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ON CERTAIN GENERALIZED CLASS OF p-VALENTLY PARABOLIC STARLIKE FUNCTIONS BASED ON AN INTEGRAL OPERATOR

SH. NAJAFZADEH, S. R. KULKARNI and G. MURUGUSUNDARAMOORTHY

Abstract. By using an integral operator, we introduce a class $p - SP_\xi(\alpha, \beta)$ of parabolic starlike functions in the unit disk Δ and investigate the interesting properties of this class.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A} = \{f \mid f \text{ analytic in } \Delta\}$, $\Delta = \{z : |z| < 1\}$ and $\mathcal{A}_0 = \{f \in \mathcal{A} \mid f(0) = f'(0) - 1 = 0\}$. Also let A_p the class of multivalent function f of the form $f(z) = z^p + \sum\limits_{k=n+p}^{\infty} a_k z^k$ and normalized by $f(0) = f^{(p)}(0) - p! = 0$.

Definition 1.1. A function $f \in \mathcal{A}_0$ is said to be in the class of parabolic starlike functions denoted by SP if (see [1])

$$\left|\frac{zf'}{f} - 1\right| < \operatorname{Re}\left(\frac{zf'}{f}\right) \quad z \in \Delta. \tag{1}$$

We can extend this definition to multivalent functions as follows:

Definition 1.2. A multivalent function $f \in \mathcal{A}_p$ is said to be in the class $p - SP$ p-valently parabolic starlike functions if

$$\left|\frac{zf'}{f} - p\right| < \operatorname{Re}\left(\frac{zf'}{f}\right) \quad z \in \Delta. \tag{2}$$

Definition 1.3. If $f(z) \in \mathcal{A}_p$ we define an integral operator from $\mathcal{A}_p$ to $\mathcal{A}_p$ by

$$F_{\xi,p}(z) = (1-\xi)z^p + \xi p \int_\epsilon^z \frac{f(s)}{s} \mathrm{d}s \ (0 \le \xi < 1,\ \epsilon \to 0^+). \tag{3}$$

Remark 1.1. When $f(z) = z^p + \sum\limits_{k=1}^{\infty} a_{p+k} z^{p+k}$ then

$$F_{\xi,p}(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k},$$

where $b_{p+k} = \frac{\xi p}{p+k} a_{p+k}$.

DEFINITION 1.4. Let $p - SP_\xi(\alpha,\beta)$ $(0 \le \xi < 1, 0 \le \alpha < 1, 0 < \beta < \infty, p \in \mathbb{Z}^+)$ be the class of functions $f \in \mathcal{A}_p$ for which

$$\left|\frac{zF''_{\xi,p}(z)}{F'_{\xi,p}(z)} + 1 - p(\alpha+\beta)\right| < p(\beta-\alpha) + \operatorname{Re}\left[1 + \frac{zF''_{\xi,p}(z)}{F'_{\xi,p}(z)}\right] \tag{4}$$

We say $p - SP_\xi(\alpha,\beta)$ be the class of parabolic p-valent starlike functions.

For particular values of ξ, α, β, p we obtain some interesting subclasses. For example:

(i) $1 - SP_\xi(\frac{1}{2},\frac{1}{2})$ $(\xi \to 1)$ is the class of parabolic starlike functions in Δ and denoted by SP and $p - SP_\xi(\frac{1}{2},\frac{1}{2})$ $(\xi \to 1)$ is the class of parabolic p-valent starlike functions (denote by $p - SP$) studied by Rønning [2].

(ii) $1 - SP_\xi(\frac{1+\alpha}{2},\frac{1-\alpha}{2})$ $(\xi \to 1)$ is the class of parabolic starlike functions of order α that is denoted by $SP(\alpha)$ $(0 \le \alpha < 1)$ and studied by Rønning [1, 2] and $p - SP_\xi(\frac{1+\alpha}{2},\frac{1-\alpha}{2})$ $(\xi \to 1)$ is the class of parabolic starlike p-valent functions of order $\alpha(p - SP(\alpha))$.

(iii) $1 - SP_\xi(\frac{1}{2},\frac{1}{2})$ is the class consisting of functions f such that $(F_{\xi,p}(z))'$ is parabolic starlike function and denoted by SP_ξ and $p - SP_\xi(\frac{1}{2},\frac{1}{2}) = p - SP_\xi$ is the class of parabolic starlike p-valent functions. This class was studied by Srivastava and Mishra [3].

