MATHEMATICA, Tome 49 (72), N° 1, 2007, pp. 29–34

FEKETE-SZEGÖ INEQUALITY FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

M. DARUS, T. N. SHANMUGAM and S. SIVASUBRAMANIAN

Abstract. In this present investigation, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disk for which $zf'(z)/f(z)+\alpha z^2 f''(z)/f(z)$ $(\alpha \geq 0)$ lies in a region starlike with respect to 1 and symmetric with respect to the real axis. Also certain application of the main result for a class of functions defined by convolution is given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives is obtained.

MSC 2000. 30C45.
Key words. Analytic functions, Starlike functions, Subordination, Coefficient problem, Fekete-Szegö inequality.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} \mid |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which $\frac{zf'(z)}{f(z)} \prec \phi(z)$ $(z \in \Delta)$ and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$ $(z \in \Delta)$, where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [10]. They have obtained the Fekete-Szegö inequality for the function in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions see the recent paper by Srivastava *et al.* [7].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_\alpha(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M_\alpha^\lambda(\phi)$ of functions defined by fractional derivatives.

DEFINITION 1.1. *Let $\phi(z)$ be a univalent starlike function with respect to* (1) *which maps the unit disk Δ onto a region in the right half plane which is* *symmetric with respect to the real axis,* $\phi(0) = 1$ *and* $\phi'(0) > 0$. *A function* $f \in \mathcal{A}$ *is in the class* $M_\alpha(\phi)$ *if* $\frac{zf'(z)}{f(z)} + \alpha z^2 \frac{f''(z)}{f(z)} \prec \phi(z) \quad (\alpha \geq 0)$. *For fixed* $g \in \mathcal{A}$, *we define the class* $M^g_\alpha(\phi)$ *to be the class of functions* $f \in \mathcal{A}$ *for which* $(f * g) \in M^g_\alpha(\phi)$.

To prove our main result, we need the following:

Lemma 1.2. [10] *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part in* Δ, *then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ *or* $v > 1$, *the equality holds if and only if* $p_1(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$, *then equality holds if and only if* $p_1(z)$ *is* $(1+z^2)/(1-z^2)$ *or one of its rotations. If* $v = 0$, *the equality holds if and only if* $p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$ *or one of its rotations. If* $v = 1$, *the equality holds if and only if* p_1 *is the reciprocal of one of the functions such that the equality holds in the case of* $v = 0$.

Also the above upper bound is sharp, it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

2. FEKETE-SZEGÖ PROBLEM

Our main result is the following:

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by* (1) *belongs to* $M_\alpha(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{B_2}{2(1+3\alpha)} - \dfrac{\mu}{(1+2\alpha)^2}B_1^2 + \dfrac{1}{2(1+3\alpha)(1+2\alpha)}B_1^2 & \text{if } \mu \leq \sigma_1 \\ \dfrac{B_1}{2(1+3\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ -\dfrac{B_2}{2(1+3\alpha)} + \dfrac{\mu}{(1+2\alpha)^2}B_1^2 - \dfrac{1}{2(1+3\alpha)(1+2\alpha)}B_1^2 & \text{if } \mu \geq \sigma_2, \end{cases}$$

where $\sigma_1 := \frac{(1+2\alpha)^2(B_2 - B_1) + (1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}$, $\sigma_2 := \frac{(1+2\alpha)^2(B_2 + B_1) + (1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}$. *The result is sharp.*

Proof. For $f(z) \in M_\alpha(\phi)$, let

$$p(z) := \frac{zf'(z)}{f(z)} + \alpha z^2 \frac{f''(z)}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2}$$

From (2), we obtain $(1+2\alpha)a_2 = b_1$ and $(2+6\alpha)a_3 = b_2 + (1+2\alpha)a_2^2$. Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1+\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{3}$$

and, from this equation (3), we obtain $b_1 = \frac{1}{2}B_1 c_1$ and $b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2$. Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4(1+3\alpha)}\left(c_2 - v c_1^2\right) \tag{4}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{(2\mu-1)+\alpha(6\mu-2)}{(1+2\alpha)^2}B_1\right].$$

