some classes of mappings which satisfy the conditions (2) and (3).

Remark 2.2. Let $q \in [0,1)$ and $h : B \times [0,\infty) \to \mathbb{C}^n$ be defined by

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z) \tag{4}$$

where the mapping E satisfies
(i) $E(z,t) \in L(\mathbb{C}^n),\ z \in B,\ t \in [0,\infty)$
(ii) $E(\cdot,t) : B \to \mathcal{L}(\mathbb{C}^n)$ is an holomorphic mapping
(iii) $E(0,t) = 0,\ \|E(z,t)\| \le q < 1.$
Then h satisfies (2) and (3).

Proof. By using the Schwarz lemma (see [9]) we easily obtain

$$\|E(z,t)\| \le q\|z\|, \quad z \in B.$$

The previous inequality and Definition 2.1 imply that

$$\Big| \|h(z,t)\| - \|z\| \Big| \le \|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \tag{5}$$
$$\le q\|z\|(\|h(z,t)\| + \|z\|)$$

and hence

$$\|h(z,t)\| \le \|z\| \frac{1 + q\|z\|}{1 - q\|z\|} < \frac{1+q}{1-q}.$$

We obtain that (3) holds with $q_1 = \frac{1+q}{1-q}$.

The right inequality in (2) is an immediate consequence of the following inequality

$$\|h(z,t)\| \le \|z\| \frac{1 + q\|z\|}{1 - q\|z\|}.$$

In order to prove the left part of (2) we shall first prove that

$$\|z\| \frac{1 - q\|z\|}{1 + q\|z\|} \le \|h(z,t)\|,\ z \in B. \tag{6}$$

From the definition of h we have

$$\|h(z,t) - z\|^2 \le q^2\|z\|^2\|h(z,t) + z\|^2$$

and hence

$$\|h(z,t)\|^2 + \|z\|^2 - 2\mathrm{Re}\ \langle h(z,t), z\rangle \le q^2\|z\|^2(\|h(z,t)\|^2 + \|z\|^2 + 2\mathrm{Re}\ \langle h(z,t), z\rangle).$$

By using the previous two inequalities we obtain that

$$(1 + q^2\|z\|^2)\mathrm{Re}\ \langle h(z,t), z\rangle \ge (1 - q^2\|z\|^2)(\|h(z,t)\|^2 + \|z\|^2)$$
$$\ge \frac{1 - q^2\|z\|^2}{(1 + q\|z\|)^2}(1 + q^2\|z\|^2)\|z\|^2$$

where from the left part (3) is an easily consequence. □

In the next remark we shall present a large class of mappings which satisfy (3).

REMARK 2.3. Let $h : B \times [0, \infty) \to \mathbb{C}^n$ such that
(i) $h(\cdot, t) \in \mathcal{H}(B),\ h(0, t) = 0,\ Dh(0, t) = I,\ t \in [0, \infty)$.
(ii) There exists $q \in [0, 1)$ such that

$$\left| \frac{\langle h(z,t), z\rangle}{\|z\|^2} - \frac{1+q^2}{1-q^2} \right| \leq \frac{2q}{1-q}, \quad z \in B,\ t \in [0, \infty). \tag{7}$$

Then h satisfies the inequality (3).

Proof. Let $z \in B \setminus \{0\}$, $t \geq 0$ and let $p : U \to \mathbb{C}$ be defined by

$$p(\zeta) = \frac{1}{\zeta} \left\langle h\left(\zeta \frac{z}{\|z\|}, t\right), \frac{z}{\|z\|} \right\rangle, \text{ if } \zeta \neq 0$$

and

$$p(0) = \lim_{\zeta \to 0} p(\zeta).$$

Since $p(0) = 1$ and $\left| p(\zeta) - \frac{1+q^2}{1-q^2} \right| \leq \frac{2q}{1-q^2}$ we have $p(\zeta) \prec \frac{1+q\zeta}{1-q\zeta}$, and hence

$$\frac{1-q|\zeta|}{1+q|\zeta|} \leq \operatorname{Re} p(\zeta) \leq \frac{1+q|\zeta|}{1-q|\zeta|}, \quad \zeta \in U.$$

If we take $\zeta = \|z\|$ in the previous inequality we easily obtain that (3) holds. □

We now are able to present our main result.

