MATHEMATICA, Tome 49 (72), N° 1, 2007, pp. 35–47

INVERSE PROBLEM OF DYNAMICS IN NON-FLAT SPACES: SOLVABLE CASES OF THE TWO BASIC EQUATIONS

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Abstract. We study several solvable cases of a first-order partial differential equation given by Mertens (1981). This equation combines the potential function $V = V(u, v)$ with a mono-parametric family of regular orbits $f(u, v) = c$ on a given surface S submersed in $\mathbb{E}^3$ and the function of energy-dependence $E = E(f)$ is given in advance. In the generic case it is shown that two differential conditions must be hold for the "*slope function*" $\gamma = f_v/f_u$ in order the above equation has solution. Moreover, in the above solvable cases, the second order PDE given by Bozis and Mertens (1985) reduces to its canonical form and can be solved too. Pertinent examples are given.

MSC 2000. 70F17, 70M20, 53A05, 35F05, 35G05, 35G20.

Key words. Inverse Problems, Monoparametric families of orbits, 2D Potentials, ODEs and PDEs.

1. INTRODUCTION

In 1974 V. Szebehely published a partial differential equation for the potential function $V = V(x, y)$ which produces a mono-parametric family of planar orbits $f(x, y) = c$ and the energy E of them is given in advance as a function of the constant c namely $E = E(c)$. Bozis (1984) presented a second order linear partial differential equation giving the potential functions $V = V(x, y)$ which give rise to a preassigned family of planar curves $f(x, y) = c$. Bozis' equation does not include the energy E and consequently no assumption about the energy dependence $E = E(f)$ needs to be made. Anisiu (2004) derived in a unified manner the two basic equations of the inverse problem of dynamics, and the region where real motion of the particle takes place.

Mertens(1981) studied a family of curves $f(u, v) = c$ on a surface S in 3D space using Szebehely's method and obtained a linear partial differential equation in the potential function $V(u, v)$. Furthermore, Bozis and Mertens (1985) derived a second order partial differential equation of hyperbolic type for the potential V in which all the coefficients are known functions of the coordinates u, v and gave some examples. Borghero (1986) determined the expressions for the covariant components Q_1, Q_2 of forces acting on a test particle which describes orbits on a given surface, using the procedure of Dainelli (Whittaker, 1994). Bozis and Borghero (1995) introduced the notion of the family boundary curves (FBC) for that version of the inverse problem of dynamics which combines the potential $V(u, v)$ with a mono-parametric family of regular orbits $f(u, v) = c$ on the configuration manifold $(M_2,\ g)$ of a conservative holonomic system with n=2 degrees of freedom. Several examples were given there. Puel (2002) gave a geometrical interpretation for the deflection at the origin of rectilinear orbits in a central field. This interpretation was based on the correspondence between the plane orbits of a conservative force field and the geodesics of a certain surface. Recently, Kotoulas (2005a) studied the case of a generalized force field which gives rise to a *two*-parametric family of curves on a given surface. Among other curves, helical lines were also studied there. A solvable version of the inverse problem of dynamics was studied by the same author (Kotoulas, 2005b). A review on basic facts of inverse problem in dynamics was made by Bozis (1995) and recently by Anisiu (2003).

In the present work we shall deal with the first-order PDE given by Mertens (1981) and find solvable cases of it for *any* energy dependence. Moreover, in these cases, the second order PDE given by Bozis and Mertens (1985) can be solved analytically too. In Section 2 we give a full description of this problem. In Section 3 we classify solvable cases of the above equation and in Section 4 we give pertinent examples. In the generic case it is shown that two differential conditions must be hold for the "*slope function*" $\gamma = f_v/f_u$ in order the above equation has solution. We conclude in Section 5.

2. ANALYSIS OF THE PROBLEM

In an Euclidean 3D-space $\mathbb{E}^3$ with an orthonormal Cartesian system of reference $Oxyz$ we assign a smooth surface S:

$$P = P(u,v) \Leftrightarrow \{x = x(u,v),\ y = y(u,v),\ z = z(u,v)\} \tag{1}$$

with $u,\ v$ as curvilinear coordinates on S. On this surface we also consider a mono-parametric family of regular curves given in the solved form

$$f(u,v) = c, \tag{2}$$

where c is the parameter of the family (2).

