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EXISTENCE, UNIQUENESS AND APPROXIMATION FOR THE SOLUTION OF A SECOND ORDER NEUTRAL DIFFERENTIAL EQUATION WITH DELAY IN BANACH SPACES

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Abstract. In order to obtain the existence, uniqueness and approximation of the solution of the initial value problem, associated to second order neutral differential equation in Banach spaces, Perov's fixed point theorem is used. The associated numerical method use the sequence of successive approximations and a recent trapezoidal type inequality for Lipschitzian functions with values in Banach space.

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1. INTRODUCTION

Consider the initial value problem:

$$(1)\qquad \begin{cases} x''(t) = f(t, x(t), x(g(t)), x'(t), x'(g(t))), & t \in [a, b] \\ x(t) = \varphi(t), & t \in [a_1, a] \end{cases}$$

with

$$\varphi \in C^1([a_1, a], X),\ \ g \in C([a, b], [a_1, b])$$

given, where X is a Banach space.

Suppose that $f \in C([a, b] \times X \times X \times X \times X, X)$ and $a_1 \le g(t) \le t, \forall t \in [a, b]$.

The initial value problem (1) is equivalent in $C^1([a_1, b], X) \cap C^2([a, b], X)$ with the neutral Volterra integro-differential equation:

$$(2)\qquad \begin{cases} x(t) = \varphi(a) + \varphi'(a)(t - a) + \\ \qquad + \int_a^t (t - s) f(s, x(s), x(g(s)), x'(s), x'(g(s)))\, \mathrm{d}s, \quad t \in [a, b] \\ x(t) = \varphi(t), \quad t \in [a_1, a]. \end{cases}$$

Indeed, for

$$x \in C^1([a_1, b], X) \cap C^2([a, b], X)$$

a solution of (1), integrating on $[a, t]$ for any $t \in [a, b]$, we obtain

$$x'(t) - \varphi'(a) = \int_a^t f\big(s, x(s), x(g(s)), x'(s), x'(g(s))\big)\, \mathrm{d}s, \quad \forall t \in [a, b].$$

Integrating again, follows

$$x(t)=\varphi(a)+(t-a)\varphi'(a)+\int\limits_a^t\left(\int\limits_a^s f\left(v,x(v),x\left(g(v)\right),x'(v),x'\left(g(v)\right)\right)\mathrm{d}v\right)\mathrm{d}s.$$

Computing by parts the above integral, we obtain

$$\begin{aligned}x(t) &= \varphi(a)+(t-a)\varphi'(a)+\\ &\quad+\int\limits_a^t (t-s)f\left(s,x(s),x\left(g\left(s\right)\right),x'\left(s\right),x'\left(g(s)\right)\right)\mathrm{d}s,\quad \forall t\in[a,b],\end{aligned}$$

and then x is solution of (2).

Let

$$x\in C^1\left([a_1,b],X\right)\cap C^2\left([a,b],X\right)$$

be a solution of (2). Then, since $f\in C\left([a,b]\times X\times X\times X\times X,X\right)$, deriving by t, it follows:

$$x'\left(t\right)=\varphi'\left(a\right)+\int\limits_a^t f\left(s,x\left(s\right),x\left(g\left(s\right)\right),x'\left(s\right),x'\left(g\left(s\right)\right)\right)\mathrm{d}s,\ \forall t\in[a,b].$$

Deriving again,

$$x''\left(t\right)=f\left(t,x\left(t\right),x\left(g\left(t\right)\right),x'\left(t\right),x'\left(g\left(t\right)\right)\right),\ \forall t\in[a,b],$$

hence x is solution of (1).

Since the solution of (2) $x\in C^1\left([a_1,b],X\right)$ and $C^1\left([a_1,b],X\right)$ is not complete with the metric generated by the Chebyshev's norm, we will derive equation (2) with respect to t and we will use the Perov's fixed point theorem (see [1], [3], [7], [10], [11], [12], [13]) on $C\left([a_1,b],X\right)$ to the pair (x,x'). In this way, since

$$x'\left(t\right)=\varphi'\left(a\right)+\int\limits_a^t f\left(s,x\left(s\right),x\left(g\left(s\right)\right),x'\left(s\right),x'\left(g\left(s\right)\right)\right)\mathrm{d}s,\ \forall t\in[a,b]$$

and

$$\varphi\in C^1\left([a,b],X\right)$$

the pair $(x,x')\in C\left([a_1,b],X\right)\times C\left([a_1,b],X\right)$ is solution of the system:

$$(3)\qquad\begin{cases} x(t)=\varphi(a)+\varphi'(a)(t-a)\\ \qquad +\int\limits_a^t (t-s)\,f\left(s,x\left(s\right),x\left(g\left(s\right)\right),y\left(s\right),y\left(g\left(s\right)\right)\right)\mathrm{d}s\\ y\left(t\right)=\varphi'\left(a\right)+\int\limits_a^t f\left(s,x\left(s\right),x\left(g\left(s\right)\right),y\left(s\right),y\left(g\left(s\right)\right)\right)\mathrm{d}s,\quad t\in[a,b]\\ x\left(t\right)=\varphi\left(t\right),\ y\left(t\right)=\varphi'\left(t\right),\ t\in[a_1,a].\end{cases}$$

Conversely, if $(x, y) \in C([a_1, b], X) \times C([a_1, b], X)$ is solution of (3), then $x \in C^1([a_1, b], X)$ is the solution of (2) and $y = x'$.

For the boundary value problems associated to neutral second order differential equations with deviating argument was obtained results on existence, existence and uniqueness of the solution in [8]. Numerical method using spline approximation for second order neutral delay differential equations can be found in [2] and [9].