THEOREM 2.4. *Let $q \in [0, 1)$ and $L : B \times [0, \infty) \to \mathbb{C}^n$ be a normalized q-subordination chain. Assume that the following conditions are satisfied:*
(i) *There exist $M > 0$ and $\alpha \in [0, 1)$ such that*

$$\|DL(z, t)\| \leq \frac{e^t M}{(1 - \|z\|)^\alpha}, \quad z \in B,\ t \in [0, \infty) \tag{8}$$

(ii) *There exists $K > 0$ such that $L(\cdot, t)$ is K-quasiconformal for each $t \geq 0$.*

Further, suppose that there exist a sequence $\{t_m\}_{m \in \mathbb{N}}$, $t_m > 0$, $\lim_{m \to \infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that

$$\lim_{m \to \infty} \frac{L(z, t_m)}{e^{t_m}} = F(z), \tag{9}$$

locally uniformly on B. Then $f(z) = L(z, 0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

The proof is based on several lemmas which will be first presented.

Lemma 2.5 (see [11]) is the n-dimensional version of Hardy's and Littlewood's Theorem [6]. This result will be applied in order to extend to $\overline{B}$ the mappings $L(\cdot, t)$ $(t \geq 0)$ given in Theorem 2.4.

Lemma 2.5. *Suppose that $\alpha \in [0,1]$ and g is a complex valued holomorphic function for $z \in B$ such that*

$$\left|\frac{\partial g(z)}{\partial z_j}\right| \le \frac{M_j}{(1-\|z\|)^\alpha}, \quad j=1,\dots,n,\ z \in B. \tag{10}$$

Then g has a continuous extension to $\overline{B}$ and there is $A>0$ such that

$$|g(z)-g(w)| \le A\|z-w\|^{1-\alpha}, \quad z,w \in \overline{B}. \tag{11}$$

Lemma 2.6. [11] *Let $f \in \mathcal{H}(B)$, $M>0$ and $\alpha \in [0,1)$ be such that*

$$\|Df(z)\| \le \frac{M}{(1-\|z\|)^\alpha}, \quad z \in B. \tag{12}$$

Then f has a continuous extension to $\overline{B}$ (also denoted by f) and there exists $A>0$ such that

$$\|f(z)-f(w)\| \le A\|z-w\|^{1-\alpha}, \quad z,w \in B. \tag{13}$$

Lemma 2.7. *Let $v : B \times [0,\infty)^2 \to \mathbb{C}^n$ be the transition mapping associated to a q-normalized subordination chain. Then the following inequalities hold:*

$$\frac{\mathrm{e}^t\|v(z,s,t)\|}{(1+q\|v(z,s,t)\|)^2} \ge \frac{\mathrm{e}^s\|z\|}{(1+q\|z\|)^2}, \quad z \in B,\ t \ge s, \tag{14}$$

$$\frac{\mathrm{e}^t\|v(z,s,t)\|}{(1-q\|v(z,s,t)\|)^2} \le \frac{\mathrm{e}^s\|z\|}{(1-q\|z\|)^2}, \quad z \in B,\ t \ge s. \tag{15}$$

Also, for all $t \ge s$ we have

$$\overline{v(B,s,t)} \subseteq B. \tag{16}$$

Proof. For all $s \ge 0$ and a.e. $t \ge s$ we have (see [2])

$$\frac{\partial v}{\partial t}(z,s,t) = -h(v(z,s,t)), \quad z \in B$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\| = \frac{1}{\|v(t)\|}\mathrm{Re}\,\left\langle \frac{\mathrm{d}v}{\mathrm{d}t}(t), v(t)\right\rangle.$$

By using the previous inequalities and (2) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\| = -\frac{1}{\|v(t)\|}\mathrm{Re}\,\langle h(v(t),t), v(t)\rangle, \text{ a.e. } t \ge s. \tag{17}$$

$$-\|v(t)\|\frac{1+q\|v(t)\|}{1-q\|v(t)\|} \le \frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\| \le -\|v(t)\|\frac{1-q\|v(t)\|}{1+q\|v(t)\|}, \text{ a.e. } t \ge s.$$

We may integrate the inequality (17) and make a change of variable to obtain (14).