2. MAIN RESULTS

DEFINITION. A function $g \in \mathcal{A}_p$ is said to be in the class $p - UCVP$ of parabolic p-valent uniformly convex functions in Δ if

$$\left|\frac{zg''}{g'} + 1 - p\right| < \operatorname{Re}\left(1 + \frac{zg''}{g'}\right). \tag{5}$$

THEOREM 2.1. *Let $f(z) \in \mathcal{A}_p$ then $F_{\xi,p}(z)$ $(\xi \to 1)$ is in $p - UCVP$ if and only if $f(z) \in p - SP$.*

Proof. Suppose $\lim_{\xi\to 1}(F'_{\xi,p}) = F'_{1,p}, \lim_{\xi\to 1}(F''_{\xi,p}) = F''_{1,p}$. Since $F_{1,p}(z) \in p - UCVP$, then by (5)

$$\left|\frac{zF''_{1,p}(z)}{F'_{1,p}(z)} + 1 - p\right| < \operatorname{Re}\left(1 + \frac{zF''_{1,p}(z)}{F'_{1,p}(z)}\right)$$

or, equivalently, by putting (3) in above inequality, we have

$$\left|\frac{z\left(p\frac{f(z)}{z}\right)'}{p\frac{f(z)}{z}} + 1 - p\right| < \operatorname{Re}\left(1 + \frac{z\left(p\frac{f(z)}{z}\right)'}{\frac{f(z)}{z}}\right)$$

or, equivalently, $\left|\frac{zf'}{f} - p\right| < \operatorname{Re}\left(\frac{zf'}{f}\right)$; then, by definition of $p - SP$, $f(z) \in p - SP$. □

THEOREM 2.2. *$f \in p - SP_\xi(\alpha, \beta)$ if and only if, for every $z \in \Delta$, the values of $\frac{z(F''_{\xi,p}(z))}{F'_{\xi,p}(z)} + 1$ lie in the interior of the parabolic region.*

Proof. By definition of the class $p - SP_\xi(\alpha, \beta)$ if we put values of $\frac{z(F''_{\xi,p}(z))}{F'_{\xi,p}(z)} + 1$ equal to w we have

$$|w - p(\alpha + \beta)| < p(\beta - \alpha) + \operatorname{Re}(w)$$

or

$$[\operatorname{Re}(w) - p(\alpha + \beta)]^2 + (\operatorname{Im}(w))^2 < (p(\beta - \alpha) + \operatorname{Re}(w))^2$$
$$(\operatorname{Re}(w))^2 + p^2(\alpha + \beta)^2 - 2p(\alpha + \beta)\operatorname{Re} w + (\operatorname{Im}(w))^2$$
$$< p^2(\beta - \alpha)^2 + (\operatorname{Re}(w))^2 + 2p(\beta - \alpha)\operatorname{Re}(w)$$

or

$$[\operatorname{Im}(w)]^2 < [2p(\alpha + \beta) + 2p(\beta - \alpha)]\operatorname{Re}(w) - 4p^2\alpha\beta$$

or

$$[\operatorname{Im}(w)]^2 < 4p\beta[\operatorname{Re}(w) - p\alpha]$$

and that is the interior of the parabolic region in the half-plane (right side) with vertex at $(p\alpha, 0)$ and $4p\beta$ is the length of the latus rectum. □

REMARK. We denote the parabolic region that was found in last theorem by

$$\Omega(p, \alpha, \beta) = \{w : w \in \mathbb{C} \text{ and } [\operatorname{Im}(w)]^2 < 4p\beta[\operatorname{Re}(w) - p\alpha]\}. \tag{6}$$

REMARK. Taking $p = 1$ in Theorem 2.2, we get a region defined by Srivastava, Mishra and Das [4].

THEOREM 2.3. *If $f(z) \in \mathcal{A}_p$ and $F_{\xi,p}(z)$ defined by (3), then f is p-valently starlike of order γ if and only if $F_{\xi,p}(z)$ $(\xi \to 1)$ is p-valently convex of order γ.*

Proof. Let $F_{\xi,p}$ be p-valently convex of order γ then $\operatorname{Re}\left\{\frac{zF''_{\xi,p}}{F'_{\xi,p}} + 1\right\} > \gamma$. But by (3) we have

$$F'_{\xi,p}(z) = p(1 - \xi)z^{p-1} + \xi p\frac{f(z)}{z}$$

and

$$F''_{\xi,p}(z) = p(p-1)(1-\xi)z^{p-2} + \xi p \frac{zf' - f}{z^2}$$

and when $\xi \to 1$ we obtain $F'_{1,p} = p\frac{f(z)}{z}$ $F''_{1,p} = p\frac{zf'-f}{z^2}$ and

$$\mathrm{Re}\left\{\frac{zp\frac{zf'-f}{z^2}}{p\frac{f(z)}{z}} + 1\right\} = \mathrm{Re}\left\{\frac{zf'-f}{f} + 1\right\} = \mathrm{Re}\left\{\frac{zf'}{f}\right\} > \gamma$$

and so $f(z)$ is p-valently starlike. All the relations are reversible and so proof is complete. □