For the given family of orbits we define γ as follows: $\gamma = f_v/f_u$ and the subscripts denote partial differentiation. The "*slope function*" γ represents the family (2) in the sense that if the family (2) is given, then γ is determined uniquely. On the other hand, if γ is given, we can obtain a unique family (2). The inverse problem of dynamics consists in finding potentials V which can give rise to this family of orbits (2) on a given surface (1).

The line-element on the surface S in this system of parameters is given by:

$$\mathrm{d}s^2 = g_{11}\mathrm{d}u^2 + 2g_{12}\mathrm{d}u\mathrm{d}v + g_{22}\mathrm{d}v^2, \tag{3}$$

where $g_{11},\ g_{12},\ g_{22}$ are known functions of $u,\ v$.

Now, we consider a particle of unit mass which describes any member of the given family (2). The kinetic energy (T) of the test particle is given by

$$T = \frac{1}{2}(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2), \tag{4}$$

where the dot denotes differentiation with respect to time.

2.1. Mertens' PDE (1981). Mertens (1981) produced a *linear*, first order partial differential equation for the potential function $V = V(u,v)$ for any preassigned dependence $E = E(f)$, of the total energy E of the given family $f = f(u,v)$. This equation is the following one:

$$(g_{22}f_u - g_{12}f_v)V_u + (g_{11}f_v - g_{12}f_u)V_v = 2W(E - V), \tag{5}$$

where W is given in the *Appendix* I.

Using the "*slope function*" γ and the notation $\Gamma = \gamma\gamma_u - \gamma_v$, the equation (5) takes a simpler form:

$$(g_{22} - \gamma g_{12})V_u + (\gamma g_{11} - g_{12})V_v + \frac{2\Delta}{A_1}(E - V) = 0, \tag{6}$$

where A_1 and Δ are given in the *Appendix* II.

The subsidiary system of equations for (6) is:

$$\frac{\mathrm{d}u}{g_{22} - \gamma g_{12}} = \frac{\mathrm{d}v}{\gamma g_{11} - g_{12}} = \frac{A_1 \mathrm{d}v}{2\Delta(V - E)} \tag{7}$$

or, equivalently, we have to solve two ODEs

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\gamma g_{11} - g_{12}}{g_{22} - \gamma g_{12}} \tag{8}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}u} - \mathcal{K}V + \mathcal{K}E = 0, \tag{9}$$

where $\mathcal{K} = 2\Delta/(A_1(g_{22} - \gamma g_{12}))$. The general solution of (6) is of the form: $F(d_1,\ d_2 = 0)$ (F is an arbitrary function of two arguments, $d_1,\ d_2 = \text{const.}$). We assume that we can find a first independent integral of (8) namely an expression of the form $F(u,v) = d_1$ or, equivalently, $v = v(u, d_1)$ (d_1=const.). If we insert it into (9), then we have to proceed and calculate the potential function $V = V(u,v)$. Thus, the function $f = f(u,v) = c$ must be known in advance in order to determine the energy dependence. We end up to the conclusion that the following ODE must be solvable

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{1}{\gamma(u,v)}. \tag{10}$$

The solution of (10) gives us the mono-parametric family of orbits (2).

2.2. Bozis and Mertens' PDE (1985). Bozis and Mertens (1985) produced a *linear*, second order partial differential equation in $V = V(u,v)$ *which is independent of the total energy* E and gives all the potential functions generating family (2) on the given surface (1). The total energy E must be constant along each orbit, so $E = E(f)$. Thus, we have $E_v = E_f f_v$ and $E_u = E_f f_u$. Assuming that $W \neq 0$ and with the use of the fact that $E_v = \gamma E_u$, Bozis and Mertens (1985) obtained the following equation:

$$k_1 V_{uu} + k_2 V_{uv} - \beta V_{vv} + k_3 V_u + k_4 V_v = 0, \tag{11}$$

where

$$k_1 = \alpha\gamma,\ k_2 = \beta\gamma - \alpha,\ k_3 = \gamma + \gamma\alpha_u - \alpha_v,\ k_4 = \gamma\beta_u - \beta_v - 1 \tag{12}$$

and the coefficients α, β, γ are given in the *Appendix* I. The subscripts denote partial differentiation with respect to the corresponding variable.