In order to obtain (15) we use the inequality

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\|v(\tau)\|}{\|v(\tau)\|} \leq -\frac{1-q\|v(\tau)\|}{1+q\|v(\tau)\|} \leq -\frac{1-q}{1+q} \text{ a.e. } \tau \in [s,t].$$

We integrate the previous inequality and obtain that

$$\|v(z,s,t)\| \leq \|z\|\mathrm{e}^{-\frac{1-q}{1+q}(t-s)}$$

which shows that (16) holds. □

LEMMA 2.8. *Let $L : B \times [0,\infty) \to \mathbb{C}^n$ be a q-normalized subordination chain and let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence with $t_m > 0$, $\lim_{m\to\infty} t_m = \infty$, $F \in \mathcal{H}(B)$, such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{\mathrm{e}^{t_m}} = F(z)$$

locally uniformly on B. Then the following inequalities hold:

$$\frac{\mathrm{e}^s\|z\|}{(1+q\|z\|)^2} \leq \|L(z,s)\| \leq \frac{\mathrm{e}^s\|z\|}{(1-q\|z\|)^2}, \quad z \in B,\ s \geq 0. \tag{18}$$

Proof. The inequalities (18) are easily consequence of (14), (15) and of the fact that (see [2])

$$L(z,s) = \lim_{t\to\infty} \mathrm{e}^t v(z,s,t)$$

locally uniformly on B. □

LEMMA 2.9. *Let $L : B \times [0,\infty) \to \mathbb{C}^n$ be a q-normalized subordination chain and let $M > 0$ and $\alpha \in [0,1)$ be such that*

$$\|DL(z,t)\| \leq \frac{\mathrm{e}^t M}{(1-\|z\|)^\alpha}, \quad z \in B,\ t \in [0,\infty). \tag{19}$$

Then the following statements hold:

(i) *For each $t \geq 0$ the mapping $L(\cdot,t)$ has a continuous and univalent extension to $\overline{B}$ (also denoted by $L(\cdot,t)$).*

(ii) *There exist $K, L > 0$ such that*

$$\mathrm{e}^{-t}\|L(z,t) - L(w,t)\| \leq K\|z-w\|^{1-\alpha}, \quad z,w \in \overline{B},\ t \geq 0 \tag{20}$$

and

$$\|L(z,t) - L(z,s)\| \leq L\mathrm{e}^t(t-s)^{1-\alpha}, \quad z \in \overline{B},\ 0 \leq s < t. \tag{21}$$

Proof. By using Lemmas 2.5 and 2.6 and the assumption (19) we deduce that the mapping $\mathrm{e}^{-t}L(\cdot,t)$ has a continuous extension to $\overline{B}$ and

$$\mathrm{e}^{-t}\|L(z,t) - L(w,t)\| \leq K\|z-w\|^{1-\alpha}, \quad z,w \in \overline{B},\ t \geq 0.$$

Hence, the condition (21) is fulfilled.

Since $L(z,s) = L(v(z,s,t),t)$ for $0 \le s < t$ and $L(\cdot,s)$ is continuous on $\overline{B}$, by using (16) we have $L(\overline{B},s) \subset L(B,t)$ for $0 \le s < t$. Then $v(z,s,t) = L^{-1}(L(z,s),t)$, $z \in \overline{B}$, defines a continuous extension of v to $\overline{B}$.

