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SURVEY ON TRANSLATIONAL REGULARLY VARYING FUNCTIONS

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Abstract. In this paper we introduce three new classes of functions under names translational slowly varying, translational regularly varying and translational rapidly varying functions. All classes have important applications in the study of asymptotic processes. In this sense, Uniform Convergence Theorem, Characterization Theorem and Representation Theorem are the main results of this paper.

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1. INTRODUCTION AND HISTORY

We shall say that a positive, finite and measurable function R, defined on $I_a := [a, \infty)$ for some $a > 0$, is a *regularly varying* function at infinity (denoted by RV) in the sense of Karamata if the limit

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = r(\lambda) \tag{1}$$

is positive and finite for each $\lambda > 0$. It follows immediately that $r(\lambda) = \lambda^\rho$ for some $\rho \in \mathbb{R}$. The number ρ is the *index* of R.

The RV functions of index $\rho = 0$ are called *slowly varying* (denoted by SV) functions and are denoted by L. Their interest lies in the fact that R is a RV function of index ρ if and only if $R(x) = x^\rho L(x)$ on some I_a.

Classes SV and RV of slowly and regularly varying functions were introduced by Jovan Karamata in 1930. In this respect we refer to the books of E. Seneta [10] and Bingham-Goldie-Teugels [2]. Both classes have important consequences in the study of asymptotic processes.

In connection with the preceding, the most important properties of RV functions may be stated as follows:

Dedicated to adventure in year 1930 when Jovan Karamata published in Mathematica (Cluj) a survey on regularly varying functions.

(a) (Characterization Theorem). *If R is a regularly varying function, then the limit $r(\lambda)$ in* (1) *is necessarily of the form λ^ρ for some $-\infty < \rho < \infty$ and for each $\lambda > 0$.*

(b) (Uniform Convergence Theorem). *The relation* (1) *holds uniformly for λ in any compact interval $I \subset (0, \infty)$.*

(c) (Representation Theorem). *There exists a number $b \geq a$ such that for $x \geq b$ we have*

$$R(x) = \exp\left(\alpha(x) + \int_b^x \frac{\beta(t)}{t}\mathrm{d}t\right),$$

where α and β are bounded measurable functions on I_b such that $\alpha(x)$ converges to a real number and $\beta(x) \to \rho$ as $x \to \infty$.

We noticee that RV functions have been introduced by J. Karamata [6]. He proved for continuous function R the crucial of the here mentioned results.

The Uniform Convergence Theorem for measurable SV functions was proved by T. van Aardennee-Ehrenfest, N.G. de Bruijn and J. Korevaar [11], H. Delange [5], W. Matuszewska [8], and Bojanić-Seneta [3].

The Representation Theorem for SV funictions L such that $\log L$ is integrable on every compact subinterval of (a, ∞) was proved in [11]. Finally, the Representation Theorem in the present form, for arbitrary measurable SV functions, was established by N. G. de Bruijn [4]; also and Bojanić-Seneta [3].

In this paper, we shall introduce some new classes of functions which have further applications in the study of asymptotic processes. This facts are closely connected with the Karamata's theory of regularly varying functions.

2. TRANSLATIONAL SLOWLY VARYING FUNCTIONS

A positive, finite and measurable function A, defined on I_a for some $a > 0$, is said to be *translational slowly varying* at infinity (denoted this class by $Tr(SV)$) if the limit

$$\lim_{x\to\infty} \frac{A(x+\lambda)}{A(x)} = 1 \tag{2}$$

for each $\lambda \geq 0$. The most important properties of $Tr(SV)$ functions may be stated as follows:

Theorem 1. (Uniform Convergence Theorem). *If A is a $Tr(SV)$ function, then for every $[a, b]$, $0 < a < b < \infty$, the relation* (2) *holds uniformly with respect to $\lambda \in [a, b]$.*

Proof. Let A be a $Tr(SV)$ function and let f be defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ \log A(x) & \text{if} \quad x \geq a. \end{cases}$$