If we apply the condition $E_v = \gamma E_u$ to the equation (6), we obtain again the equation (11) but now the coefficients α and β are given as follows:

$$\alpha = -\frac{A_1(g_{22} - \gamma g_{12})}{2\Delta},\ \ \beta = -\frac{A_1(\gamma g_{11} - g_{12})}{2\Delta}. \tag{13}$$

From (11) it is easy to check that if V is a solution, then $V' = d_1 V + d_2$ is a solution too ($d_1,\ d_2$ are constants). So, without loss of generality, we shall omit these constants below. In the present work we shall consider the previous equation (11) and $\alpha,\ \beta$ are given in (13).

Firstly, we see that equation (11) is a partial differential equation of second order in *hyperbolic type* of the potential function $V = V(u, v)$. Indeed, we consider the trinominal

$$k_1\lambda^2 + k_2\lambda - \beta = 0 \tag{14}$$

and the discriminant of (14) is:

$$\Delta = k_2^2 + 4k_1\beta = (\beta\gamma + \alpha)^2 > 0. \tag{15}$$

The roots of (14) are:

$$\lambda_1 = -\frac{\beta}{\alpha},\ \ \lambda_2 = \frac{1}{\gamma}. \tag{16}$$

From (13) we observe that

$$\frac{\beta}{\alpha} = \frac{\gamma g_{11} - g_{12}}{g_{22} - \gamma g_{12}}. \tag{17}$$

So, the roots of equation (14) are written as:

$$\lambda_1 = -\frac{\gamma g_{11} - g_{12}}{g_{22} - \gamma g_{12}},\ \ \lambda_2 = \frac{1}{\gamma}. \tag{18}$$

If we select an appropriate transformation

$$\eta = f_1(u, v),\ \ \xi = f_2(u, v). \tag{19}$$

where $f_1(u, v) = c_1$ and $f_2(u, v) = c_2$ are the solutions of the ordinary differential equations of first order,

$$\frac{\mathrm{d}v}{\mathrm{d}u} + \lambda_1(u, v) = 0,\ \ \frac{\mathrm{d}v}{\mathrm{d}u} + \lambda_2(u, v) = 0, \tag{20}$$

then the equation (11) can be reduced into its canonical form in any fixed point of the region where the equation is defined and *may* be solved. We should state here that the curves $\eta = c_1$ and $\xi = c_2$ are the *characteristic* ones of (11). Someone can find a classification of second order PDEs with linear principal part in [13] (Chapter 4). In view of (18), the relations (20) are written as:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\gamma g_{11} - g_{12}}{g_{22} - \gamma g_{12}}, \quad \frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{1}{\gamma}. \tag{21}$$

Now, we see that the equations (21) are the same ones with (8) and (10).

REMARK 1. If we consider planar orbits, then we can use the Cartesian coordinates $\{x,\ y\}$ instead of the curvilinear coordinates $\{u,\ v\}$ and the elements of metric tensor are: $g_{11} = 1,\ g_{12} = 0,\ g_{22} = 1$. Then the equations (20) become:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \gamma, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\gamma} \tag{22}$$

Grigoriadou *et al.* (1999) solved the equations (22) by quadratures for some specific forms of the "*slope function*" γ. Thus, Szebehely's equation is solvable.

REMARK 2. Generally speaking, if we consider "*Liouville's surfaces*", i.e. surfaces with the following coefficients of first fundamental form:

$$g_{11} = g_{22}, \quad g_{12} = 0, \tag{23}$$

then the equation (21) take again the form (22). Thus, the solvable cases of the planar inverse problem of dynamics (see also Grigoriadou *et al.*, 1999) are very useful for the study of regular orbits on Liouville's surfaces. So, in the present study we shall consider the generic case in which $g_{11} \neq g_{22}$.

Now we set the question: *In which cases are the equations* (21) *solved*?

If for appropriate $\alpha,\ \beta,\ \gamma$ the equations (21) are solved, then the equation (11) is reduced to its *canonical form* and finally may be solved. These solvable cases are going to be studied in the following section.