We have used the Perov's fixed point theorem for a first order neutral delay differential equation in [4]. The Perov's fixed point theorem was also used for two point boundary value problems associated to second order differential equations (see [3]) and in the study of smooth dependence by parameters of the solution of integral equations (see [12] and [13]).

In this paper, using Perov's fixed point theorem we will obtain existence, uniqueness and approximation of the solution in $C([a_1, b], X) \times C([a_1, b], X)$. Afterward, we construct a numerical method for (3) using the sequence of successive approximations and a recent error estimation in the trapezoidal quadrature rule for Lipschitzian functions on Banach spaces from [6].

2. EXISTENCE AND UNIQUENESS

Consider the product functional space

$$Y = C([a_1, b], X) \times C([a_1, b], X)$$

and define the metric

$$d : Y \times Y \longrightarrow \mathbb{R}^2,$$

by

$$d((u_1, v_1), (u_2, v_2)) = (\|u_1 - u_2\|_B, \|v_1 - v_2\|_B),$$

where,

$$\|u\|_B = \max\left\{\|u(t)\|_X \mathrm{e}^{-\theta(t-a_1)} : t \in [a_1, b]\right\}.$$

It is easy to see that (Y, d) is complete metric space.

We impose the conditions:

(i) $f \in C([a, b] \times X \times X \times X \times X, X)$, $\varphi \in C^1([a, b], X)$;

(ii) $g \in C([a, b], [a_1, b])$ and $a_1 \le g(t) \le t$, $\forall t \in [a, b]$;

(iii) there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that for every $s \in [a, b]$ and every $z_1, z_2, u_1, u_2, v_1, v_2, w_1, w_2 \in X$ the following inequality holds

$$\begin{aligned} &\|f(s, z_1, u_1, v_1, w_1) - f(s, z_2, u_2, v_2, w_2)\|_X \\ &\le \alpha_1 \|z_1 - z_2\|_X + \alpha_2 \|u_1 - u_2\|_X + \beta_1 \|v_1 - v_2\|_X + \beta_2 \|w_1 - w_2\|_X, \end{aligned} \tag{4}$$

(iv) $\exists M > 0$ such that $\|f(s, z, u, v, w)\|_X \le M$, $\forall s \in [a, b]$, $\forall z, u, v, w \in X$.

We will use Perov's fixed point theorem:

Theorem 1. (Perov, see [13], [7]) *Let* (X,d) *a complete generalized metric space (i.e.,* $d(x,y)\in\mathbb{R}^n$ *for* $x,y\in X$*). Let* $A:X\longrightarrow X$ *be a function with the property*

$$d(A(x),A(y))\le Qd(x,y),\ \forall x,y\in X,$$

where $Q\in M_n(\mathbb{R}_+)$. *If all the eigenvalues of* Q *lies in the open unit disc from* $\mathbb{R}^2$*, then the operator* A *has an unique fixed point* $x^*\in X$*. Moreover, the sequence of successive approximations* $x_m=A(x_{m-1})$ *converges to* x^* *in* X *for any* $x_0\in X$ *and the following estimation holds:*

$$d(x_m,x^*)\le Q^m(I_n-Q)^{-1}d(x_0,x_1),\ \forall m\in\mathbb{N}^*, \tag{5}$$

where I_n *is the unit matrix in* $M_n(\mathbb{R})$.

We define the operator

$$A:Y\longrightarrow Y,\ A=(A_1,A_2)$$

by

$$\begin{cases} A_1(x,y)(t)=\varphi(t), & t\in[a_1,a]\\ A_2(x,y)(t)=\varphi'(t), & t\in[a_1,a],\end{cases} \tag{6}$$

where

$$\begin{aligned}A_1(x,y)(t)&=\varphi(a)+\varphi'(a)(t-a)+\\&+\int_a^t(t-s)f(s,x(s),x(g(s)),y(s),y(g(s)))\,\mathrm{d}s,\ t\in[a,b],\end{aligned} \tag{7}$$

and

$$\begin{aligned}A_2(x,y)(t)&=\varphi'(a)+\\&+\int_a^t f(s,x(s),x(g(s)),y(s),y(g(s)))\,\mathrm{d}s,\ t\in[a,b].\end{aligned} \tag{8}$$

Theorem 2. *Under the conditions* (i), (ii), (iii) *the operator* A *has an unique fixed point* $(x^*,y^*)\in Y$ *such that*

$$x^*\in C^1([a,b],X),\ y^*=(x^*)'$$

and x^* *is the unique solution of the equation* (1). *Moreover, the sequence of successive approximations given by*

$$x_0(t)=\begin{cases}\varphi(t), & t\in[a_1,a]\\ \varphi(a)+\varphi'(a)(t-a), & t\in[a,b],\end{cases} \tag{9}$$

$$y_0(t)=\begin{cases}\varphi'(t), & t\in[a_1,a]\\ \varphi'(a), & t\in[a,b],\end{cases} \tag{10}$$

and, for $t\in[a,b]$,

$$\begin{aligned}x_m(t)&=\varphi(a)+\varphi'(a)(t-a)+\\&+\int_a^t(t-s)f(s,x_{m-1}(s),x_{m-1}(g(s)),y_{m-1}(s),y_{m-1}(g(s)))\,\mathrm{d}s,\end{aligned} \tag{11}$$

$$y_m(t)=\varphi'(a)+\int_a^t f\left(s,x_{m-1}(s),x_{m-1}\left(g(s)\right),y_{m-1}(s),y_{m-1}\left(g(s)\right)\right)\mathrm{d}s, \tag{12}$$

$$(x_m(t),y_m(t))=\left(\varphi(t),\varphi'(t)\right),\ \ t\in[a_1,a]\,,\ m\in\mathbb{N}^*, \tag{13}$$

converges in Y *to* (x^*,y^*) *and the following error estimation holds:*

$$d\left((x_m,y_m),(x^*,y^*)\right)\le Q^m\left(I_n-Q\right)^{-1}d\left((x_0,y_0),(x_1,y_1)\right), \tag{14}$$

