MATHEMATICA, Tome 49 (72), N° 1, 2007, pp. 21–28

EXISTENCE OF VIABLE SOLUTIONS FOR A CLASS OF NONCONVEX DIFFERENTIAL INCLUSIONS WITH MEMORY

AURELIAN CERNEA and VASILE LUPULESCU

Abstract. We prove the existence of viable solutions for an autonomus differential inclusion with memory in the case when the multifunction that define the inclusion is upper semicontinuous compact valued and contained in the Fréchet subdifferential of a ϕ-convex function of order two.

MSC 2000. 34A60.

Key words. Differential inclusion with memory, ϕ-convex function of order two, viable solutions.

1. INTRODUCTION

Differential inclusions with memory, known also as functional differential inclusions, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential inclusions with memory encompasses a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential-difference inclusions and the Volterra inclusions. For a detailed discussion on this topic we refer to [1].

Let $\mathbb{R}^m$ be the m-dimensional euclidean space with the norm $||\cdot||$ and the scalar product $\langle\cdot,\cdot\rangle$. Let σ be a positive number and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma,0],\mathbb{R}^m)$ the Banach space of continuous functions from $[-\sigma,0]$ into $\mathbb{R}^m$ with the norm given by $||x(\cdot)||_\sigma := \sup\{||x(t)||; t\in[-\sigma,0]\}$. For each $t\in[0,\tau]$, we define the operator $T(t) : \mathcal{C}([-\sigma,\tau],\mathbb{R}^m)\to\mathcal{C}_\sigma$ as follows: $(T(t)x)(s) := x(t+s)$, $s\in[-\sigma,0]$. If K is a given nonempty subset in $\mathbb{R}^m$ then we introduce the following set $\mathcal{K} := \{\varphi\in\mathcal{C}_\sigma; \varphi(0)\in K\}$.

For a given multifunction $F : \mathcal{C}_\sigma\to\mathcal{P}(\mathbb{R}^m)$ we consider the following differential inclusion with memory

$$x' \in F(T(t)x) \tag{1.1}$$

and we are interested to find sufficient conditions such that for each $\varphi\in\mathcal{K}$ there exist $\tau>0$ and a solution $x(\cdot) : [-\sigma,\tau]\to\mathbb{R}^m$ of (1.1) satisfying the initial condition

$$T(0)x = \varphi \quad \text{on } [-\sigma,0] \tag{1.2}$$

and the viability constraint

$$x(t)\in K \quad \forall t\in[0,\tau]. \tag{1.3}$$

We recall that a continuous function $x(\cdot) : [-\sigma, \tau] \to \mathbb{R}^m$ is said to be a solution of (1.1) if $x(\cdot)$ is absolutely continuous on $[0, \tau]$ and $x'(t) \in F(T(t)x)$ for almost all $t \in [0, \tau]$.

The existence of solutions of problem (1.1)–(1.3), well known as viable solutions, in the case when F is single valued was studied by many authors. For results and references in this framework we refer to [10].

In general, the results concerning differential inclusions defined by upper semicontinuous multifunctions can be extended to functional differential inclusions. The first viability result for functional differential inclusions was given by Haddad ([8], [9]) in the case when F is upper semicontinuous with convex compact values.

Recently in [5], the situation when the multifunction is not convex valued is considered. More exactly, in [5] it is proved the existence of solutions of problem (1.1)–(1.3) when $F(\cdot)$ is an upper semicontinuous multifunction contained in the subdifferential of a proper convex function $V(\cdot)$.

The aim of the present paper is to relax the convexity assumption on the function $V(\cdot)$ that appear in [5], in the sense that we assume that $F(\cdot)$ is contained in the Fréchet subdifferential of a ϕ-convex function of order two. Since the class of proper convex functions is strictly contained into the class of ϕ-convex functions of order two, our result generalizes the one in [5].

We note that the corresponding viability result for differential inclusions was obtained in [4]. The proof of our main result follows the general ideas in [3] and [9].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. PRELIMINARIES

For $x \in \mathbb{R}^m$ and $r > 0$ let $B(x, r) := \{y \in \mathbb{R}^m; ||y - x|| < r\}$ be the open ball centered in x with radius r, and let $\overline{B}(x, r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; ||\psi - \varphi||_\sigma < r\}$ and $\overline{B}_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; ||\psi - \varphi||_\sigma \le r\}$. For $x \in \mathbb{R}^m$ and for a closed subset $A \subset \mathbb{R}^m$ we denote by $\mathrm{d}(x, A)$ the distance from x to A given by $\mathrm{d}(x, A) := \inf\{||y - x||; y \in A\}$.