Then, as is easy to see, f is a measurable function on $\mathbb{R}$ and

$$f(x+\lambda) - f(x) \to 0 \quad (x \to \infty) \tag{3}$$

for every $\lambda \in \mathbb{R}$. If we show that the following fact holds that

$$\sup_{0\le\lambda\le 1} |f(x+\lambda) - f(x)| \to 0 \quad (x \to \infty) \tag{4}$$

the statement will clearly be proved. Suppose that (3) holds and that (4) is not true. Then we can find $\delta > 0$ and sequences (λ_n) and (x_n) such that $\lambda_n \in [0,1]$, $x_n \ge n$, and

$$\big|f(x_n+\lambda_n) - f(x_n)\big| \ge \delta \quad \text{for } n \in \mathbb{N}.$$

Let m^* be the outer measure of subsets of $\mathbb{R}$ and for $0 < \varepsilon < \delta/4$ let

$$M_n := \Big\{t : \sup_{x\ge n} |f(x+t) - f(x)| \le \varepsilon\Big\}.$$

Since $(M_n \cap [0,3])$ is an increasing sequence of subsets of $\mathbb{R}$ converging to $[0,3]$ we obtain

$$\lim_{n\to\infty} m^*(M_n \cap [0,3]) = 3,$$

and hence we can find $s \in \mathbb{N}$ such that $m^*(M_s \cap [0,3]) \ge 5/2$. Let

$$B = \big\{t : |f(t) - f(x_s)| \le \varepsilon\big\} \cap [x_s, x_s+4]$$
$$C = \big\{t : |f(t) - f(x_s+\lambda_s)| \le \varepsilon\big\} \cap [x_s, x_s+4],$$

then B and C are disjoint measurable subsets of $[x_s, x_s+4]$ and $m(B)+m(C) \le 4$.

If we denote by X and Y the set $M_s \cap [0,3]$ translated by x_s and $x_s+\lambda_s$, respectively, then it is easy to see that $X \subset B$ and $Y \subset C$. Hence, consequently,

$$\frac{5}{2} \le m^*\big(M_s \cap [0,3]\big) = m^*(X) \le m^*(B),$$
$$\frac{5}{2} \le m^*\big(M_s \cap [0,3]\big) = m^*(Y) \le m^*(C),$$

and thus so $m(B)+m(C) \ge 5$, which is impossible in view of $m(B)+m(C) \le 4$.

For the case of an arbitrary interval $[a,b]$ define $\tilde f(x) = f((b-a)x)$. Then

$$f(x+\lambda) - f(x) = \tilde f(y+\mu) - \tilde f(y) + f(x-a) - f(x)$$

where $y = (x-a)/(b-a)$, $\mu = (\lambda-a)/(b-a)$, so that $y \to \infty$ if and only if $x \to \infty$; i.e., $\lambda \in [a,b]$ if and only if $\mu \in [0,1]$. The proof is complete. □

We notice that proof of Theorem 1 follows from the same ideas as in the proof of Lemma 1 in Bojanić-Seneta [3]. For a proof of Theorem 1 see: Tasković [12].

The following statement gives an integral representation theorem for functions from the class $Tr(SV)$.

Theorem 2. (Representation Theorem). *If A is a $Tr(SV)$ function, then there exists a positive number $b \geq a$ such that for all $x \geq b$ we have*

$$A(x) = \mu(x) \exp\left(\int_b^x \varepsilon(t)\mathrm{d}t\right), \tag{5}$$

where $\mu(x)$ is a positive and measurable function on I_b such that $\mu(x) \to c \in (0, \infty)$ as $x \to \infty$, and $\varepsilon(x)$ is a continuous function on I_b such that $\varepsilon(x) \to 0$ (as $x \to \infty$). Conversely, if a function A of the form (5), then A is a $Tr(SV)$ function.