$\forall m\in\mathbb{N}^*$, *where*

$$Q=\begin{pmatrix}\frac{b-a}{\theta}\left(\alpha_1+\alpha_2\right) & \frac{b-a}{\theta}\left(\beta_1+\beta_2\right)\\ \frac{\alpha_1+\alpha_2}{\theta} & \frac{\beta_1+\beta_2}{\theta}\end{pmatrix}.$$

Proof. From (i) and (ii) we infer that

$$A\left(Y\right)\subset Y.$$

In a classical way, using (iii) we obtain, for all $t\in[a,b]$,

$$\begin{aligned}&\|A_1(u_1,v_1)(t)-A_1(u_2,v_2)(t)\|_X\\&\le(b-a)\left[\frac{\alpha_1}{\theta}\|u_1-u_2\|_B\int_a^t\theta\mathrm{e}^{\theta(s-a_1)}\mathrm{d}s+\frac{\alpha_2}{\theta}\|u_1-u_2\|_B\int_a^t\theta\mathrm{e}^{\theta(g(s)-a_1)}\mathrm{d}s\right.\\&\left.+\frac{\beta_1}{\theta}\|v_1-v_2\|_B\int_a^t\theta\mathrm{e}^{\theta(s-a_1)}\mathrm{d}s+\frac{\beta_2}{\theta}\|v_1-v_2\|_B\int_a^t\theta\mathrm{e}^{\theta(g(s)-a_1)}\mathrm{d}s\right].\end{aligned}$$

Since $a_1\le g\left(t\right)\le t$, $\forall t\in[a,b]$, from the above, we infer that

$$\begin{aligned}&\left\|A_1\left(u_1,v_1\right)(t)-A_1\left(u_2,v_2\right)(t)\right\|_X\mathrm{e}^{-\theta(t-a_1)}\\&\le(b-a)\left[\frac{\alpha_1+\alpha_2}{\theta}\left\|u_1-u_2\right\|_B+\frac{\beta_1+\beta_2}{\theta}\left\|v_1-v_2\right\|_B\right],\ \ \forall t\in[a,b]\,.\end{aligned}$$

Analogous, we obtain

$$\begin{aligned}&\left\|A_2\left(u_1,v_1\right)(t)-A_2\left(u_2,v_2\right)(t)\right\|_X\mathrm{e}^{-\theta(t-a_1)}\\&\le\frac{\alpha_1+\alpha_2}{\theta}\left\|u_1-u_2\right\|_B+\frac{\beta_1+\beta_2}{\theta}\left\|v_1-v_2\right\|_B,\ \ \forall t\in[a,b]\,.\end{aligned}$$

Consequently,

$$d\left(A\left(u_1,v_1\right),A\left(u_2,v_2\right)\right)\le Qd\left(\left(u_1,v_1\right),\left(u_2,v_2\right)\right),\ \ \forall\left(u_1,v_1\right),\left(u_2,v_2\right)\in Y,$$

where

$$Q=\begin{pmatrix}\frac{b-a}{\theta}\left(\alpha_1+\alpha_2\right) & \frac{b-a}{\theta}\left(\beta_1+\beta_2\right)\\ \frac{\alpha_1+\alpha_2}{\theta} & \frac{\beta_1+\beta_2}{\theta}\end{pmatrix}.$$

Taking

$$\theta>(b-a)\left(\alpha_1+\alpha_2\right)+\beta_1+\beta_2,$$

we have $0 < \lambda_1, \lambda_2 < 1$ and with the corresponding metric d, A is a Q-contraction, that is

$$Q^m \longrightarrow 0, \text{ for } m \longrightarrow \infty.$$

Using Perov's fixed point theorem, the map A has an unique fixed point $(x^*, y^*) \in Y$, and the estimation (14) follows. Using the conditions (i) and (ii), after elementary calculus, it follows

$$x^* \in C^1\left([a_1, b], X\right) \cap C^2([a, b], X), \quad y^* = (x^*)'$$

and x^* is the unique solution of (1). □

3. THE NUMERICAL METHOD

Suppose that

$$g(t) = t - \tau, \ \forall t \in [a, b] \text{ for } \tau > 0, \ a_1 = a - \tau$$

and there exist

$$l \in \mathbb{N}^* \text{ such that } b - a = l\tau.$$

To compute the terms of the sequence of successive approximations we will use in the calculus of integrals from (11), (12) the trapezoidal quadrature rule for Lipschitzian functions with values in Banach spaces, obtained in ([6]):

$$\text{(c1)} \qquad \int_a^b F(x)\,\mathrm{d}x = \frac{b-a}{2n}\left[F(a) + 2\sum_{i=1}^{n-1} F\left(a + \frac{i(b-a)}{n}\right) + F(b)\right] + R_n(F),$$

with

$$\text{(c2)} \qquad \left\|R_n(F)\right\|_X \leq \frac{L(b-a)^2}{4n},$$

where L is the Lipschitz constant of $F : [a, b] \longrightarrow X$.