Let $\Omega \subset \mathbb{R}^m$ be an open set and let $V : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in \mathbb{R}^m;\ V(x) < +\infty\}$.

Definition 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(\mathbb{R}^m)$, defined as:

$$\partial_F V(x) = \left\{\alpha \in \mathbb{R}^m, \liminf_{y \to x} \frac{V(y) - V(x) - \langle \alpha, y - x\rangle}{||y - x||} \ge 0\right\} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the *Fréchet subdifferential* of V.

We also put $D(\partial_F V) = \{x \in \mathbb{R}^m; \partial_F V(x) \neq \emptyset\}$.

According to [6] the values of $\partial_F V(\cdot)$ are closed and convex.

Definition 2.2. Let $V : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order 2* if there exists a continuous map

$\phi_V : (D(V))^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + \langle \alpha, x - y \rangle - \phi_V(x, y, V(x), V(y))(1 + ||\alpha||^2)||x - y||^2. \tag{2.1}$$

In [3], [6] there are several examples and properties of such maps. For example, according to [3], if $K \subset \mathbb{R}^2$ is a closed and bounded domain, whose boundary is a C^2 regular Jordan curve, the indicator function of K

$$V(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

is ϕ-convex of order 2.

If $K \subset \mathbb{R}^m$, as above, we denote $\mathcal{K} := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}$. We say that a multifunction $F : \mathcal{K} \to \mathcal{P}(\mathbb{R}^m)$ is upper semicontinuous if for every $\varphi \in \mathcal{K}$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\psi) \subset F(\varphi) + B(0, \varepsilon), \quad \forall \psi \in \mathcal{K} \cap B_\sigma(\varphi, \delta).$$

This definition of upper semicontinuous multifunctions is less restrictive than the usual (e.g. Definition 1.1.1 in [1]) and it is equivalent with the upper semicontinuity for compact valued multifunctions (e.g. Proposition 1.1 in [7]).

For a multifunction $F : \mathcal{K} \to \mathcal{P}(\mathbb{R}^m)$ we consider the differential inclusion with memory (1.1) under the following assumptions.

Hypothesis 2.1. (a) K is a locally closed subset in $\mathbb{R}^m$ and $F : \mathcal{K} \to \mathcal{P}(\mathbb{R}^m)$ is upper semicontinuous with compact values.

(b) There exists a proper lower semicontinuous ϕ-convex function of order two $V : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ such that for any $\psi \in \mathcal{K}$

$$F(\psi) \subset \partial_F V(\psi(0)). \tag{2.2}$$

(c) For any $\varphi \in \mathcal{K}$ and for any $v \in F(\varphi)$ the following tangential condition holds:

$$\liminf_{h \downarrow 0} \frac{1}{h} d(\varphi(0) + hv, K) = 0. \tag{2.3}$$

The next technical result proved in [5] is a key tool in the proof of our main result.

Lemma 2.1. *Assume that the Hypotheses* 2.1 (a) *and* (c) *are satisfied. Then, for any $\varphi \in \mathcal{K}$ there exist $r > 0$ and $\tau > 0$ such that $K \cap B(\varphi(0), r)$ is closed and for each $k \in \mathbb{N}^*$ there exist $m(k) \in \mathbb{N}^*$, t_k^p, y_k^p, u_k^p and a continuous function $x_k : [-\sigma, \tau] \to \mathbb{R}^m$ such that for every $p \in \{0, 1, ..., m(k) - 1\}$ we have*

(i) $h_k^p := t_k^{p+1} - t_k^p < \frac{1}{k}$ *and* $t_k^{m(k)-1} \leq \tau < t_k^{m(k)}$,

(ii) $x_k(t) = y_k^p + (t - t_k^p) u_k^p$ *for every* $t \in [t_k^p, t_k^{p+1}]$ *and* $x_k(t) = \varphi(t)$ *for every* $t \in [-\sigma, 0]$,

(iii) $u_k^p \in F(T(t_k^p) x_k) + \frac{1}{k} B$,

(iv) $y_k^p \in K \cap B(\varphi(0), r)$ *and* $T(t_k^p) x_k \in \mathcal{K} \cap B_\sigma(\varphi, r)$.