3. SOLVABLE CASES OF EQUATIONS (21)

In this section we shall study solvable cases of the equations (21). We will start with the

- Case 1: $\gamma = \sigma(u)$.

The first of equations (21) is written as:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\sigma(u) g_{11} - g_{12}}{g_{22} - \sigma(u) g_{12}}. \tag{24}$$

From (24) we observe that if

$$g_{11} = g_{11}(u), \quad g_{12} = g_{12}(u), \quad g_{22} = g_{22}(u) \tag{25}$$

then the equation (24) is written as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \tau(u), \tag{26}$$

where

$$\tau(u) = \frac{\sigma(u)g_{11}(u) - g_{12}(u)}{g_{22}(u) - \sigma(u)g_{12}(u)}. \tag{27}$$

We integrate (26) and we get

$$v - \phi(u) = d_1, \quad \phi(u) = \int \tau(u)\mathrm{d}u. \tag{28}$$

On the other hand, the second of equations (21) is solved directly by quadratures, i.e.

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{1}{\sigma(u)} \Longleftrightarrow v + \int \frac{\mathrm{d}u}{\sigma(u)} = d_2. \tag{29}$$

• Case 2: $\gamma = \sigma(a_1 u + a_2 v + a_3)$ where $a_1, \ a_2, \ a_3 = \text{const.} \neq 0$. For simplicity reasons we shall set $w = a_1 u + a_2 v + a_3$.

The first of equations (21) is written as:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\sigma(w)g_{11} - g_{12}}{g_{22} - \sigma(w)g_{12}}. \tag{30}$$

From (30) we see that, if

$$g_{12} = \phi_0(w), \quad g_{11} = \phi_1(w), \quad g_{22} = \phi_2(w) \tag{31}$$

then the above equation is written as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \tau(w) \tag{32}$$

with $\tau(w) = \frac{\sigma(w)\phi_1(w) - \phi_0(w)}{\phi_2(w) - \sigma(w)\phi_0(w)}$ and it is integrated as follows:

$$\int \frac{\mathrm{d}w}{a_2\tau(w) + a_1} - u = d_1. \tag{33}$$

On the other hand, the second of equations (21) is solved directly by quadratures, i.e.

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{1}{\sigma(w)} \Longleftrightarrow \int \frac{\sigma(w)}{a_1\sigma(w) + a_2}\mathrm{d}w - u = d_2. \tag{34}$$

• Case 3: $\gamma = \sigma(w)$ with $w = \frac{v}{u}$.

The first of equations (21) is written as:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\sigma(w)g_{11} - g_{12}}{g_{22} - \sigma(w)g_{12}}. \tag{35}$$

From (36) we see that if

$$g_{12} = \phi_0(w), \quad g_{11} = \phi_1(w), \quad g_{22} = \phi_2(w) \tag{36}$$

then the above equation is written as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \tau(w) \tag{37}$$

with $\tau(w) = \frac{\sigma(w)\phi_1(w)-\phi_0(w)}{\phi_2(w)-\sigma(w)\phi_0(w)}$ and it is integrated as follows:

$$\int \frac{\mathrm{d}w}{\tau(w)-w} - \log u = d_1. \tag{38}$$

On the other hand, the second of equations (21) is solved directly by quadratures, i.e.

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{1}{\sigma(w)} \Longleftrightarrow \int \frac{\sigma(w)}{w\sigma(w)+1}\mathrm{d}w + \log u = d_2. \tag{39}$$

• Case 4: *The Generic Case*. We shall study here the case in which both the equations (21) are exact differential equations after we multiply them with suitable factors (the so-called "*Euler's multipliers*"). Let $\tau(u,v)$ and $\rho(u,v)$ be the multipliers for the first and the second of equations (21) respectively. Then we have:

$$\tau(\gamma g_{11} - g_{12})\mathrm{d}u - \tau(g_{22} - \gamma g_{12})\mathrm{d}v = dJ \tag{40}$$