Proof. Let A be a $Tr(SV)$ function and let f be defined by $f(t) = \log A(t)$. Then, as is easy to see, f is a measurable function for $t \geq b$, where I_b is the domain of A, and f satisfies the following condition that

$$f(t+\lambda) - f(t) \to 0 \quad (t \to \infty)$$

uniformly with respect to $\lambda \in [0, 1]$. Define the function $f_1(t)$ by

$$f_1(t) = f(n) + 6\big[f(n+1) - f(n)\big] \int_0^{t-n} y(1-y)dy$$

for $n \leq t \leq n+1$, and all $n \in \mathbb{N} \cup \{0\}$. Since

$$f_1'(t) = 6\big[f(n+1) - f(n)\big](t-n)(n+1-t)$$

for $n \leq t \leq n+1$, it follows that, for all $n \in \mathbb{N} \cup \{0\}$, $f_1'(n) = 0$. Hence, $f_1'(t)$ is continuous and

$$\big|f_1'(t)\big| \leq \frac{3}{2}\big|f(n+1) - f(n)\big|$$

for $n \leq t \leq n+1$. Also we obtain that

$$\begin{aligned}\big|f_1(t) - f(t)\big| &\leq \big|f(n) - f(t)\big| + 6\big|f(n+1) - f(n)\big|\left|\int_0^{t-n} y(1-y)dy\right| \\ &= \big|f(n) - f(t)\big| + \big|f(n+1) - f(n)\big|(t-n)^2(2n+3-2t) \\ &\leq \big|f(t) - f(n)\big| + 3\big|f(n+1) - f(n)\big|,\end{aligned}$$

and thus, as $t \to \infty$, we have

$$f_1'(t) \to 0 \quad \text{and} \quad f_1(t) - f(t) \to 0.$$

Hence, since the function $x \mapsto f_1(x)$ it has continuous derivative for $x \geq b$, we obtain

$$f_1(x) = \int_b^x \varphi(t)\mathrm{d}t + \text{ constant}, \tag{6}$$

where $x \mapsto \varphi(x)$ is a contunuous function for $x \geq b$. If to differentiate (6) we have

$$f_1'(x) = \varphi(x) \quad \text{for } x \geq b,$$

i.e., from the preceding facts, we obtain

$$\varphi(x) = \varepsilon(x) \quad \text{for } x \geq b, \tag{7}$$

with the function $x \mapsto \varepsilon(x)$ which is a continuous function on I_b such that $\varepsilon(x) \to 0$ as $x \to \infty$. From (6) and (7) we obtain

$$A_1(x) = \exp(f_1(x)) = C \exp\left(\int_b^x \varepsilon(t)\mathrm{d}t\right),$$

where C is a constant. Also, from the preceding facts, we have

$$\begin{aligned}\mu(x) = \frac{A_1(x)}{A(x)} &= \frac{C\exp\big(f_1(x)\big)}{\exp\big(f(x)\big)}\\ &= C\exp\big(f_1(x) - f(x)\big) \to C \quad (x \to \infty).\end{aligned}$$

Hence

$$A(x) = \mu(x) \exp\left(\int_b^x \varepsilon(t)\mathrm{d}t\right),$$

where $\mu(x)$ and $\varepsilon(t)$ are as required the Representation Theorem.

Conversely, according to these conditions, every the function $A(x)$, with the representation (5), is a measurable function on I_b and for every $\lambda \geq 0$ holds

$$\begin{aligned}\left|\frac{A(x+\lambda)}{A(x)} - 1\right| &= \left|\exp\left(\int_x^{x+\lambda} \varepsilon(t)\mathrm{d}t\right) - 1\right|\\ &\leq \exp\left(\lambda \max_{t\geq x} \varepsilon(t)\right) - 1 \to 0 \quad (x \to \infty);\end{aligned} \tag{8}$$

and with this the proof is complete. □

Remark 1. We notice that from the preceding proof of part (8) we have a directly and a simple proof, in the proper manner, of the Theorem 1.