Our result now follows by an application of Lemma 1.2. To show that the bounds are sharp, we define the functions $K_\alpha^{\phi_n}$ $(n = 2, 3, \ldots)$ by

$$\frac{z[K_\alpha^{\phi_n}]'(z)}{\left[K_\alpha^{\phi_n}(z)\right]} + \alpha z^2 \frac{z[K_\alpha^{\phi_n}]''(z)}{\left[K_\alpha^{\phi_n}(z)\right]} = \phi(z^{n-1}), \quad K_\alpha^{\phi_n}(0) = 0 = [K_\alpha^{\phi_n}]'(0) - 1$$

and the function F_α^λ and G_α^λ $(0 \le \lambda \le 1)$ by

$$\frac{z[F_\alpha^\lambda]'(z)}{F_\alpha^\lambda(z)} + \alpha z^2 \frac{z[F_\alpha^\lambda]''(z)}{F_\alpha^\lambda(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F^\lambda(0) = 0 = (F^\lambda)'(0) - 1$$

and

$$\frac{z[G_\alpha^\lambda]'(z)}{G_\alpha^\lambda(z)} + \alpha z^2 \frac{z[G_\alpha^\lambda]''(z)}{G_\alpha^\lambda(z)} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G^\lambda(0) = 0 = (G^\lambda)'(0).$$

Clearly the functions $K_\alpha^{\phi n}, F_\alpha^\lambda, G_\alpha^\lambda \in M_\alpha(\phi)$. Also we write $K_\alpha^\phi := K_\alpha^{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_α^λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations. □

REMARK 2.2. *If $\sigma_1 \le \mu \le \sigma_2$, then, in view of Lemma 1.2, the Theorem 2.1 can be improved. Let σ_3 be given by*

$$\sigma_3 := \frac{(1+2\alpha)^2 B_2 + (1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+2\alpha)^2}{2(1+3\alpha)B_1^2}\left[B_1 - B_2 + \frac{(2\mu-1)+\alpha(6\mu-2)}{(1+2\alpha)^2}B_1^2\right]|a_2|^2 \le \frac{B_1}{2(1+3\alpha)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+2\alpha)^2}{2(1+3\alpha)B_1^2}\left[B_1 + B_2 - \frac{(2\mu-1)+\alpha(6\mu-2)}{(1+2\alpha)^2}B_1^2\right]|a_2|^2 \le \frac{B_1}{2(1+3\alpha)}.$$

3. APPLICATION TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

In order to introduce the class $M_\alpha^\lambda(\phi)$, we need the following:

DEFINITION 3.1 (see [3, 4]; see also [8, 9]). *Let the function $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The* fractional derivative *of f of order λ is defined by*

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)}\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}\mathrm{d}\zeta \quad (0 \le \lambda < 1),$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) \quad (\lambda \ne 2, 3, 4, \ldots).$$

The class $M_\alpha^\lambda(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M_\alpha(\phi)$. Note that $M_0^0(\phi) \equiv S^*(\phi)$ and $M_\alpha^\lambda(\phi)$ is the special case of the class $M_\alpha^g(\phi)$ when

$$g(z) = z + \sum_{n=2}^\infty \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n. \tag{5}$$

Let $g(z) = z + \sum_{n=2}^\infty g_n z^n$ $(g_n > 0)$. Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in M_\alpha^g(\phi)$ if and only if $(f*g) = z + \sum_{n=2}^\infty g_n a_n z^n \in M_\alpha(\phi)$, we obtain the coefficient estimate for functions in the class $M_\alpha^g(\phi)$, from the corresponding estimate for functions in the class $M_\alpha(\phi)$. Applying Theorem 2.1 for the function $(f*g)(z) = z + g_2a_2z^2 + g_3a_3z^3 + \cdots$, we get the following Theorem 3.2 after an obvious change of the parameter μ:

THEOREM 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_\alpha^g(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{1}{g_3} & \left[\frac{B_2}{2(1+3\alpha)} - \frac{\mu g_3}{(1+2\alpha)^2 g_2^2}B_1^2 + \frac{1}{2(1+3\alpha)(1+2\alpha)}B_1^2\right] \\ & \text{if } \mu \le \sigma_1 \\ \dfrac{1}{g_3} & \frac{B_1}{2(1+3\alpha)} \text{ if } \sigma_1 \le \mu \le \sigma_2 \\ \dfrac{1}{g_3} & \left[-\frac{B_2}{2(1+3\alpha)} + \frac{\mu g_3}{(1+2\alpha)^2 g_2^2}B_1^2 - \frac{1}{2(1+3\alpha)(1+2\alpha)}B_1^2\right] \\ & \text{if } \mu \ge \sigma_2, \end{cases}$$

where $\sigma_1 := \frac{g_2^2}{g_3}\frac{(1+2\alpha)^2(B_2-B_1)+(1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}$, $\sigma_2 := \frac{g_2^2}{g_3}\frac{(1+2\alpha)^2(B_2+B_1)+(1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}$.

The result is sharp.

Since $(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n$, we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{6}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{7}$$

For g_2 and g_3 given by (6) and (7), Theorem 3.2 reduces to the following:

THEOREM 3.3. *Let the function* $\phi(z)$ *be given by*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots .$$

If $f(z)$ *given by* (1) *belongs to* $M_\alpha^\lambda(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \leq \sigma_1 \\ \dfrac{(2-\lambda)(3-\lambda)}{6}\dfrac{B_1}{2(1+3\alpha)} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \gamma &:= \frac{B_2}{2(1+3\alpha)} - \frac{3(2-\lambda)}{2(3-\lambda)}\frac{\mu B_1^2}{(1+2\alpha)^2} + \frac{1}{2(1+3\alpha)(1+2\alpha)} B_1^2 \\ \sigma_1 &:= \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(1+2\alpha)^2(B_2 - B_1) + (1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2} \\ \sigma_2 &:= \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(1+2\alpha)^2(B_2 + B_1) + (1+2\alpha)B_1^2}{2(1+3\alpha)B_1^2}. \end{aligned} \tag{8}$$

The result is sharp.

REMARK 3.4. *When* $\alpha = 0$, $B_1 = 8/\pi^2$ *and* $B_2 = 16/(3\pi^2)$, *the above Theorem* 3.2 *reduces to a recent result of Srivastava and Mishra* [6, Theorem 8, p. 64] *for a class of functions for which* $\Omega^\lambda f(z)$ *is a parabolic starlike function* [2, 5].

REFERENCES

[1] B. C. CARLSON and D. B. SHAFFER, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15** (1984), 737–745.

[2] A. W. GOODMAN, *Uniformly convex functions*, Ann. Polon. Math., **56** (1991), 87–92.

[3] S. OWA and H. M. SRIVASTAVA, *Univalent and starlike generalized bypergeometric functions*, Canad. J. Math., **39** (1987), 1057–1077.

[4] S. OWA, *On the distortion theorems I*, Kyungpook Math. J., **18** (1978), 53–58.

[5] F. RØNNING, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993), 189–196.

[6] H. M. SRIVASTAVA and A. K. MISHRA, *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Computer Math. Appl., **39** (2000), 57–69.

[7] H. M. Srivastava, A. K. Mishra and M. K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl., **44** (2001), 145–163.
[8] H. M. Srivastava and S. Owa, *An application of the fractional derivative*, Math. Japon., **29** (1984), 383–389.
[9] H. M. Srivastava and S. Owa, *Univalent functions, Fractional Calculus, and Their Applications*, Halsted Press/John Wiley and Songs, Chichester/New York (1989).
[10] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press (1994), 157–169.

Received June 04, 2005

T.N. Shanmugam
Department of Mathematics
College of Engineering
Anna University, Chennai-600 025
Tamilnadu, India
E-mail: shan@annauniv.edu
S. Sivasubramanian
Department of Mathematics
Easwari Engineering College
Ramapuram, Chennai-600 089
Tamilnadu, India
E-mail: sivasaisastha@rediffmail.com
and
M. Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Malaysia
E-mail: maslina@pkrisc.cc.ukm.my