and

$$\rho(\mathrm{d}u + \gamma \mathrm{d}v) = dH. \tag{41}$$

The first of equations (21) is written as

$$\kappa_1 \mathrm{d}u + \kappa_2 \mathrm{d}v = 0 \tag{42}$$

where $\kappa_1 = \gamma g_{11} - g_{12}$, $\kappa_2 = \gamma g_{12} - g_{22}$. Then the relation (40) is written as $\tau\kappa_1 \mathrm{d}u + \tau\kappa_2 \mathrm{d}v = dJ$ and it is integrable when

$$\frac{\partial(\tau\kappa_1)}{\partial v} = \frac{\partial(\tau\kappa_2)}{\partial u} \tag{43}$$

or equivalently

$$\kappa_1\tau_v - \kappa_2\tau_u = (\kappa_{2u} - \kappa_{1v})\tau. \tag{44}$$

Also, from (40) we have:

$$J_u = \tau\kappa_1, \quad J_v = \tau\kappa_2. \tag{45}$$

From (45) we compute the partial derivatives of second order for the function J. Thus, we have:

$$J_{uu} = \tau_u\kappa_1 + \tau\kappa_{1u}, \quad J_{vv} = \tau_v\kappa_2 + \tau\kappa_{2v} \tag{46}$$

and the *Laplacian* of J is:

$$\nabla^2 J = J_{uu} + J_{vv} = \tau(\kappa_{1u} + \kappa_{2v}) + \kappa_1\tau_u + \kappa_2\tau_v. \tag{47}$$

From (47) we get:

$$\kappa_1\tau_u + \kappa_2\tau_v = \nabla^2 J - \tau(\kappa_{1u} + \kappa_{2v}). \tag{48}$$

Combining the equations (44) and (48), we compute the partial derivatives of first order of τ namely τ_v and τ_u:

$$\tau_v = \frac{\Delta_1}{\Delta_0}, \quad \tau_u = \frac{\Delta_2}{\Delta_0} \tag{49}$$

where

$$\begin{aligned}\Delta_0 &= \kappa_1^2+\kappa_2^2,\\ \Delta_1 &= \kappa_2\nabla^2 J+(\kappa_1\kappa_{2u}-\kappa_1\kappa_{1v}-\kappa_2\kappa_{1u}-\kappa_2\kappa_{2v})\tau,\\ \Delta_2 &= \kappa_1\nabla^2 J-(\kappa_1\kappa_{1u}+\kappa_1\kappa_{2v}+\kappa_2\kappa_{2u}-\kappa_2\kappa_{1v})\tau.\end{aligned} \tag{50}$$

The total differential of τ is:

$$\mathrm{d}\tau=\tau_v\mathrm{d}v+\tau_u\mathrm{d}u. \tag{51}$$

Now, we set

$$\begin{aligned}M &= \kappa_1\kappa_{2u}-\kappa_1\kappa_{1v}-\kappa_2\kappa_{1u}-\kappa_2\kappa_{2v},\\ N &= \kappa_1\kappa_{1u}+\kappa_1\kappa_{2v}+\kappa_2\kappa_{2u}-\kappa_2\kappa_{1v}.\end{aligned} \tag{52}$$

With the use of (49), (50) and (52), the equation (51) becomes:

$$\mathrm{d}\tau=\frac{1}{\Delta_0}[(\kappa_1\mathrm{d}u+\kappa_2\mathrm{d}v)\nabla^2 J+\tau M\mathrm{d}v-\tau N\mathrm{d}u]. \tag{53}$$

Using the relation (42) the equation (53) is written as:

$$\frac{\mathrm{d}\tau}{\tau}=\frac{1}{\Delta_0}[M\mathrm{d}v-N\mathrm{d}u]. \tag{54}$$

So, the equation (54) is integrable when

$$\frac{\partial}{\partial u}\Big(\frac{M}{\Delta_0}\Big)=-\frac{\partial}{\partial v}\Big(\frac{N}{\Delta_0}\Big) \tag{55}$$

or, equivalently,

$$(M_u+N_v)\Delta_0=M\Delta_{0u}+N\Delta_{0v}. \tag{56}$$

The lhs of (56) become, after some straightforward algebra,

$$M_u+N_v=\kappa_1\kappa_{2uu}-\kappa_2\kappa_{1uu}+\kappa_1\kappa_{2vv}-\kappa_2\kappa_{1vv}. \tag{57}$$