3. THE MAIN RESULT

We are now able to prove our main result.

THEOREM 3.1. *We assume that Hypothesis* 2.1 *is satisfied. Then, for any* $\varphi \in \mathcal{K}$ *there exists a solution to* (1.1)–(1.3).

Proof. Let φ be arbitrary fixed in $\mathcal{K}$. Since K is locally closed in $\mathbb{R}^m$, there exists $r > 0$ such that $K \cap B(\varphi(0), r)$ is closed. By Proposition 1.1.3 in [1], F is locally bounded; therefore, we can assume that there exists $M > 0$ such that

$$\sup\{||v||; \quad v \in F(\psi),\ \psi \in \mathcal{K} \cap B_\sigma(\varphi, r)\} \leq M. \tag{3.1}$$

We prove that the sequence $\{x_k(\cdot)\}_k$, constructed in Lemma 2.1, has a subsequence that converges to a solution of (1.1).

First, we define the functions $\theta_k : [0,\tau] \to [0,\tau]$ by $\theta_k(t) = t_k^p$ for every $t \in [t_k^p, t_k^{p+1}]$. Since $|\theta_k(t) - t| \leq \frac{1}{k}$ for every $t \in [0,\tau]$, then $\theta_k(t) \to t$ uniformly on $[0,\tau]$. Also, by (ii), (iii) and (iv), for every $k \geq 1$, we have

$$x_k'(t) \in F(T(\theta_k(t))x_k) + \frac{1}{k}B \quad a.e.\ ([0,\tau]), \tag{3.2}$$

$$x_k(\theta_k(t)) \in K \cap B\left(\varphi(0), \frac{r}{4}\right) \quad \forall\, t \in [0,\tau] \tag{3.3}$$

and

$$T(\theta_k(t))x_k \in \mathcal{K} \cap B_\sigma(\varphi, r) \quad \forall\, t \in [0,\tau].$$

Moreover, by (3.1) and (3.2) we have

$$||x_k'(t)|| \leq M + 1 \quad \forall t \in [0,\tau],\ \forall k \geq 1 \tag{3.4}$$

and so $\{x_k'(\cdot)\}_k$ is bounded in $L^2([0,\tau], \mathbb{R}^m)$.

Further on, by (ii), (iii) and (3.3) we have that, for k large enough,

$$\begin{aligned} ||x_k(t) - \varphi(0)|| &\leq ||x_k(t) - x_k(\theta_k(t))|| + ||x_k(\theta_k(t) - \varphi(0)|| \\ &\leq (M+1)|\theta_k(t) - t| + \frac{r}{4} < \frac{r}{4} + \frac{r}{4} < r, \end{aligned}$$

thus $x_k(t) \in B(\varphi(0), r)$, for every $t \in [0,\tau]$ and for every $k \geq 1$. Hence, $\{x_k(\cdot)\}_k$ is bounded in $C([0,\tau], \mathbb{R}^m)$. Moreover, by (3.4), for every $t, s \in [0,\tau]$ we have

$$||x_k(t) - x_k(s)|| \leq \left| \int_s^t ||x_k'(u)|| \mathrm{d}u \right| \leq (M+1)|t-s|$$

and we infer that the sequence $\{x_k(\cdot)\}_k$ is equi-uniformly continuous.

Therefore, by Theorem 0.3.4 in [1] there exists a subsequence, still denote by $\{x_k(\cdot)\}_k$, and an absolutely continuous function $x : [0,\tau] \to \mathbb{R}^m$ such that

(j) $x_k(\cdot)$ converges uniformly to $x(\cdot)$,

(jj) $x_k'(\cdot)$ converges weakly in $L^2([0,\tau], \mathbb{R}^m)$ to $x'(\cdot)$.

Moreover, since for all $k \geq 1$ $x_k = \varphi$ on $[-\sigma, 0]$, we can obviously say that $x_k \to x$ on $[-\sigma, \tau]$, if we extend x in such a way that $x = \varphi$ on $[-\sigma, 0]$. By the fact that x_k converges uniformly to x on $[0,\tau]$ and θ_k converges uniformly to t on $[0,\tau]$ we deduce that $x_k(\theta_k(t)) \to x(t)$ uniformly on $[0,\tau]$. Also, it is clearly that $T(0)x = \varphi$ on $[-\sigma, 0]$.