For $z \in B$, $t > s \ge 0$, we have

$$\|z - v(z,s,t)\| = \left\| \int_s^t \frac{\partial}{\partial \tau} v(z,s,\tau) \mathrm{d}\tau \right\| \tag{22}$$

$$= \left\| \int_s^t h(v(z,s,\tau),\tau) \mathrm{d}\tau \right\| \le q_1(t-s).$$

Since v is continuous on $\overline{B}$, the previous relation holds for $z \in \overline{B}$. Next, we shall prove that $L(\cdot,s)$ is univalent on $\overline{B}$. Suppose that $L(z_1,s) = L(z_2,s)$, for $z_1, z_2 \in \overline{B}$. Then for $t > s$ we have

$$L(v(z_1,s,t),t) = L(v(z_2,s,t),t).$$

Since $v(z_1,s,t), v(z_2,s,t) \in B$ for $0 \le s < t$ and $L(\cdot,t)$ is univalent on B, we obtain $v(z_1,s,t) = v(z_2,s,t)$. If we let $t \to s$, $v(z_1,s,t) = v(z_2,s,t)$ we obtain that $z_1 = z_2$. Here we also use (22).

From (22) and (19) we easily obtain that

$$\|L(z,s) - L(z,t)\| = \|L(v(z,s,t),t) - L(z,t)\| \le \mathrm{e}^t M \|z - v(z,s,t)\|^{1-\alpha}$$
$$\le \mathrm{e}^t M q_1^{1-\alpha} (t-s)^{1-\alpha}, \quad z \in \overline{B},\ t > s \ge 0,$$

which means that (21) holds with $L = M q_1^{1-\alpha}$. □

We are now able to prove the main result.

Proof of Theorem 2.4. Let

$$F(z) = \begin{cases} L(z,0), & \|z\| \le 1 \\ L\left(\dfrac{z}{\|z\|}, \log\|z\|\right), & \|z\| > 1. \end{cases}$$

First, we will show that F is a homeomorphism of $\mathbb{R}^{2n}$ onto itself. Since for every $t \ge 0$, the mapping $L(\cdot,t)$ is univalent on $\overline{B}$ and for all $0 \le s < t$ we have $L(\overline{B},s) \subseteq L(B,t)$ we obtain that F is univalent on $\mathbb{C}^n$ ($\mathbb{R}^{2n}$).

The continuity in $\mathbb{C}^n$ ($\mathbb{R}^{2n}$) of the extension F follows since (20) and (21) yield that $L(z,t)$ is continuous in $\overline{B} \times [0,\infty)$. The left-hand inequality (18) shows that $F(z) \to \infty$ as $z \to \infty$ and hence that F is a homeomorphism of $\mathbb{R}^{2n}$. It remains to show that F is quasiconformal in $\mathbb{R}^{2n}$. We shall do this by using an approximation argument similar to Becker's [1] and Pfaltzgraff [11].

Let $r > 1$ and let

$$L_r(z,t) = rL\left(\frac{z}{r},t\right), \quad h_r(z,t) = rh\left(\frac{z}{r},t\right), \quad t \ge 0, \tag{23}$$

$$F_r(z) = \begin{cases} L_r(z,0), & \|z\| \le 1 \\ L_r\left(\dfrac{z}{\|z\|}, \log\|z\|\right), & \|z\| \ge 1. \end{cases} \tag{24}$$

Clearly, $L_r(z,t)$ satisfies the differential equation

$$\frac{\partial}{\partial t}L_r(z,t) = DL_r(z,t)h_r(z,t) \text{ a.e. } t\geq 0, \text{ for all } \|z\|<r \tag{25}$$

and hence $\|z\|\leq 1$.

On the other hand, since

$$\begin{aligned}\|L_r(z,t)-L(z,t)\| &\leq \left\|rL\left(\frac{z}{r},t\right)-L\left(\frac{z}{r},t\right)\right\| + \left\|L\left(\frac{z}{r},t\right)-L(z,t)\right\| \\ &\leq (1-r)\left\|L\left(\frac{z}{r},t\right)\right\| + M\mathrm{e}^t\left\|\frac{z}{r}-z\right\|^{1-\alpha} \\ &\leq \frac{\mathrm{e}^{t\frac{\|z\|}{r}}}{\left(1-\frac{q\|z\|}{r}\right)^2}(1-r) + M\mathrm{e}^t\frac{\|z\|^{1-\alpha}}{r^{1-\alpha}}(1-r)^{1-\alpha} \\ &\leq \frac{\mathrm{e}^{\frac{T}{r}}}{\left(1-\frac{q}{r}\right)^2}(1-r) + M\mathrm{e}^T\frac{1}{r^{1-\alpha}}(1-r)^{1-\alpha},\end{aligned}$$

for all $\|z\|\leq 1$, $0<t\leq T$, we deduce that $L_r(z,t)\to L(z,t)$, uniformly in $\|z\|\leq 1$, $0\leq t\leq T$, as r decreases to 1. Hence F_r converges to F uniformly in $\mathbb{R}^{2n}$ as r decreases to 1.