Moreover, the rhs of (56) are written as:

$$M\Delta_{0u}+N\Delta_{0v}=2(\kappa_1^2-\kappa_2^2)(\kappa_{1u}\kappa_{2u}+\kappa_{1v}\kappa_{2v})+2\kappa_1\kappa_2(\kappa_{2u}^2-\kappa_{1u}^2+\kappa_{2v}^2-\kappa_{1v}^2). \tag{58}$$

So, the equation (56), with the use of (57) and (58), reads:

$$\begin{aligned}&(\kappa_1\kappa_{2uu}-\kappa_2\kappa_{1uu}+\kappa_1\kappa_{2vv}-\kappa_2\kappa_{1vv})(\kappa_1^2+\kappa_2^2)\\ &= 2(\kappa_1^2-\kappa_2^2)(\kappa_{1u}\kappa_{2u}+\kappa_{1v}\kappa_{2v})+2\kappa_1\kappa_2(\kappa_{2u}^2-\kappa_{1u}^2+\kappa_{2v}^2-\kappa_{1v}^2).\end{aligned} \tag{59}$$

If for the mono-parametric family of orbits (2) on the given surface (1) the relation (59) holds, then the first of equations (21) is integrable.

Furthermore, if we work with the relation (41) in a similar way, then we obtain:

$$\nabla^2(\arctan\gamma)=0. \tag{60}$$

Now, we can formulate the following

PROPOSITION 1. *If for the given family of regular orbits* (2) *on a certain surface* (1)*, the slope function* γ *satisfies the differential relations* (59) *and* (60) *and the function of the energy-dependence is given in advance, then the equation* (6) *is solved. The equation* (11) *given by Bozis and Mertens* (1985) *is reduced to its canonical form and can be solved analytically.*

REMARK 3. *If we consider planar orbits, then we can use the cartesian coordinates* x, y *instead of the curvilinear coordinates and the elements of metric tensor are:* $g_{11} = 1$, $g_{12} = 0$, $g_{22} = 1$. *Thus the equation* (59) *coincides with* (60) *and the "*slope function γ*" has to satisfy only one condition, as it was shown by Grigoriadou* et al. (1999).

4. PERTINENT EXAMPLES

In this section we shall offer some examples in which selecting an appropriate transformation (ξ, η) of (18), the second order PDE is reduced to its canonical form and it is solved. In the first two examples we have selected surfaces with $g_{12} \neq 0$ and in the last ones we have taken an isothermic net of parameters on the given surface (ex. 3, 4). Let us start with

EXAMPLE 1. We assign the surface S: $\vec{r}(u,v) = \{u,\ v,\ uv\}$ and we consider the mono-parametric family of hyperbolas $f = uv = c$ on it. Then we have:

$$g_{11} = 1 + v^2,\ \ g_{12} = uv,\ \ g_{22} = 1 + u^2,\ \ \gamma = \frac{u}{v},\ \ \frac{\beta}{\alpha} = \frac{u}{v}. \tag{61}$$

So, from (18) we select the transformation

$$\xi = uv,\ \ \eta = u^2 - v^2. \tag{62}$$

Thus the equation (11), in the new variables ξ, η, reads:

$$(\eta^2 + 4\xi^2)V_{\xi\eta} - 4\xi V_\eta + 2\eta V_\xi = 0. \tag{63}$$

If we integrate (63) with respect to the variable η, we will obtain

$$(\eta^2 + 4\xi^2)V_\xi - 4\xi V = F(\xi), \tag{64}$$

where F in (64) is an arbitrary C^2-function of its argument. Then we find the general solution of (64) and it is:

$$V(\xi,\eta) = (\eta^2 + 4\xi^2)^{1/2}\left[G(\eta) + \int \frac{F(\xi)\mathrm{d}\xi}{(\eta^2 + 4\xi^2)^{3/2}}\right]. \tag{65}$$