A brief proof of this statement may be found in: Tasković [12].

A subclass of $Tr(SV)$. A positive measurable function f belongs to the class $Tr(Z)$ if for every $\delta > 0$, $x \mapsto \mathrm{e}^{\delta x} f(x)$ is an increasing, and $x \mapsto \mathrm{e}^{-\delta x} f(x)$ is a decreasing function for x large enough. We notice that is $Tr(Z) \subset Tr(SV)$. Indeed, let $f \in Tr(Z)$ and let $\lambda \geq 0$, then for every $\delta > 0$ and for sufficiently large x we obtain

$$\mathrm{e}^{-\delta\lambda} \leq \mathrm{e}^{-\delta\lambda}\frac{\mathrm{e}^{\delta(x+\lambda)}f(x+\lambda)}{\mathrm{e}^{\delta x}f(x)} = \frac{f(x+\lambda)}{f(x)} = \mathrm{e}^{\delta\lambda}\frac{\mathrm{e}^{-\delta(x+\lambda)}f(x+\lambda)}{\mathrm{e}^{-\delta x}f(x)} \leq \mathrm{e}^{\delta\lambda},$$

and thus

$$\mathrm{e}^{-\delta\lambda} \leq \liminf_{x\to\infty} \frac{f(x+\lambda)}{f(x)} \leq \limsup_{x\to\infty} \frac{f(x+\lambda)}{f(x)} \leq \mathrm{e}^{\delta\lambda},$$

i.e., as $\delta \to 0$ we have

$$\lim_{x\to\infty} \frac{f(x+\lambda)}{f(x)} = 1,$$

i.e., we have $f \in Tr(SV)$. This means that functions of the class $Tr(Z)$ are a subclass of the class of all translational slowly varying functions.

Remark 2. Here the role of the preceding class of functions $Tr(Z)$ is similar with the role Zygmund's class of functions in the Karamata's theory of regularly varying functions, see: Seneta [10].

3. TRANSLATIONAL SLOWLY VARYING FUNCTIONS WITH REMAINDER TERM

We notice that a typical result of the Abelian nature can be stated as follows. Let k be a measurable function such that

$$\int_0^1 t^{-\delta}\big|k(t)\big|\mathrm{d}t < \infty \quad \text{and} \quad \int_1^\infty \mathrm{e}^{\delta t}\big|k(t)\big|\mathrm{d}t < \infty$$

for some $0 < \delta < \infty$, then for every translational slowly varying function A we have

$$\lim_{x\to\infty}\int_0^\infty k(t)\frac{A(x+t)}{A(x)}\mathrm{d}t = \int_0^\infty k(t)\mathrm{d}t. \tag{9}$$

If more precise result than (9) is desired, then the class of $Tr(SV)$ functions must be suitably restricted. Namely, if one considers a specific $Tr(SV)$ function then a more precise asymptotic relation than (2) is usually available, in most cases.

In this sense, let ψ be a positive decreasing function on $[0,\infty)$ such that $\psi(x) \to 0$ (as $x \to \infty$) and $\mathrm{e}^{-\delta x}/\psi(x)$ is *eventually decreasing* (i.e., there exists $x_0 \geq 0$ such that $x_2 \geq x_1 \geq x_0$ implies $f(x_2) \leq f(x_1)$, where $f(x) = \mathrm{e}^{-\delta x}/\psi(x)$) for some $0 < \delta < \infty$.

A positive measurable function A on $[0,\infty)$ is called a $Tr(SV)$ function with remainder ψ (denoted this class by $Tr(SVr)$) if

$$\frac{A(x+\lambda)}{A(x)} = 1 + O\big(\psi(x)\big), \quad \text{as } x \to \infty, \tag{10}$$

for every $\lambda \geq 0$.

The basic result, the totally analog of Uniform Convergence Theorem, can be stated as follows.