Consider the uniform partitions $\Delta' \in Div\,[a - \tau, a]$,

$$\Delta' : a - \tau = t_0 < t_1 < \cdots < t_{n-1} < t_n = a,$$

with

$$t_i = t_0 + \frac{\tau i}{n}, \ \forall i = \overline{1, n},$$

and $\Delta'' \in Div\,[a, b]$,

$$\Delta'' : a = t_n < t_{n+1} < \cdots < t_{q-1} < t_q = b,$$

$$t_{i+1} - t_i = \frac{\tau}{n}, \ \forall i = \overline{n, q-1}.$$

Let $\Delta = \Delta' \cup \Delta''$. Then $\Delta \in Div\,[a - \tau, b]$,

$$\text{(c3)} \qquad \Delta : a - \tau = t_0 < t_1 < \cdots < t_{q-1} < t_q = b.$$

On the knots of the partition Δ'' we have for every $i \in \{1, \ldots, n\}$ and every $m \in \mathbb{N}^*$

$$\begin{aligned} x_m(t_i) &= \varphi(a) + \varphi'(a)(t_i - a) \\ &+ \int_a^{t_i} (t_i - s) f(s, x_{m-1}(s), x_{m-1}(s-\tau), y_{m-1}(s), y_{m-1}(s-\tau)) \, ds, \end{aligned} \tag{15}$$

$$\begin{aligned} y_m(t_i) &= \varphi'(a) + \\ &+ \int_a^{t_i} f(s, x_{m-1}(s), x_{m-1}(s-\tau), y_{m-1}(s), y_{m-1}(s-\tau)) \, ds. \end{aligned} \tag{16}$$

Applying the quadrature rule (c1)+(c2) to (15) and (16) we obtain the following numerical method:

$$x_m(t_i) = \varphi(t_i), \ y_m(t_i) = \varphi'(t_i), \ \forall i = \overline{0, n}, \ \forall m \in \mathbb{N}, \tag{17}$$

$$x_0(t_i) = \begin{cases} \varphi(t_i), & \forall i = \overline{0, n} \\ \varphi(a) + \varphi'(a)(t_i - a), & \forall i = \overline{n+1, q}, \end{cases} \tag{18}$$

$$y_0(t_i) = \begin{cases} \varphi'(t_i), & \forall i = \overline{0, n} \\ \varphi'(a), & \forall i = \overline{n+1, q}, \end{cases} \tag{19}$$

and for every $m \in \mathbb{N}^*$ and every $i \in \{n+1, \ldots, q\}$

$$\begin{aligned} & x_m(t_i) = \varphi(a) + \varphi'(a)(t_i - a) + \tfrac{b-a}{2nl}[(t_i - a) \cdot \\ & f(a, x_{m-1}(a), x_{m-1}(a-\tau), y_{m-1}(a), y_{m-1}(a-\tau)) + 2 \sum_{j=n+1}^{i-1} (t_i - t_j) \cdot \\ & f(t_j, x_{m-1}(t_j), x_{m-1}(t_j - \tau), y_{m-1}(t_j), y_{m-1}(t_j - \tau))] \\ & + R_{m,i}, \end{aligned} \tag{20}$$

$$\begin{aligned} y_m(t_i) &= \varphi'(a) \\ &+ \tfrac{b-a}{2nl}[f(a, x_{m-1}(a), x_{m-1}(a-\tau), y_{m-1}(a), y_{m-1}(a-\tau)) \\ &+ 2 \sum_{j=n+1}^{i-1} f(t_j, x_{m-1}(t_j), x_{m-1}(t_j - \tau), y_{m-1}(t_j), y_{m-1}(t_j - \tau)) \\ &+ f(t_i, x_{m-1}(t_i), x_{m-1}(t_i - \tau), y_{m-1}(t_i), y_{m-1}(t_i - \tau))] + \omega_{m,i}. \end{aligned} \tag{21}$$

4. LIPSCHITZ PROPERTIES

We define the functions:

$$F_{m,i}, G_m : [a, b] \longrightarrow X, \ m \in \mathbb{N}, \ i = \overline{n, q},$$

$$\begin{aligned} F_{m,i} &= (t_i - s) f(s, x_m(s), x_m(s-\tau), y_m(s), y_m(s-\tau)), \\ G_m(s) &= f(s, x_m(s), x_m(s-\tau), y_m(s), y_m(s-\tau)), \ \forall s \in [a, b]. \end{aligned} \tag{22}$$

Definition 1. (see [5]) A set $Z \subset C([a, b], X)$ is equally Lipschitz if there exists $L \geq 0$ such that $\forall h \in Z$,

$$\left\| h(t) - h(t') \right\|_X \leq L \left| t - t' \right|, \ \forall t, t' \in [a, b].$$

We impose the conditions: there exist $\gamma > 0,\ \ \delta > 0$ such that for every $s \in [a,b]$ and every $z,u,v,w \in X$

$$\left\| f\left(s_1,z,u,v,w\right) - f\left(s_2,z,u,v,w\right)\right\|_X \leq \gamma\left|s_1 - s_2\right|, \tag{23}$$

$$\left\|\varphi\left(t\right) - \varphi\left(t'\right)\right\|_X \leq \delta\left|t - t'\right|,\ \forall t,t' \in [a-\tau,a]. \tag{24}$$