Further on, let us denote the modulus of continuity of a function ψ on the interval $I \subset \mathbb{R}$ by

$$\omega(\psi, I, \varepsilon) := \sup\{||\psi(t) - \psi(s)||; \quad s,t \in I,\ |s-t| < \varepsilon\}, \quad \varepsilon > 0.$$

Then we have:

$$\begin{aligned} ||T(\theta_k(t))x_k - T(t)x_k||_\sigma &= \sup_{-\sigma \le s \le 0} ||x_k(\theta_k(t)+s) - x_k(t+s)|| \\ &\le \omega\left(x_k, [-\sigma,\tau], \frac{1}{k}\right) \\ \le \omega\left(\varphi, [-\sigma, 0], \frac{1}{k}\right) + \omega\left(x_k, [0,\tau], \frac{1}{k}\right) &\le \omega\left(\varphi, [-\sigma,0], \frac{1}{k}\right) + \frac{M+1}{k} \end{aligned}$$

hence

$$||T(\theta_k(t))x_k - T(t)x_k||_\sigma \le \delta_k \quad \forall k \ge 1, \tag{3.5}$$

where $\delta_k := \omega(\varphi, [-\sigma,0], \frac{1}{k}) + \frac{M+1}{k}$.

Thus, by continuity of φ, we have $\delta_k \to 0$ as $k \to \infty$, hence $||T(\theta_k(t))x_k - T(t)x_k||_\sigma \to 0$ as $k \to \infty$ and so, since the uniform convergence of x_k to x on $[-\sigma,\tau]$ implies

$$T(t)x_k \to T(t)x \quad \text{uniformly on} \quad [0,\tau], \tag{3.6}$$

we deduce that

$$T(\theta_k(t))x_k \to T(t)x \quad \text{in} \quad \mathcal{C}_\sigma. \tag{3.7}$$

Since $T(\theta_k(t))x_k \in \mathcal{K} \cap B_\sigma(\varphi, r)$ for every $t \in [0,\tau]$ and for every $k \ge 1$, thus by (3.7) we have $T(t)x \in \mathcal{K} \cap B_\sigma(\varphi, r)$.

Therefore, by (3.2) and (3.5) we have

$$\mathrm{d}((T(t)x_k, x_k'(t)), \mathrm{graph}(F)) \le \delta_k + \frac{1}{k} \quad \forall k \ge 1. \tag{3.8}$$

By (jj), (3.6) and Theorem 1.4.1. in [1] we obtain that

$$x'(t) \in \mathrm{co}F(T(t)x) \subset \partial_F V(x(t)) \quad a.e.\ ([0,\tau]), \tag{3.9}$$

where co stands for the closed convex hull.

Since the function $t \to x(t)$ is absolutely continuous we apply Theorem 2.2 in [3] and we deduce that there exists $\tau_1 > 0$ such that the mapping $t \to V(x(t))$ is absolutely continuous on $[0, \min\{\tau, \tau_1\}]$ and

$$(V(x(t)))' = \langle x'(t), x'(t) > \quad a.e.\ ([0, \min\{\tau,\tau_1\}]).$$

Without loss of generality we may assume that $\tau = \min\{\tau, \tau_1\}$. Therefore,

$$V(x(\tau)) - V(x(0)) = \int_0^\tau ||x'(t)||^2 \mathrm{d}t. \tag{3.10}$$

On the other hand, since $x_k'(t)=u_k^p$ for every $t\in[t_k^p,t_k^{p+1}]$, by (iii), there exists $b_k^p\in\frac{1}{k}B$ such that

$$u_k^p-b_k^p\in F(T(t_k^p)x_k)\subset\partial_F V(x_k(t_k^p)),\quad \forall k\in\mathbb{N}^* \tag{3.11}$$

and so the properties of Fréchet subdifferential of a ϕ-convex function of order two imply that, for every $p\leq m(k)-2$, and for every $k\in\mathbb{N}^*$ we have

$$\begin{aligned}&V(x_k(t_k^{p+1}))-V(x_k(t_k^p))\geq\langle u_k^p-b_k^p,x_k(t_k^{p+1})-x_k(t_k^p)\rangle-\phi_V(x_k(t_k^{p+1}),\\&x_k(t_k^p),V(x_k(t_k^{p+1})),V(x_k(t_k^p)))(1+||u_k^p-b_k^p||^2)||x_k(t_k^{p+1})-x_k(t_k^p)||^2=\\&=\langle u_k^p,\textstyle\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\rangle-\langle b_k^p,\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\rangle-\phi_V(x_k(t_k^{p+1}),x_k(t_k^p),\\&V(x_k(t_k^{p+1})),V(x_k(t_k^p)))(1+||u_k^p-b_k^p||^2)||x_k(t_k^{p+1})-x_k(t_k^p)||^2,\end{aligned}$$