Theorem 2.4. *Let* $f_k \in p - SP_\xi(\alpha_k, \beta_k)$ *with* $(0 < \xi < 1,\ 0 \le \alpha_k < 1,\ \sum_{k=1}^n \alpha_k < 1,\ 0 < \beta_k < \infty,\ k = 1, \dots, n)$ *and* $t_k > 0$ $(k = 1, \dots, n)$ *and* $\sum\limits_{k=1}^n t_k = 1$. *Then* $g(z) = \prod\limits_{k=1}^n (f_k)^{t_k}$ *is in* $p - SP_\xi(\alpha, \beta)$, *where* $\alpha = \sum\limits_{k=1}^n t_k\alpha_k$ *and* $\beta = \sum\limits_{k=1}^n t_k\beta_k$.

Proof. We prove this theorem when $\xi \to \overline{1}$. Let

$$F^k_{\xi,p}(z) = (1-\xi)z^p + \int_\in^z \frac{f_k(z)}{z}$$

and

$$G_{\xi,p}(z) = (1-\xi)z^p + \int_\in^z \frac{g(z)}{z}\ (\epsilon \to 0^+).$$

Since $f_k \in p - SP_\xi(\alpha_k, \beta_k)$ $(k = 1, 2, \dots, n)$ then by definition of $p - SP_\xi(\alpha, \beta)$ we have

$$\left|\frac{z(F^k_{\xi,p}(z))''}{(F^k_{\xi,p}(z))'} + 1 - p(\alpha_k + \beta_k)\right| < \mathrm{Re}\left(1 + \frac{z(F^k_{\xi,p}(z))''}{(F^k_{\xi,p}(z))'}\right) + p(\beta_k - \alpha_k). \tag{7}$$

Now we must show

$$\left|\frac{zG''_{\xi,p}(z)}{G'_{\xi,p}(z)} + 1 - p(\alpha + \beta)\right| < \mathrm{Re}\left(1 + \frac{zG''_{\xi,p}(z)}{G'_{\xi,p}(z)}\right) + p(\beta - \alpha).$$

But when $\xi \to 1$ by direct computation we obtain

$$\begin{aligned}\left|\frac{zG''_{\xi,p}}{G'_{\xi,p}} + 1 - p(\alpha + \beta)\right| &= \left|\frac{zg'}{g} - p(\alpha + \beta)\right| \\ &= \left|\sum_{k=1}^n t_k\left(\frac{zf'_k}{f_k} - p(\alpha_k + \beta_k)\right)\right| \\ &\le \sum_{k=1}^n \left[t_k\left|\frac{zf'_k}{f_k} - p(\alpha_k + \beta_k)\right|\right]\end{aligned}$$

with a simple calculation on (7) when $\xi \to 1^-$ we obtain

$$\left|\frac{zf'}{f} - p(\alpha_k + \beta_k)\right| < \mathrm{Re}\left(\frac{zf'}{f}\right) + p(\beta_k - \alpha_k) \tag{8}$$

and so

$$\begin{aligned}\left|\frac{zG''_{\xi,p}}{G'_{\xi,p}} + 1 - p(\alpha+\beta)\right| &< \sum_{k=1}^{n}\left[t_k\left(\mathrm{Re}\left(\frac{zf'_k}{f_k}\right) + p(\alpha_k+\beta_k)\right)\right] \\ &= \mathrm{Re}\left(\frac{zg'}{g}\right) + p(\beta - \alpha).\end{aligned}$$

So $g \in p{-}SP_\xi(\alpha,\beta)$ (when $\xi \to \overline{1}$). The proof of Theorem 2.4 is completed. □

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Department of Mathematics, Fergusson College,
Pune - 411004, India
E-mail: Najafzadeh1234@yahoo.ie

Department of Mathematics, Fergusson College,
Pune - 411004, India
E-mail: kulkarni_ferg@yahoo.com

School of Science and Humanities, VIT University,
Vellore - 632014, Tamil Nadu, India.
E-mail: gmsmoorthy@yahoo.com