Theorem 3. *Under the conditions* (i), (ii), (iii), (iv), (23) *and* (24) *the subsets*

$$\begin{aligned} \left\{F_{m,i}\right\}_{m\in\mathbb{N}\ i=\overline{n,q}} &\subset C\left(\left[a,b\right],X\right), \\ \left\{G_m\right\}_{m\in\mathbb{N}} &\subset C\left(\left[a,b\right],X\right) \end{aligned}$$

defined in (22) *are equally Lipschitz.*

Proof. For $s_1, s_2 \in [a,b]$, we have for $m = 0$ and for all $i \in \{n,\ldots,q\}$

$$\begin{aligned} &\left\|F_{0,i}\left(s_1\right) - F_{0,i}\left(s_2\right)\right\|_X \\ &\leq M\left|s_1 - s_2\right| + (b-a)\left[\gamma\left|s_1 - s_2\right| + \alpha_1\left\|x_0\left(s_1\right) - x_0\left(s_2\right)\right\|_X\right. \\ &+\alpha_2\left\|x_0\left(s_1-\tau\right) - x_0\left(s_2-\tau\right)\right\|_X + \beta_1\left\|y_0\left(s_1\right) - y_0\left(s_2\right)\right\|_X \\ &\left.+\beta_2\left\|y_0\left(s_1-\tau\right) - y_0\left(s_2-\tau\right)\right\|_X\right] \\ &\leq \left[M + (b-a)\left(\gamma + \left\|\varphi'\right\|_C\left(\alpha_1+\alpha_2\right) + \delta\left(\beta_1+\beta_2\right)\right)\right]\left|s_1 - s_2\right|, \end{aligned} \tag{25}$$

and

$$\left\|G_0\left(s_1\right) - G_0\left(s_2\right)\right\|_X \leq \left[\gamma + \left\|\varphi'\right\|_C\left(\alpha_1+\alpha_2\right) + \delta\left(\beta_1+\beta_2\right)\right]\left|s_1 - s_2\right|. \tag{26}$$

For $m \in \mathbb{N}^*$ we obtain

$$\begin{aligned} &\left\|F_{m,i}\left(s_1\right) - F_{m,i}\left(s_2\right)\right\|_X \\ &\quad\leq M\left|s_1 - s_2\right| + (b-a)\left[\gamma\left|s_1 - s_2\right|\right. \\ &\quad+ \alpha_1\left\|x_m\left(s_1\right) - x_m\left(s_2\right)\right\|_X + \alpha_2\left\|x_m\left(s_1-\tau\right) - x_m\left(s_2-\tau\right)\right\|_X \\ &\quad\left.+ \beta_1\left\|y_m\left(s_1\right) - y_m\left(s_2\right)\right\|_X + \beta_2\left\|y_m\left(s_1-\tau\right) - y_m\left(s_2-\tau\right)\right\|_X\right] \\ &\quad\leq M\left|s_1 - s_2\right| + (b-a)\left[\gamma\left|s_1 - s_2\right|\right. \\ &\quad+ \alpha_1\left(\left\|\varphi'\left(a\right)\right\|_X\left|s_1 - s_2\right| + 2M\left(b-a\right)\left|s_1 - s_2\right|\right) \\ &\quad+ \alpha_2\max\left\{\left\|\varphi'\right\|_C, \left\|\varphi'\left(a\right)\right\|_X + 2M\left(b-a\right)\right\}\left|s_1 - s_2\right| \\ &\quad\left.+ \beta_1 M\left|s_1 - s_2\right| + \beta_2\max\left\{M,\delta\right\}\left|s_1 - s_2\right|\right] \\ &\quad\leq \left[M + (b-a)\left(\gamma + \alpha_1\left(\left\|\varphi'\left(a\right)\right\|_X + 2M\left(b-a\right)\right)\right.\right. \\ &\quad+ \alpha_2\max\left\{\left\|\varphi'\right\|_C, \left\|\varphi'\left(a\right)\right\|_X + 2M\left(b-a\right)\right\} \\ &\quad\left.\left.+ \beta_1 M + \beta_2\max\left\{M,\delta\right\}\right)\right]\left|s_1 - s_2\right|,\ \forall s_1,s_2 \in [a,b],\ \forall i = \overline{n,q}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\left\|G_m\left(s_1\right) - G_m\left(s_2\right)\right\|_X \leq \gamma\left|s_1 - s_2\right| + \alpha_1\left\|x_m\left(s_1\right) - x_m\left(s_2\right)\right\|_X \\ &\quad+\alpha_2\left\|x_m\left(s_1-\tau\right) - x_m\left(s_2-\tau\right)\right\|_X + \beta_1\left\|y_m\left(s_1\right) - y_m\left(s_2\right)\right\|_X \\ &\quad+\beta_2\left\|y_m\left(s_1-\tau\right) - y_m\left(s_2-\tau\right)\right\|_X \\ &\quad\leq \left[\gamma + \alpha_1\left(\left\|\varphi'\left(a\right)\right\|_X + 2M\left(b-a\right)\right)\right. \\ &\quad+\alpha_2\max\left\{\left\|\varphi'\right\|_C, \left\|\varphi'\left(a\right)\right\|_X + 2M\left(b-a\right)\right\} \\ &\quad\left.+\beta_1 M + \beta_2\max\left\{M,\delta\right\}\right]\left|s_1 - s_2\right|,\ \forall s_1,s_2 \in [a,b]. \end{aligned} \tag{28}$$

Let

$$L_1 = \max\{M + (b-a)\left[\gamma + \left\|\varphi'\right\|_C(\alpha_1+\alpha_2) + \delta(\beta_1+\beta_2)\right],$$
$$M + (b-a)\left[\gamma + \alpha_1\left(\left\|\varphi'(a)\right\|_X + 2M(b-a)\right) + \alpha_2 \max\{\left\|\varphi'\right\|_C,\right.$$
$$\left.\left\|\varphi'(a)\right\|_X + 2M(b-a)\} + \beta_1 M + \beta_2 \max\{M,\delta\}\right]\}$$

and

$$\begin{aligned} L_2 \;&= \max\{\gamma + \|\varphi'\|_C(\alpha_1+\alpha_2) + \delta(\beta_1+\beta_2),\\ &\gamma + \alpha_1(\|\varphi'(a)\|_X + 2M(b-a))\\ &+\alpha_2\max\{\|\varphi'\|_C, \|\varphi'(a)\|_X + 2M(b-a)\}\\ &+\beta_1 M + \beta_2\max\{M,\delta\}\}.\end{aligned}$$