We see that two arbitrary functions $F(\xi)$, $G(\eta)$ appear in the general solution of (65). Moreover, in the calculation of the integral in (65) the variable η should be considered as constant. So, with the aid of (62), we come back to the variables u, v and the potential function is:

$$V(u,v) = (u^2 + v^2)\Big[G(u^2 - v^2) + H(u,v)\Big]. \tag{66}$$

Example 2. We assign the surface S: $\vec{r}(u,v) = \{u+v,\ u-v,\ u^2+v^2\}$ and we consider the mono-parametric family of circles $f = u^2+v^2 = c$ on it. Then we have:

$$g_{11} = 2+4u^2,\ \ g_{12} = 4uv,\ \ g_{22} = 2+4v^2,\ \ \gamma = \frac{v}{u},\ \ \frac{\beta}{\alpha} = \frac{v}{u}. \tag{67}$$

So, from (18) we select the transformation

$$\xi = u^2+v^2,\ \ \eta = \frac{v}{u}. \tag{68}$$

Thus the equation (11), in the new variables ξ, η, reads:

$$\xi V_{\xi\eta} + V_\eta = 0. \tag{69}$$

If we integrate (69) with respect to the variable η, then we obtain

$$\xi V_\xi + V = F(\xi), \tag{70}$$

where F in (70) is an arbitrary C^2-function of its argument. Then we find the general solution of (70) and it is:

$$V(\xi,\eta) = H(\xi) + \frac{G(\eta)}{\xi}. \tag{71}$$

We see that two arbitrary functions $H(\xi)$, $G(\eta)$ appear in the general solution of (71). The function $H(\xi)$ is related to $F(\xi)$ as follows: $H(\xi) = \frac{1}{\xi}\int F(\xi)d\xi$. So, with the aid of (68), we come back to the variables u, v and the potential function is:

$$V(u,v) = H(u^2+v^2) + \frac{G(\frac{v}{u})}{u^2+v^2}. \tag{72}$$

Example 3. We assign the surface S: $\vec{r}(u,v) = \{u - \frac{u^3}{3} + uv^2,\ -v + \frac{v^3}{3} - vu^2,\ u^2 - v^2\}$ ("Enneper's" surface) and we consider the mono-parametric family of circles $f = u^2+v^2 = c$ on it. Then we have:

$$\begin{aligned} g_{11} &= g_{22} = 1 + 2(u^2+v^2) + (u^2+v^2)^2,\ \ g_{12} = 0, \\ \gamma &= \frac{v}{u},\ \ \frac{\beta}{\alpha} = \frac{v}{u} \end{aligned} \tag{73}$$

So, from (18) we select the transformation

$$\xi = u^2+v^2,\ \ \eta = \frac{v}{u}. \tag{74}$$

Thus the equation (11), in the new variables ξ, η, reads:

$$(1+3\xi)V_\eta + (\xi+\xi^2)V_{\xi\eta} = 0. \tag{75}$$

If we integrate (75) with respect to the variable η, then we obtain

$$(1+3\xi)V + \xi(\xi+1)V_\xi = F(\xi), \tag{76}$$

where F in (76) is an arbitrary C^2-function of its argument. Then we find the general solution of (76) and it is:

$$V(\xi,\eta)=H(\xi)+\frac{G(\eta)}{\xi(\xi+1)^2}. \tag{77}$$

We see that two arbitrary functions $F(\xi)$, $G(\eta)$ appear in the general solution of (75). The function $H(\xi)$ is related to $F(\xi)$ as follows: $H(\xi)=\frac{1}{\xi(\xi+1)^2}\int(\xi+1)F(\xi)\mathrm{d}\xi$. So, with the aid of (77), we come back to the variables u, v and the potential function is:

$$V(u,v)=H(u^2+v^2)+\frac{G(\frac{v}{u})}{(u^2+v^2)(u^2+v^2+1)^2}. \tag{78}$$

EXAMPLE 4. We assign the sphere S: $\vec{r}(u,v)=\{\frac{u^2+v^2-1}{u^2+v^2+1},\ \frac{2u}{u^2+v^2+1},\ \frac{2v}{u^2+v^2+1}\}$ and we consider the mono-parametric family of circles $f=u^2+v^2=c$ on it. Then we have:

$$g_{11}=g_{22}=\frac{4}{(1+u^2+v^2)^2},\ \ g_{12}=0,\ \ \gamma=\frac{v}{u},\ \ \frac{\beta}{\alpha}=\frac{v}{u}. \tag{79}$$