Theorem 3. *Let $A \in Tr(SVr)$, then the relation* (10) *holds uniformly in λ on every compact subinterval of* $(0,\infty)$*, i.e.,*

$$\sup_{a\leq\lambda\leq b}\left|\frac{A(x+\lambda)}{A(x)} - 1\right| = O\big(\psi(x)\big), \quad \text{as } x \to \infty,$$

where $0 < a < b < \infty$.

The proof of this result is a totally analogous with the proof of Theorem 1. For a brief proof of this statement see: Tasković [12].

Another basic property of $Tr(SVr)$ functions with remainder term is the following Representation Theorem.

Theorem 4. *A positive measurable function A on $[0,\infty)$ is in the class $Tr(SVr)$ if and only if there exists a number $b>0$ such that for $x\geq b$ holds*

$$A(x)=\mu(x)\exp\left(\int_b^x \varepsilon(t)\mathrm{d}t\right),$$

where $\mu(x)$ is a positive and measurable function on I_b such that $\mu(x)\to c\in(0,\infty)$ as $x\to\infty$, and $\varepsilon(x)$ is a continuous function on I_b such that $\varepsilon(x)\to 0$ as $x\to\infty$, and satisfying as $x\to\infty$ the following asymptotic relations

$$\mu(x)=c+O\big(\psi(x)\big)\quad \textit{and}\quad \varepsilon(x)=O\big(\psi(x)\big).$$

The proof of this statements is a totally analogous with the preceding proof of Theorem 2.

4. TRANSLATIONAL REGULARLY VARYING FUNCTIONS

A positive, finite and measurable function f, defined on I_a for some $a>0$, is said to be *translational regularly varying* at infinity (denoted this class by $Tr(RV)$) if the limit

$$\lim_{x\to\infty}\frac{f(x+\lambda)}{f(x)}=h(\lambda) \tag{11}$$

is positive and finite for each $\lambda\geq 0$.

A function f is said to be *translational regularly varying* at zero if $f(1/x)$ is translational regularly varying at infinity.

Translational regular variation can now be defined at any finite point a by shifting the origin of the function to this point.

It is thus apparent that it suffices to develop the theory of translation regularly variation at infinity, which we shall do, frequently omitting the words "at infinity" in the sequel.

The fundamental statement of this section is the following, since it shows that $h(\lambda)$ must have the form $\mathrm{e}^{\sigma\lambda}$, and so the f considered must be translational regularly varying in the previously defined sense.

Theorem 5. (Characterization Theorem)*. If f is a translational regularly varying function (i.e., $f\in Tr(RV)$), then the limit $h(\lambda)$ in* (11) *is necessarily of the form $\mathrm{e}^{\sigma\lambda}$ for some $-\infty<\sigma<\infty$ and for each $\lambda\geq 0$.*

The number σ is the *index* of f. The $Tr(RV)$ functions of index $\sigma=0$ are called *translational slowly varying* ($Tr(SV)$) functions and are denoted by A. Their interest lies in the fact that f is a $Tr(RV)$ function of index σ if and only if $f(x)=\mathrm{e}^{\sigma x}A(x)$ on some I_b.

We shall proceed by proving Theorem 5 via a well-know variant statement of Cauchy in the following form.

Theorem 6. *Let $r(y)$ be a real measurable function defined on I_b for some $b>0$. If*

$$r(y+\mu)-r(y)\to\rho(\mu)\quad \textit{as } y\to\infty,$$

with finite $\rho(\mu)$ *and for each* $\mu \in \mathbb{R}$*, then*

$$\frac{r(y)}{y} \to \frac{\rho(\mu)}{\mu} = \sigma \quad \text{as } y \to \infty$$

for each $\mu \neq 0$*. Consequently, then* $\rho(\mu) = \sigma\mu$ *for* $\mu \in \mathbb{R}$*.*

A brief variant proof of this statement based on Césaro limit of a sequence may be found in E. Seneta [10].