Then, from (25), (26), (27), (28) we infer that

$$\|F_{m,i}(s_1) - F_{m,i}(s_2)\|_X \le L_1|s_1-s_2|,\ \forall s_1,s_2\in[a,b], \forall i=\overline{n,q},\ \forall m\in\mathbb{N},$$

and

$$\|G_m(s_1) - G_m(s_2)\|_X \le L_2|s_1-s_2|,\ \forall s_1,s_2\in[a,b],\ \forall m\in\mathbb{N},$$

from which we conclude the equally Lipschitz property. □

COROLLARY 1. *In the conditions of Theorem* 3*, the second derivative of the solution of* (1) *is Lipschitzian with the constant* L_2.

Proof. From inequality (28) follows,

$$\|G_m(s_1) - G_m(s_2)\|_X \le L_2|s_1-s_2|,\ \forall s_1,s_2\in[a,b],\ \forall m\in\mathbb{N}^*.$$

That is,

$$\begin{aligned}&\|f(s_1, x_m(s_1), x_m(s_1-\tau), y_m(s_1), y_m(s_1-\tau))\\ &\quad - f(s_2, x_m(s_2), x_m(s_2-\tau), y_m(s_2), y_m(s_2-\tau))\|_X \le L_2|s_1-s_2|.\end{aligned}$$

By Theorem 2

$$x_m \xrightarrow{\text{unif}} x^* \text{ and } y_m \xrightarrow{\text{unif}} y^*,$$

and since f is continuous, after passing to limit for $m \longrightarrow \infty$, we obtain:

$$\begin{aligned}&\|f(s_1, x^*(s_1), x^*(s_1-\tau), y^*(s_1), y^*(s_1-\tau))\\ &\quad - f(s_2, x^*(s_2), x^*(s_2-\tau), y^*(s_2), y^*(s_2-\tau))\|_X \le L_2|s_1-s_2|.\end{aligned}$$

Because $y^* = (x^*)'$, from the above follows

$$\begin{aligned}&\left\|f\left(s_1, x^*(s_1), x^*(s_1-\tau), (x^*)'(s_1), (x^*)'(s_1-\tau)\right)\right.\\ &\quad \left.- f\left(s_2, x^*(s_2), x^*(s_2-\tau), (x^*)'(s_2), (x^*)'(s_2-\tau)\right)\right\|_X \le L_2|s_1-s_2|,\end{aligned}$$

that is,

$$\left\|(x^*)''(s_1) - (x^*)''(s_2)\right\|_X \le L_2|s_1-s_2|,\ \forall s_1,s_2\in[a,b].$$

□

5. THE ALGORITHM

In (20) and (21) the remainder estimations are

$$\|R_{m,i}\|_X \leq \frac{L_1\,(b-a)^2}{4nl},\ \forall i=\overline{n+1,q},\ \forall m\in\mathbb{N}^*, \tag{29}$$

$$\|\omega_{m,i}\|_X \leq \frac{L_2\,(b-a)^2}{4nl},\ \forall i=\overline{n+1,q},\ \forall m\in\mathbb{N}^*. \tag{30}$$

The relations (17)–(21) lead to the following algorithm:

$$x_m\,(t_i)=\varphi\,(t_i)\,,\ y_m\,(t_i)=\varphi'\,(t_i)\,,\ \forall i=\overline{0,n},\ \forall m\in\mathbb{N},$$

$$x_0\,(t_i)=\begin{cases}\varphi(t_i), & \forall i=\overline{0,n}\\ \varphi(a)+\varphi'(a)(t_i-a), & \forall i=\overline{n+1,q},\end{cases}$$

$$y_0(t_i)=\begin{cases}\varphi'(t_i), & \forall i=\overline{0,n}\\ \varphi'(a), & \forall i=\overline{n+1,q},\end{cases}$$

$$\begin{aligned} x_1(t_i) &= \varphi\,(a)+\varphi'\,(a)\,(t_i-a)\\ &+ \tfrac{b-a}{2nl}\big[\,(t_i-a)f\,(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau))\\ &+ 2\sum_{j=n+1}^{i-1}(t_i-t_j)f\,(t_j,x_0(t_j),x_0(t_{j-n}),y_0(t_j),y_0(t_{j-n}))]\\ &+ R_{1,i}=\\ &= \overline{x}_1(t_i)+R_{1,i},\ \forall i=\overline{n+1,q}, \end{aligned} \tag{31}$$

$$\begin{aligned} y_1(t_i) &= \varphi'(a)+\tfrac{b-a}{2nl}\,[f\,(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau))\\ &+ 2\sum_{j=n+1}^{i-1}f\,(t_j,x_0(t_j),x_0(t_{j-n}),y_0(t_j),y_0(t_{j-n}))\\ &+ f\,(t_i,x_0(t_i),x_0(t_{i-n}),y_0(t_i),y_0(t_{i-n}))]+\omega_{1,i}\\ &= \overline{y}_1(t_i)+\omega_{1,i},\ \ \forall i=\overline{n+1,q}, \end{aligned} \tag{32}$$