So, from (18) we select the transformation

$$\xi=u^2+v^2,\ \ \eta=\frac{v}{u} \tag{80}$$

Thus the equation (11), in the new variables ξ,η, reads:

$$(1-\xi)V_\eta+(\xi+\xi^2)V_{\xi\eta}=0 \tag{81}$$

If we integrate (81) with respect to the variable η, then we obtain

$$(1-\xi)V+\xi(1+\xi)V_\xi=F(\xi), \tag{82}$$

where F in (82) is an arbitrary C^2-function of its argument. Then we find the general solution of (82) and it is:

$$V(\xi,\eta)=H(\xi)+\frac{(\xi+1)^2}{\xi}G(\eta). \tag{83}$$

We see that two arbitrary functions $H(\xi)$, $G(\eta)$ appear in the general solution of (83). The functions $H(\xi)$ and $F(\xi)$ are combined with the relation: $H(\xi)=\frac{(\xi+1)^2}{\xi}\int\frac{F(\xi)}{(\xi+1)^3}\mathrm{d}\xi$. So, with the aid of (80), we come back to the variables u, v and the potential function is:

$$V(u,v)=H(u^2+v^2)+\frac{(u^2+v^2+1)^2}{u^2+v^2}G\left(\frac{v}{u}\right). \tag{84}$$

5. CONCLUDING COMMENTS

We considered a mono-parametric family of curves $f(u,v)=c$ on a given surface S submersed in $\mathbb{E}^3$. We studied the PDE given by Mertens (1981) and we found several solvable cases of it. In general, this problem has no solution. It is not expected to find a solution for *any* mono-parametric family of orbits (2) on a given surface (1) unless the "*slope function*" γ satisfies the **two** differential conditions (59) and (60). These conditions are the basic results of our study. In Section 4 we gave several examples in which the second order PDE given by Bozis and Mertens (1985) is reduced into canonical form and it is solved. All the potentials found are real. The examples are completely new and found with the aid of the program MATHEMATICA 5.2.

ACKNOWLEDGMENTS

I would like to express my thanks to Prof. G. Bozis for many useful discussions. The financial support of the scientific program "*EPEAEK II, PYTHAGORAS*", No. 21878., of the Greek Ministry of Education and E.U. is also acknowledged.

Appendix I (General case of Section 2)

$$
\begin{aligned}
\alpha &= \frac{1}{2W}(g_{22}f_u - g_{12}f_v),\ \beta = \frac{1}{2W}(-g_{12}f_u + g_{11}f_v),\ \gamma = \frac{f_v}{f_u},\\
W &= \frac{1}{A}[g(f_v^2 f_{uu} - 2f_u f_v f_{uv} + f_u^2 f_{vv}),\\
&\quad -B_1(g_{22}f_u - g_{12}f_v) - B_2(g_{11}f_v - g_{12}f_u)],\\
A &= g_{11}f_v^2 - 2g_{12}f_u f_v + g_{22}f_u^2,\\
B_1 &= \frac{1}{2}(g_{11})_u f_v^2 + [(g_{12})_v - \frac{1}{2}(g_{22})_u]f_u^2 - (g_{11})_v f_u f_v,\\
B_2 &= [(g_{12})_u - \frac{1}{2}(g_{11})_v]f_v^2 + \frac{1}{2}(g_{22})_v f_u^2 - (g_{22})_u f_u f_v,\\
g &= g_{11}g_{22} - (g_{12})^2.
\end{aligned}
$$

Appendix II

$$
\begin{aligned}
\Delta &= g\Gamma + B_1'(g_{22} - \gamma g_{12}) + B_2'(\gamma g_{11} - g_{12}),\\
A_1 &= g_{11}\gamma^2 - 2g_{12}\gamma + g_{22},\\
B_1' &= \frac{1}{2}(g_{11})_u\gamma^2 + [(g_{12})_v - \frac{1}{2}(g_{22})_u] - (g_{11})_v\gamma,\\
B_2' &= [(g_{12})_u - \frac{1}{2}(g_{11})_v]\gamma^2 + \frac{1}{2}(g_{22})_v - (g_{22})_u\gamma.
\end{aligned}
$$

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Received February 17, 2006

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