hence

$$\begin{aligned}&V(x_k(t_k^{p+1})-V(x_k(t_k^p))\geq\textstyle\int_{t_k^p}^{t_k^{p+1}}||x'(t)||^2\mathrm{d}t-\langle b_k^p,\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\rangle\\&-\phi_V(x_k(t_k^{p+1}),x_k(t_k^p),V(x_k(t_k^{p+1})),V(x_k(t_k^p)))\\&\cdot(1+||u_k^p-b_k^p||^2)||x_k(t_k^{p+1})-x_k(t_k^p)||^2.\end{aligned}\tag{3.12}$$

Similarly, if $t\in[t_k^{m(k)-1},\tau]$, then by (i) we have

$$\begin{aligned}&V(x_k(\tau))-V(x_k(t_k^{m(k)-1}))\geq\textstyle\int_{t_k^{m(k)-1}}^{\tau}||x_k'(t)||^2\mathrm{d}t\\&-\langle b_k^{m(k)-1},\textstyle\int_{t_k^{m(k)-1}}^{\tau}x_k'(t)\mathrm{d}t\rangle\\&-\phi_V(x_k(\tau),x_k(t_k^{m(k)-1}),V(x_k(\tau)),V(x_k(t_k^{m(k)-1})))\\&\cdot(1+||u_k^{m(k)-1}-b_k^{m(k)-1}||^2)||x_k(\tau)-x_k(t_k^{m(k)-1})||^2.\end{aligned}\tag{3.13}$$

By adding the $m(k)-1$ inequalities from (3.12) and the inequality from (3.13), we get

$$V(x_k(\tau))-V(x(0))\geq\int_0^\tau||x_k'(t)||^2\mathrm{d}t+\alpha(k)+\beta(k),\tag{3.14}$$

where

$$\alpha(k)=-\sum_{p=0}^{m(k)-2}\langle b_k^p,\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\rangle-\langle b_k^{m(k)-1},\int_{t_k^{m(k)-1}}^{\tau}x_k'(t)\mathrm{d}t\rangle,$$

$$\begin{aligned}&\beta(k)=-\textstyle\sum_{p=0}^{m(k)-2}\phi_V(x_k(t_k^{p+1}),x_k(t_k^p),V(x_k(t_k^{p+1})),V(x_k(t_k^p)))(1+||u_k^p\\&-b_k^p||^2)||x_k(t_k^{p+1})-x_k(t_k^p)||^2-\phi_V(x_k(\tau),x_k(t_k^{m(k)-1}),V(x_k(\tau)),\\&V(x_k(t_k^{m(k)-1})))(1+||u_k^{m(k)-1}-b_k^{m(k)-1}||^2)||x_k(\tau)-x_k(t_k^{m(k)-1})||^2.\end{aligned}$$

We have

$$\begin{aligned}&|\alpha(k)|\leq\textstyle\sum_{p=0}^{m(k)-2}\left|\langle b_k^p,\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\rangle\right|+\left|\langle b_k^{m(k)-1},\int_{t^{m(k)-1}}^{\tau}x_k'(t)\mathrm{d}t\rangle\right|\\&\leq\textstyle\sum_{p=0}^{m(k)-2}||b_k^p||\left|\left|\int_{t_k^p}^{t_k^{p+1}}x_k'(t)\mathrm{d}t\right|\right|+||b_k^{m(k)-1}||\,||\int_{t^{m(k)-1}}^{\tau}x_k'(t)\mathrm{d}t||\leq\frac{(M+1)\tau}{k}.\end{aligned}$$

Denote

$$S := \sup\{\phi_V(x_1, x_2, y_1, y_2);\quad x_i \in K \cap B(\varphi(0), \tfrac{r}{4}),\\ y_i \in [V(\varphi(0)) - 1, V(\varphi(0)) + 1],\quad i = 1, 2\} < +\infty.$$