$$
\begin{aligned}
x_2(t_i) &= \varphi(a)+\varphi'(a)(t_i-a)\\
&+ \tfrac{b-a}{2nl}\Big[\,(t_i-a)f\left(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)\right)\\
&+ 2\sum_{j=n+1}^{i-1}(t_i-t_j)f\left(t_j,\overline{x}_1(t_j)+R_{1,j},\overline{x}_1(t_{j-n})\right.\\
&+ \left.R_{1,j-n},\overline{y}_1(t_j)+\omega_{1,j},\overline{y}_1\left(t_{j-n}\right)+\omega_{1,j-n}\right)\Big]+R_{2,i}\\
&= \varphi(a)+\varphi'(a)(t_i-a)\\
&+ \tfrac{b-a}{2nl}\Big[\,(t_i-a)f\left(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)\right)\\
&+ 2\sum_{j=n+1}^{i-1}(t_i-t_j)f\left(t_j,\overline{x}_1(t_j),\overline{x}_1(t_{j-n}),\overline{y}_1(t_j),\overline{y}_1(t_{j-n})\right)\Big]\\
&+ \overline{R}_{2,i}\\
&= \overline{x}_2(t_i)+\overline{R}_{2,i},\quad \forall i=\overline{n+1,q},
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
y_2(t_i) &= \varphi'(a)+\tfrac{b-a}{2nl}\Big[\,f\left(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)\right)\\
&+ 2\sum_{j=n+1}^{i-1}f\left(t_j,\overline{x}_1(t_j)+R_{1,j},\overline{x}_1(t_{j-n})+R_{1,j-n},\overline{y}_1(t_j)\right.\\
&+ \left.\omega_{1,j},\overline{y}_1(t_{j-n})+\omega_{1,j-n}\right)+f\left(t_i,\overline{x}_1(t_i)+R_{1,i},\overline{x}_1(t_{i-n})\right.\\
&+ \left.R_{1,i-n},\overline{y}_1(t_i)+\omega_{1,i},\overline{y}_1(t_{i-n})+\omega_{1,i-n}\right)\Big]+\omega_{2,i}\\
&= \varphi'(a)+\tfrac{b-a}{2nl}\Big[\,f\left(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)\right)\\
&+ 2\sum_{j=n+1}^{i-1}f\left(t_j,\overline{x}_1(t_j),\overline{x}_1(t_{j-n}),\overline{y}_1(t_j),\overline{y}_1(t_{j-n})\right)\\
&+ f\left(t_i,\overline{x}_1(t_i),\overline{x}_1(t_{i-n}),\overline{y}_1(t_i),\overline{y}_1(t_{i-n})\right)\Big]+\overline{\omega}_{2,i}\\
&= \overline{y}_2(t_i)+\overline{\omega}_{2,i}.
\end{aligned}
\tag{34}
$$

For $m\geq 3$, we obtain by induction:

$$
\begin{aligned}
x_m(t_i) &= \varphi(a)+\varphi'(a)(t_i-a)\\
&+\frac{b-a}{2nl}[(t_i-a)f(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau))\\
&+2\sum_{j=n+1}^{i-1}(t_i-t_j)f(t_j,\overline{x}_{m-1}(t_j)+\overline{R}_{m-1,j},\overline{x}_{m-1}(t_{j-n})\\
&+\overline{R}_{m-1,j-n},\overline{y}_{m-1}(t_j)+\overline{\omega}_{m-1,j},\overline{y}_{m-1}(t_{j-n})+\overline{\omega}_{m-1,j-n})]+R_{m,i}\\
&=\varphi(a)+\varphi'(a)(t_i-a)\\
&+\frac{b-a}{2nl}[(t_i-a)f(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau))\\
&+2\sum_{j=n+1}^{i-1}(t_i-t_j)f(t_j,\overline{x}_{m-1}(t_j),\overline{x}_{m-1}(t_{j-n}),\overline{y}_{m-1}(t_j),\overline{y}_{m-1}(t_{j-n}))]\\
&+\overline{R}_{m,i}=\overline{x}_m(t_i)+\overline{R}_{m,i},\quad \forall i=\overline{n+1,q},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
y_m(t_i) &= \varphi'(a) + \frac{b-a}{2nl}[f(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)) \\
&+ 2\sum_{j=n+1}^{i-1} f(t_j,\overline{x}_{m-1}(t_j)+\overline{R}_{m-1,j},\overline{x}_{m-1}(t_{j-n})+\overline{R}_{m-1,j-n}, \\
&\overline{y}_{m-1}(t_j)+\overline{\omega}_{m-1,j},\overline{y}_{m-1}(t_{j-n})+\overline{\omega}_{1,j-n}) \\
&+ f(t_i,\overline{x}_{m-1}(t_i)+\overline{R}_{m-1,i},\overline{x}_{m-1}(t_{i-n})+\overline{R}_{m-1,i-n}, \\
&\overline{y}_{m-1}(t_i)+\overline{\omega}_{m-1,i},\overline{y}_{m-1}(t_{i-n})+\overline{\omega}_{m-1,i-n})]+\omega_{m,i} \\
&= \varphi'(a) + \frac{b-a}{2nl}[f(a,\varphi(a),\varphi(a-\tau),\varphi'(a),\varphi'(a-\tau)) \\
&+ 2\sum_{j=n+1}^{i-1} f(t_j,\overline{x}_{m-1}(t_j),\overline{x}_{m-1}(t_{j-n}),\overline{y}_{m-1}(t_j),\overline{y}_{m-1}(t_{j-n})) \\
&+ f(t_i,\overline{x}_{m-1}(t_i),\overline{x}_{m-1}(t_{i-n}),\overline{y}_{m-1}(t_i),\overline{y}_{m-1}(t_{i-n}))]+\overline{\omega}_{m,i} \\
&= \overline{y}_m(t_i)+\overline{\omega}_{m,i}, \quad \forall i=\overline{n+1,q}.
\end{aligned}
\tag{36}
$$