Next, we shall show that F_r (as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$) is ACL, differentiable a.e., and has outer dilatation bounded a.e. by a bound independent of r. Then it will follow [12] that F is quasiconformal.

We show that $\mathrm{e}^{-t}L_r(z,t)$ satisfies a Lipschitz condition on $\overline{B}$ with exponent 1. Indeed, we have

$$\|DL_r(z,t)\| = \left|DL\left(\frac{z}{r},t\right)\right\| \leq \frac{\mathrm{e}^tM}{\left(1-\frac{\|z\|}{r}\right)^\alpha} \leq \frac{\mathrm{e}^tM}{\left(1-\frac{1}{r}\right)^\alpha}$$

and hence

$$\|L_r(z,t)-L_r(w,t)\| \leq \frac{\mathrm{e}^tM}{\left(1-\frac{1}{r}\right)^\alpha}\|z-w\| = \mathrm{e}^tM(r)\|z-w\|, \quad z,w\in\overline{B}. \tag{26}$$

By using (26) and the fact that L is a Loewner chain we get

$$\begin{aligned}\|L_r(z,t)-L_r(z,s)\| &= r\left\|L\left(\frac{z}{r},t\right)-L\left(\frac{z}{r},s\right)\right\| \\ &= r\left\|L\left(\frac{z}{r},t\right)-L\left(v\left(\frac{z}{r},s,t\right),t\right)\right\| \\ &\leq \mathrm{e}^tM(r)r\left|\frac{z}{r}-v\left\|\frac{z}{r},s,t\right)\right\| \\ &\leq \mathrm{e}^tM(r)rq_1(t-s) \\ &= \mathrm{e}^tL(r)(t-s), \quad z\in\overline{B},\ 0\leq s<t.\end{aligned} \tag{27}$$

Next, we will show that F_r satisfies a local Lipschitz condition (with exponent one) on $\mathbb{C}^n$. It is sufficient to prove this condition for $z, w \in \mathbb{C}^n$ with $\|z - w\| < 1$. We prove this condition in the following 3 cases:
i) $z, w \in \overline{B}$;
ii) $z, w \in \mathbb{C}^n \setminus B,\ \|z\| \le \|w\|$ and $\|w - z\| < 1$;
iii) $z \in B,\ w \in \mathbb{C}^n \setminus B$.
i) If $z, w \in \overline{B}$ we obtain by (26) that:

$$\begin{aligned} \|F_r(z) - F_r(w)\| &= \|L_r(z,0) - L_r(w,0)\| \\ &\le M(r)\|z - w\|. \end{aligned} \tag{28}$$

ii) If $z, w \in \mathbb{C}^n$, $\|z\| \le \|w\|$ and $\|w - z\| < 1$ we obtain by (26) and (27) that

$$\begin{aligned} \|F_r(z) - F_r(w)\| &= \left\| L_r\left(\frac{z}{\|z\|}, \log\|z\|\right) - L_r\left(\frac{w}{\|w\|}, \log\|w\|\right) \right\| \\ &\le \left\| L_r\left(\frac{z}{\|z\|}, \log\|z\|\right) - L_r\left(\frac{z}{\|z\|}, \log\|w\|\right) \right\| + \\ &+ \left\| L_r\left(\frac{z}{\|z\|}, \log\|w\|\right) - L_r\left(\frac{w}{\|w\|}, \log\|w\|\right) \right\| \\ &\le M(r)\left\| z - \frac{\|z\|}{\|w\|}w \right\| + \|w\| \log\frac{\|w\|}{\|z\|} L(r) \\ &\le qM(r)\|z - w\| + \frac{\|w\|}{\|z\|}(\|w\| - \|z\|)L(r) \\ &\le 2[M(r) + L(r)]\|w - z\|. \end{aligned} \tag{29}$$

