MATHEMATICA, Tome 49(72), N° 1, 2007, pp. 63–68

ON A DIFFERENTIAL SUPERORDINATION DEFINED BY RUSCHEWEYH DERIVATIVE

GH. OROS and GEORGIA IRINA OROS

Abstract. By using the Ruscheweyh operator $D^m f(z)$, $z \in U$, we obtain sharp superordinations results related to some normalized holomorphic functions in the unit disk U.

MSC 2000. Primary 30C80, Secondary 30C45, 30A20, 34A40.

Key words. Differential subordination, differential superordination, univalent function.

1. INTRODUCTION

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. In a series of articles the authors and many others [1] have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z)|\ z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of function p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z)|\ z \in U\}.$$

This problem was introduced in [2].

We let $\mathcal{H}(U)$ denote the class of holomorphic functions in the unit disk $U = \{z \in \mathbb{C} :\ |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

and

$$A_n = \{f \in \mathcal{H}(U) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}.$$

For $0 < r < 1$, we let $U_r = \{z,\ |z| < r\}$.

Definition 1. [2] Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination

$$h(z) \prec \varphi(p(z), zp'(z); z), \tag{1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1, suppose (1) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z); z) |\ z \in U\}. \tag{1'}$$

Although this more general situation is a "differential containment", the condition in (1′) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extended to this generalization.

Definition 2. [2] We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U :\ \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

In order to prove the new results we shall use the following lemma:

Lemma A. [2] *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ with* $\mathrm{Re}\,\gamma \geq 0$, *and $p \in \mathcal{H}[a, n] \cap Q$. If $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U,*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma},$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}\mathrm{d}t, \quad z \in U.$$

The function q is convex and is the best subordinant.

Lemma B. [2] *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U,$$

with $\mathrm{Re}\,\gamma \geq 0$. *If $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U,$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}\mathrm{d}t.$$

The function q is the best subordinant.

Definition 4. [3] For $f \in A_n$ and $m \geq 0$, $m \in \mathbb{N}$, the operator $D^m f$ is defined by

$$D^m f(z) = f(z) * \frac{z}{(1-z)^{m+1}} = \frac{z}{m!}[z^{m-1}f(z)]^{(m)}, \quad z \in U,$$

where $*$ stands for convolution.

Remark 1. We have

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = zf'(z)$$
$$\ldots$$
$$(m+1)D^{m+1}f(z) = z[D^m f(z)]' + mD^m f(z),\ z \in U.$$

2. MAIN RESULT

Theorem 1. *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}$$

be convex in U, with $h(0) = 1$.

Let $f \in A_n$, and suppose that $[D^{m+1}f(z)]'$ is univalent and

$$[D^m f(z)]' \in \mathcal{H}[1,n] \cap Q.$$

If

$$h(z) \prec [D^{m+1}f(z)]', \quad z \in U, \tag{2}$$

then

$$q(z) \prec [D^m f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} t^{\frac{m+1}{n}-1}\mathrm{d}t. \tag{3}$$

The function q is convex and is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of the operator $D^m f(z)$ we have

$$(m+1)D^{m+1}f(z) = z[D^m f(z)]' + mD^m f(z), \quad z \in U. \tag{4}$$

Differentiating (5), we obtain

$$(m+1)[D^{m+1}f(z)]' = (m+1)[D^m f(z)]' + z[D^m f(z)]'', \quad z \in U. \tag{5}$$

If we let $p(z) = [D^m f(z)]'$ then (6) becomes

$$[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1}zp'(z), \quad z \in U.$$

Then (3) becomes

$$h(z) \prec p(z) + \frac{1}{m+1}zp'(z), \quad z \in U.$$

By using Lemma A, we have

$$q(z) \prec p(z) = [D^m f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} t^{\frac{m+1}{n}-1} \mathrm{d}t.$$

THEOREM 2. *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}$$

be convex in U, with $h(0) = 1$. Let $f \in A_n$ and suppose that $[D^m f(z)]'$ is univalent and $\frac{D^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$.

If

$$h(z) \prec [D^m f(z)]', \quad z \in U, \tag{6}$$

then

$$q(z) \prec \frac{D^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{t} t^{\frac{1}{n}-1} \mathrm{d}t.$$

The function q is convex and is the best subordinant.

Proof. We let

$$p(z) = \frac{D^m f(z)}{z}, \quad z \in U,$$

and we obtain

$$D^m f(z) = zp(z), \quad z \in U. \tag{7}$$

By differentiating (8) we obtain

$$[D^m f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (7) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma A we have

$$q(z) \prec p(z) = \frac{D^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} \mathrm{d}t.$$

THEOREM 3. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{1}{m+1} zq'(z), \quad z \in U.$$

Let $f \in A_n$ and suppose that $[D^{m+1}f(z)]'$ is univalent in U, $[D^m f(z)]' \in \mathcal{H}[1,n] \cap Q$ and

(8) $$h(z) = q(z) + \frac{1}{m+1} zq'(z) \prec [D^{m+1}f(z)]',$$

then

$$q(z) \prec [D^m f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t)t^{\frac{m+1}{n}-1}\mathrm{d}t.$$

The function q is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of the operator $D^m f(z)$, we have

(9) $$(m+1)[D^{m+1}f(z)]' = (m+1)[D^m f(z)]' + z[D^m f(z)]''.$$

If we let $p(z) = [D^m f(z)]'$ then (10) becomes

$$[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1} zp'(z), \quad z \in U.$$

Then (9) becomes

$$q(z) + \frac{1}{m+1} zq'(z) \prec p(z) + \frac{1}{m+1} zp'(z), \quad z \in U.$$

By using Lemma B, we have

$$q(z) \prec p(z) = [D^m f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t)t^{\frac{m+1}{n}-1}\mathrm{d}t. \ \square$$

Theorem 4. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + zq'(z), \quad z \in U.$$

Let $f \in A_n$ and suppose that $[D^m f(z)]'$ is univalent in U, $\dfrac{D^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and

(10) $$h(z) = q(z) + zq'(z) \prec [D^m f(z)]', \quad z \in U,$$

then

$$q(z) \prec \frac{D^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}\mathrm{d}t.$$

Proof. We let

$$p(z) = \frac{D^m f(z)}{z}, \quad z \in U,$$

and we obtain

$$D^m f(z) = zp(z), \quad z \in U.$$

By differentiating, we obtain

$$[D^m f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (11) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma B we have

$$q(z) \prec p(z) = \frac{D^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} \mathrm{d}t, \quad z \in U.$$

REFERENCES

[1] Miller, S.S. and Mocanu, P.T., *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Miller, S.S. and Mocanu, P.T., *Subordinants of Differential Superordinations*, Complex Variables, **48** (2003), 815–826.

[3] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109–115.

Received: November 19, 2005

Department of Mathematics
University of Oradea
Str. Armatei Române, No. 5
410087 Oradea, Romania