For the remainder estimations, from (29) and (30), we have

$$
\|R_{1,i}\|_X \le \frac{L_1(b-a)^2}{4nl}
$$

$$
\|\omega_{1,i}\|_X \le \frac{L_2(b-a)^2}{4nl}, \ \forall i=\overline{n+1,q}.
$$

Using the Lipschitz inequality (iii) we obtain

$$
\begin{aligned}
\left\|\overline{R}_{2,i}\right\|_X &\le \|R_{2,i}\|_X + (b-a)^2(\alpha_1+\alpha_2)\frac{L_1(b-a)^2}{4nl} \\
&+ (b-a)^2(\beta_1+\beta_2)\frac{L_2(b-a)^2}{4nl} \\
&\le \left[1+(b-a)^2(\alpha_1+\alpha_2)\right]\frac{L_1(b-a)^2}{4nl} \\
&+ (b-a)^2(\beta_1+\beta_2)\frac{L_2(b-a)^2}{4nl}, \ \forall i=\overline{n+1,q}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\overline{\omega}_{2,i}\|_X &\le \|\omega_{2,i}\|_X + (b-a)(\alpha_1+\alpha_2)\frac{L_1(b-a)^2}{4nl} \\
&+ (b-a)(\beta_1+\beta_2)\frac{L_2(b-a)^2}{4nl} \\
&\le (b-a)(\alpha_1+\alpha_2)\frac{L_1(b-a)^2}{4nl} + [1+(b-a)(\beta_1+\beta_2)]\frac{L_2(b-a)^2}{4nl},
\end{aligned}
$$

$\forall i = \overline{n+1,q}$.

Let $L' = \max\{L_1, L_2\}$ and $D = \max\left\{(b-a), (b-a)^2\right\}$. Then,

$$\left\|\overline{R}_{2,i}\right\|_X \le \left[1 + D\left(\alpha_1+\alpha_2+\beta_1+\beta_2\right)\right]\frac{L'(b-a)^2}{4nl},\ \forall i = \overline{n+1,q}$$

and

$$\left\|\overline{\omega}_{2,i}\right\|_X \le \left[1 + D\left(\alpha_1+\alpha_2+\beta_1+\beta_2\right)\right]\frac{L'(b-a)^2}{4nl},\ \forall i = \overline{n+1,q}.$$

By induction, we obtain for $m \ge 2$:

$$\begin{aligned}\|\overline{R}_{m,i}\|_X &\le [1 + D(\alpha_1+\alpha_2+\beta_1+\beta_2) + \dots \\ &\quad \dots + D^{m-1}(\alpha_1+\alpha_2+\beta_1+\beta_2)^{m-1}]\frac{L'(b-a)^2}{4nl},\ \forall i = \overline{n+1,q}\end{aligned} \tag{37}$$

and

$$\begin{aligned}\|\overline{\omega}_{m,i}\|_X &\le [1 + D(\alpha_1+\alpha_2+\beta_1+\beta_2) + \dots \\ &\quad \dots + D^{m-1}(\alpha_1+\alpha_2+\beta_1+\beta_2)^{m-1}]\frac{L'(b-a)^2}{4nl},\ \forall i = \overline{n+1,q}.\end{aligned} \tag{38}$$

THEOREM 4. *In the conditions* (i), (ii), (iii), (iv), (23), (24), *if*

$$D\left(\alpha_1+\alpha_2+\beta_1+\beta_2\right) < 1,$$

where $D = \max\left\{(b-a), (b-a)^2\right\}$, *then solution* (x^*, y^*) *of the system* (3) *is approximated on the knots* t_i, $i = \overline{n+1,q}$, *of the partition* $\Delta \in Div\,[a-\tau, b]$, *by the terms of the sequence* $\left(\overline{x}_m(t_i), \overline{y}_m(t_i)\right)_{m\in\mathbb{N}^*}$ *given in* (31)–(36) *and the error estimation is*

$$\begin{pmatrix}\|x^*(t_i) - \overline{x}_m(t_i)\|_X \\ \|y^*(t_i) - \overline{y}_m(t_i)\|_X\end{pmatrix} \le Q^m\left(I_n - Q\right)^{-1} d\left((x_0,y_0),(x_1,y_1)\right) +$$

$$\begin{pmatrix}\frac{L'(b-a)^2}{4nl[1-D(\alpha_1+\alpha_2+\beta_1+\beta_2)]} \\ \frac{L'(b-a)^2}{4nl[1-D(\alpha_1+\alpha_2+\beta_1+\beta_2)]}\end{pmatrix},\ \forall m \in \mathbb{N}^*,\ \forall i = \overline{n+1,q}.$$

Proof. Follows from (14), (37), (38), (35), (36), since

$$\|x^*(t_i) - \overline{x}_m(t_i)\|_X \le \|x^*(t_i) - x_m(t_i)\|_X + \|x_m(t_i) - \overline{x}_m(t_i)\|_X,$$

$$\forall i = \overline{n+1,q},\ \forall m \in \mathbb{N}^*,$$

and

$$\|y^*(t_i) - \overline{y}_m(t_i)\|_X \le \|y^*(t_i) - y_m(t_i)\|_X + \|y_m(t_i) - \overline{y}_m(t_i)\|_X,$$

$\forall i = \overline{n+1,q}$, $\forall m \in \mathbb{N}^*$. □

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