As an immediate application of Theorem 6, as a directly consequence, putting $r(y) = \log f(y)$, $\lambda = \mu$ and $\rho(\mu) = \log h(\mu)$, we obtain the following essential result.

THEOREM 7. *If* $f \in Tr(RV)$*, then there exists a real number* σ *such that for every* $\lambda \geq 0$ *we have that*

$$\frac{f(x+\lambda)}{f(x)} \to \mathrm{e}^{\sigma\lambda} \quad \text{as } x \to \infty$$

and such that

$$\frac{\log f(x)}{x} \to \sigma \quad \text{as } x \to \infty.$$

In connection with preceding facts, in further, from Theorem 7 we have that every translational regularly varying function f has the representation in the form

$$f(x) = \mathrm{e}^{\sigma x} A(x) \quad \text{for } x \geq b,$$

where $b \geq a$, where $\sigma \in \mathbb{R}$ and $A(x)$ is a translational slowly varying function.

In connection with this, from the preceding section and this facts, we have the following fundamental statement.

THEOREM 8. (Representation Theorem). *A function* $f \in Tr(RV)$ *if and only if there exist* $\sigma \in \mathbb{R}$ *and a positive number* $b \geq a$ *such that for all* $x \geq b$ *we have*

$$f(x) = \mu(x) \exp\left(\sigma x + \int_b^x \varepsilon(t)\mathrm{d}t\right),$$

where $\mu(x)$ *is a positive and measurable function on* I_b *such that* $\mu(x) \to c \in (0,\infty)$ *as* $x \to \infty$*, and* $\varepsilon(x)$ *is a continuous function on* I_b *such that* $\varepsilon(x) \to 0$ *(as* $x \to \infty$*).*

Now, from Theorem 8, as an immediate consequence, we obtain the following statement on uniformity of convergence in the following sense.

The following statement, the analogue of Theorem 1, ensures, under measurability of f, uniformity of convergence of finite intervals in (11).

THEOREM 9. (Uniform Convergence Theorem). *If* f *is a* $Tr(RV)$ *function, then the relation* (11) *holds uniformly for* λ *in any compact interval* $I \subset (0,\infty)$*.*

The proof of this statement is analogous to the proof of Theorem 1.

Annotation 1. In connection with the Characterization Theorem we notice that from

$$\frac{f(x+\lambda+\gamma)}{f(x)} = \frac{f(x+\lambda+\gamma)}{f(x+\lambda)}\frac{f(x+\lambda)}{f(x)}$$

there follows, as $x \to \infty$,

$$h(\lambda+\gamma) = h(\gamma)h(\lambda)$$

for all nonnegative λ and γ.

This is a form of the Cauchy (or Hamel) functional equation on the nonnegative real numbers, for a function $h > 0$, which, being a pointwise limit of measurable functions, is measurable.

It is known (see: J. Aczél [1]) that under these conditions the only solutions are of the form $\mathrm{e}^{\sigma\lambda}$ for $-\infty < \sigma < \infty$.

Based on the above facts the proof of the preceding statement , as and Theorem 5, we can give also serve as an illustration of the use of Lusin's Theorem in the present setting, which with Egorov's Theorem and Steinhaus's Theorem, appear to be in the natural tools for the present theory.

Example 1. The function $A(x) = \log(x+3)$ for $x \geq 0$ belongs to the class $Tr(SV)$; also, the function

$$A(x) = \frac{1}{x}\int_1^x \frac{\mathrm{d}t}{1+\log t} \quad \text{for } x \geq 1$$

belongs to the class of $Tr(SV)$. On the other hand, the function $f(x) = e^x$ for $x \in \mathbb{R}$ belongs to the class $Tr(RV)$, but $\lim_{x\to\infty}\big(f(\lambda x)/f(x)\big)$ does not exist, for example, for $\lambda = 3$. Hence $f \notin RV$.