According to Remark 1.14 and Theorem 1.18 in [6] (or Theorem 2.1 in [3]) $V(\cdot)$ is continuous on $D(V)$. So, we have the following estimation

$$|\beta(k)| \leq \sum_{p=0}^{m(k)-2} S[1 + (M+2)^2] \left|\left| x_k(t_k^{p+1}) - x_k(t_k^p) \right|\right|^2 + S[1 + (M+2)^2] \cdot \\ \cdot \left|\left| x_k(\tau) - x_k(t_k^{m(k)-1}) \right|\right|^2 + S[1 + (M+2)^2] \Big(\sum_{p=0}^{m(k)-2} \left|\left| \int_{t_k^p}^{t_k^{p+1}} x_k'(t)\mathrm{d}t \right|\right|^2 + \\ \left|\left| \int_{t_k^{m(k)-1}}^{\tau} x_k'(t)\mathrm{d}t \right|\right|^2 \Big) \leq S[1 + (M+2)^2] \left(\sum_{p=0}^{m(k)-2} \frac{1}{k} \int_{t_k^p}^{t_k^{p+1}} ||x_k'(t)||^2\, \mathrm{d}t + \right. \\ \left. \frac{1}{k} \int_{t_k^{m(k)-1}}^{\tau} ||x_k'(t)||^2 \mathrm{d}t \right) \leq \frac{1}{k} S[1 + (M+2)^2]\tau(M+1)^2.$$

We infer that $\lim_{k\to\infty} \alpha(k) = \lim_{k\to\infty} \beta(k) = 0$.

Therefore, from (3.14), passing to the limit with $k \to \infty$, we obtain

$$V(x(\tau)) - V(x(0)) \geq \limsup_{k\to\infty} \int_0^{\tau} ||x_k'(t)||^2 \mathrm{d}t. \tag{3.15}$$

By (3.10) and (3.15) we find that

$$\int_0^{\tau} ||x'(t)||^2 \mathrm{d}t \geq \limsup_{k\to\infty} \int_0^{\tau} ||x_k'(t)||^2 \mathrm{d}t$$

and, since $x_k'(\cdot)$ converges weakly in $L^2([0,\tau], \mathbb{R}^m)$ to $x'(\cdot)$, applying Proposition III 30 in [2], we obtain that $x_k'(\cdot)$ converges strongly in $L^2([0,\tau], \mathbb{R}^m)$ to $x'(\cdot)$, hence a subsequence (again denote by) $x_k'(\cdot)$ converges pointwise a.e. to $x'(\cdot)$.

Since, by (3.8) $\lim_{k\to\infty} \mathrm{d}((T(t)x_k, x_k'(t)), \mathrm{graph}(F)) = 0$ and since the graph of F is closed ([1], Proposition 1.1.2), we have that

$$x'(t) \in F(T(t)x) \quad a.e.\ ([0, \tau]).$$

It remains to prove that $x(t) \in \Omega := K \cap B(\varphi(0), r)$ for every $t \in [0, \tau]$. Indeed, by (i), (ii) and (iii) we have $||x_k(t) - y_k^p|| \leq \frac{M+1}{k}$ for every $t \in [0, \tau]$ and by (j) we have $||x_k(t) - x(t)|| \to 0$ as $k \to \infty$. Therefore, since $y_k^p \in \Omega$, we have $d(x(t), \Omega) \leq ||x(t) - y_k^p|| \leq ||x(t) - x_k(t)|| + ||x_k(t) - y_k^p||$, hence, by passing to the limit for $k \to \infty$ we obtain that $d(x(t), \Omega) = 0$, $\forall t \in [0, \tau]$. Since Ω is closed, we obtain that $x(t) \in \Omega$ for all $t \in [0, \tau]$ and thus $x(t) \in K$ for all $t \in [0, \tau]$, which completes the proof. □

Remark 3.1. If in Hypothesis 2.1, $V(\cdot)$ is assumed to be a convex function then from Theorem 3.1 we obtain Theorem 2.2 in [5]. On the other hand, if in Theorem 3.1 the operator $T(t)$ is defined by $T(t)x = x$, then Theorem 3.1 yields the main result in [4].

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Received September 29, 2005

Faculty of Mathematics and Informatics
University of Bucharest
Str. Academiei 14
010014 Bucharest, Romania
e-mail: acernea@math.math.unibuc.ro
Departament of Mathematics
"Constantin Brâncuşi" University
Str. Republicii 1
210152 Târgu Jiu, Romania
e-mail: vasile@utgjiu.ro