iii) If $z \in B$ and $w \in \mathbb{C}^n \setminus \overline{B}$ then there exists a real number β with $0 < \beta < 1$ such that $u = (1 - \beta)z + \beta z \in \partial B$. By using (28) and (29) we obtain that:

$$\begin{aligned} &\|F_r(z) - F_r(w)\| \le \|F_r(z) - F_r(u)\| + \|F_r(u) - F_r(w)\| \\ &= \|L_r(z,0) - L_r(u,0)\| + \left\| L_r(u,0) - L_r\left(\frac{w}{\|w\|}, \log\|w\|\right) \right\| \\ &\le M(r)\|u - z\| + 2[M(r) + L(r)]\|u - w\| \\ &\le [3M(r) + 2L(r)]\|z - w\|. \end{aligned}$$

Thus, F_r satisfies a local Lipschitz condition. Hence F_r is ACL in $\mathbb{R}^{2n}$ and so is (real) differentiable a.e. in $\mathbb{R}^{2n}$.

It remains to prove that F_r has outer dilatation bounded a.e. by a bound independent of r.

Let $r > 1$ and let $G(z) = F_r(z)$ (in order to simplify notation).

We let $z = (x, y) = (x_1, y_1, \ldots, x_n, y_n)$, $\|z\| \ge 1$, be a point when the mapping $G = (U, V) = (U_1, V_1, U_2, V_2, \ldots, U_n, V_n)$ defined by

$$G((x_1, y_1, \ldots, x_n, y_n)) = (U_1, V_1, \ldots, U_n, V_n)$$

$$U_k = \mathrm{Re}\ G_k(x, y), \quad V_k = \mathrm{Im}\ G_k(x, y), \quad k = 1, \ldots, n,$$

is differentiable.

To compute the (real) derivative of (30) we use the chain rule on the composed mappings.

By denoting $\zeta = \dfrac{z}{r\|z\|}$, $t = \log\|z\|$, $u_k = \text{Re } L_k(\zeta,t)$, $v_k = \text{Im } L_k(\zeta,t)$ we obtain:

$$D(U,V,x,y) = \frac{1}{\|z\|}D(u,v,\xi,\eta)\left\{I + r^2\begin{bmatrix}\text{Re } (h(\zeta,t)-\zeta)\\ \text{Im } (h(\zeta,t)-\zeta)\end{bmatrix}(\xi,\eta)\right\} \tag{30}$$

If we denote by $A = r^2\begin{bmatrix}\text{Re } (h(\zeta,t)-\zeta)\\ \text{Im } (h(\zeta,t)-\zeta)\end{bmatrix}(\xi,\eta)$ by using a similar argument as in [11] we obtain that

$$\det(I+A) \geq \frac{1-q}{1+q}$$

and hence

$$D(U,V;x,y) = \frac{1}{\|z\|}D(u,v,\xi,\eta)[I+A]$$

Also, we have

$$\|D(U,V;x,y)\| \leq \frac{1}{\|z\|}\|DL(\zeta,t)\|\|I+A\|.$$

Since

$$\begin{aligned}\|A\| &\leq r^2\|h(\zeta,t)-\zeta\|\left\|\frac{z}{r\|z\|}\right\| = t\|h(\zeta,t)-\zeta\|\\ &\leq 1 + r\,\|h(\zeta,t)\| \leq 1+q_1\end{aligned}$$

and hence $\|I+A\| \leq 2+q_1$.

By using the previous inequalities and the fact that $L(z,t)$ is a quasiconformal mapping we get

$$\begin{aligned}\|D(U,V;x,y)\|^{2n} &\leq \|z\|^{-2n}\|DL(\zeta,t)\|^{2n}(2+q_1)^{2n}\\ &\leq \frac{1+q}{1-q}(2+q_1)^{2n}|J(U,V;x,y)|.\end{aligned}$$

This inequality completes the proof. □

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