Annotation 2. We notice, if f is a $Tr(RV)$ function of index σ, then, from the preceding facts and results, the following statements hold:

(a) $\displaystyle\lim_{x\to\infty}\frac{\log f(x)}{x} = \sigma$.

(b) The function $\log f(x)$ is locally bounded on I_b for some $b \geq a$.

(c) $\lim_{x\to\infty} \mathrm{e}^{-\tau x} f(x) = \infty$ for $\tau < \sigma$.

(d) $\lim_{x\to\infty} \mathrm{e}^{-\tau x} f(x) = 0$ for $\tau > \sigma$.

(e) For each pair of real numbers τ and ρ with the property $\tau < \sigma < \rho$, the following facts hold:[1]

$$\inf_{t\geq x}\left\{\mathrm{e}^{-\tau t} f(t)\right\} \sim \mathrm{e}^{-\tau x} f(x) \quad \text{as } x \to \infty,$$

$$\sup_{t\geq x}\left\{\mathrm{e}^{-\rho t} f(t)\right\} \sim \mathrm{e}^{-\rho x} f(x) \quad \text{as } x \to \infty.$$

(f) For each $\tau < \sigma$ the following fact holds that is

$$\lim_{x\to\infty}\frac{1}{\mathrm{e}^{-\tau x} f(x)}\int_b^x \mathrm{e}^{-\tau t} f(t)\mathrm{d}t = \frac{1}{\sigma-\tau}.$$

[1] $g(x) \sim r(x)$ means $g(x)/r(x) \to 1$ as $x \to \infty$.

(g) For each $\tau > \sigma$ the following fact holds that is

$$\lim_{x\to\infty} \frac{1}{\mathrm{e}^{-\tau x} f(x)} \int_x^\infty \mathrm{e}^{-\tau t} f(t)\mathrm{d}t = \frac{1}{\tau - \sigma}.$$

5. TRANSLATIONAL RAPIDLY VARYING FUNCTIONS

We say that a measurable function $f : [0, \infty) \to (0, \infty)$, is *translational rapidly varying of index* ∞ (denoted this class by $Tr(R_\infty)$) if

$$\lim_{x\to\infty} \frac{f(x+\lambda)}{f(x)} = \infty \tag{12}$$

for every $\lambda > 0$, and is *translational rapidly varying of index* $-\infty$ (denoted this class by $Tr(R_{-\infty})$) if

$$\lim_{x\to\infty} \frac{f(x+\lambda)}{f(x)} = 0$$

for every $\lambda > 0$. The most important properties of translational rapidly varying functions may be stated as follows:

Theorem 10. (Uniform Convergence Theorem for $Tr(R_\infty)$). *Let f be measurable and positive, and assume that for some $a > 0$ is*

$$\lim_{x\to\infty} \frac{f(x+\lambda)}{f(x)} = \infty \tag{13}$$

for every $\lambda \geq a$. Then (13) *holds uniformly in λ over every interval $[b, \infty)$ for $b > a$.*

The proof of this statement is a totally analogous to the proofs of Theorems 1, 3 and 9.

In connection with the preceding, likewise we denote the class of measurable f whose indices

$$r(f) := \inf\left\{r \in \mathbb{R} : \mathrm{e}^{-rx} f(x) \sim \phi(x) \quad \text{for some nonincreasing } \phi\right\}$$

and

$$d(f) := \sup\left\{d \in \mathbb{R} : \mathrm{e}^{-dx} f(x) \sim \phi(x) \quad \text{for some nondecreasing } \phi\right\}$$

are both ∞ by $Tr(KR_\infty)$. We notice that $Tr(KR_\infty) \subset Tr(R_\infty)$.

Theorem 11. (Representation Theorem for $Tr(KR_\infty)$). *A function $f \in Tr(KR_\infty)$ if and only if there exist $\sigma \in \mathbb{R}$ such that for all $x \geq 0$ we have*

$$f(x) = \exp\left(\sigma x + \mu(x) + \eta(x) + \int_0^x \xi(t)\mathrm{d}t\right) \tag{14}$$

where the measurable functions $\mu(x)$, $\eta(x)$ and $\xi(x)$ are such that $\mu(x)$ is nondecreasing, $\eta(x) \to 0$ as $x \to \infty$, and $\xi(x) \to \infty$ as $x \to \infty$.

Proof. If f is given by (14), then (12) holds and for $\lambda = 0$ is $\psi(0) = 1$, where

$$\psi(\lambda) := \limsup_{x\to\infty} \sup_{\mu\in[0,\lambda]} \frac{f(x)}{f(x+\mu)},$$

i.e., $f \in Tr(KR_\infty)$.[2] Conversely, assume in further, that $d(f) = \infty$, i.e., that is

$$\infty = \sup\Big\{d \in \mathbb{R} : \mathrm{e}^{-dx} f(x) \sim \phi(x) \text{ for some nondecreasing } \phi\Big\},$$

then for every real d we have $\mathrm{e}^{-dx}f(x) \sim \phi_d(x)$, where $\phi_d(x) := \inf_{y\geq x} \mathrm{e}^{-dy} f(y)$ is increasing. Write $h(x) := \log f(x)$ as usual, and set $k_d(x) := \log \phi_d(x)$. Then

$$h(x) = k_d(x) + b_d(x) + dx \quad (\text{for } x \geq 0), \tag{15}$$

where $k_d(x)$ is nondecreasing and $b_d(x) \to 0$ as $x \to \infty$. Let $x_0 := 0$. For each $n \in \mathbb{N}$ find $x_n > x_{n-1} + 1$ such that $|b_n(x)| \leq 2^{-n}$ for every $x \geq x_n$. Thus we have that $C := b_0(x) + \sum_{n=0}^{\infty}(b_n(x_{n+1}) - b_n(x_n))$ is finite. For $k \in \mathbb{N} \cup \{0\}$ we define

$$\mu(x) := k_0(0) + \sum_{j=0}^{k-1}\Big(k_j(x_{j+1}) - k_j(x_j)\Big) + k_k(x) - k_k(x_k) + C$$

for $x_k \leq x \leq x_{k+1}$ and

$$\eta(x) := b_0(0) + \sum_{j=0}^{k-1}\Big(b_j(x_{j+1}) - b_j(x_j)\Big) + b_k(x) - b_k(x_k) - C$$

for $x_k \leq x \leq x_{k+1}$ and $\zeta(x) := \zeta_k(x)$ for $x_k \leq x < x_{k+1}$. Then $\mu(x)$ is a nondecreasing function, $\eta(x) \to 0$ (as $x \to \infty$), and $\zeta(x) \to \infty$ (as $x \to \infty$). Thus for $k \in \mathbb{N} \cup \{0\}$, for $\sigma \in \mathbb{R}$, and $x_k \leq x < x_{k+1}$, by (15) we obtain

$$\begin{aligned} h(x) &= h(0) + \sum_{j=0}^{k-1}\Big(h(x_{j+1}) - h(x_j)\Big) + h(x) - h(x_k) = \\ &= \mu(x) + \eta(x) + \int_0^x \zeta(t)\mathrm{d}t = \\ &= \sigma x + \mu(x) + \eta(x) + \int_0^x \Big(\zeta(t) - \sigma\Big)\mathrm{d}t = \\ &= \sigma x + \mu(x) + \eta(x) + \int_0^x \xi(t)\mathrm{d}t, \end{aligned}$$

where $\xi(t) := \zeta(t) - \sigma \to \infty$ (as $x \to \infty$). Since $h(x) = \log f(x)$, thus from the preceding facts we have that the representation for f follows. The proof is complete. □

A brief similar proof of this statement may be found in: Tasković [12].

[2] We notice that $f \in Tr(KR_\infty)$ if and only if $\psi(0) = 1$ and $f \in Tr(R_\infty)$.

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