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EXISTENCE OF INVARIANT APPROXIMATION RESULT IN LOCALLY CONVEX SPACE

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Abstract. A fixed point theorem of Hadzic [5] is generalized to locally convex spaces and the new result is applied to extend a recent result on invariant approximation of Jungck and Sessa [8] and Mukherjee and Som [11] for non-convex condition of domain and without affineness condition of mappings. Some known results [1], [6], [13] and [16] are also extended and improved. A property known as *Property* Γ is defined to restore the affineness nature of mapping.

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1. INTRODUCTION

Fixed point theorems have been used at many places in approximation theory. One of the places is, while existence of best approximation is proved. By now, a number of results have been developed using fixed point theorems to prove the existence of best approximation. An excellent reference can be seen in [17].

In 1963, Meinardus [10] was the first who observed the general principle and employed a fixed point theorem to establish the existence of an invariant approximation. Afterwards in 1969, Brosowski [1] obtained the following generalization of Meinardus's result.

THEOREM 1.1. *Let X be a normed space and $T : X \to X$ be a linear and nonexpansive operator. Let M be a T-invariant subset of X and $x_0 \in F(T)$. If D, the set of best approximations of x_0 in M, is nonempty compact and convex, then there exists a y in D which is also a fixed point of T.*

Using a fixed point theorem, Subrahmanyam [18] obtained the following generalization of the above mentioned theorem of Meinardus [10].

THEOREM 1.2. *Let X be a normed space. If $T : X \to X$ is a nonexpansive operator with a fixed point x_0, leaving a finite dimensional subspace M of X invariant, then there exists a best approximation of x_0 in M which is also a fixed point of T.*

In 1979, Singh [14] observed that the linearity of mapping T and the convexity of the set D of best approximation of x_0 in Theorem 1.1, can be relaxed and proved the following extension of it.

THEOREM 1.3. *Let X be a normed space, $T : X \to X$ be a nonexpansive mapping, M be a T-invariant subset of X and $x_0 \in F(T)$. If D is nonempty compact and starshaped, then there exists a best approximation of x_0 in M which is also a fixed point of T.*

In a subsequent paper, Singh [15] also observed that only the nonexpansiveness of T on $D' = D \cup \{x_0\}$ is necessary for the validity of Theorem 1.3. Further in 1982, Hicks and Humpheries [6] have shown that Theorem 1.3 remains true, if $T : M \mapsto M$ is replaced by $T : \partial M \mapsto M$, where ∂M, denotes the boundary of M. Furthermore, Sahab, Khan and Sessa [13] generalized the result of Hicks and Humpheries [6] and Theorem 1.3 using two mappings, one linear and other nonexpansive for commuting mappings and established the following result of common fixed point for best approximation in setup of normed linear space. They took this idea from Park [12].

THEOREM 1.4. *Let I and T be self maps of X with $x_0 \in F(I) \cap F(T)$, $M \subset X$ with $T : \partial M \mapsto M$, and $p \in F(I)$. If D, the set of best approximation is compact and p-starshaped, $I(D) = D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{x_0\}$, then I and T have a common fixed point in D.*

In an other paper, Jungck and Sessa [8] further weakened the hypothesis of Sahab, Khan and Sessa [13] by replacing the condition of linearity by affineness, to prove the existence of best approximation in normed linear space. However, they used weak continuity of the mapping for such purpose in the second result.

Here it is important to remark that Dotson [3] proved the existence of fixed point for nonexpansive mapping. He further extended his result without starshapedness under non-convex condition [4]. This idea was utilized by Mukherjee and Som [11] to prove existence of fixed point and then to apply it for proving existence of best approximation. In this way, they extended the result of Singh [14] without starshapedness condition.

In this paper, we first derive a common fixed point result in locally convex space which generalizes the result of Hadzic [5]. This new result is used to prove another fixed point result for invariant approximation. By doing so, we in fact, extend and improve the results of Jungck and Sessa [8] and Mukherjee and Som [11] by increasing the number of mappings and more general nonexpansive mappings for non-convex condition of domain and without affineness condition of mappings. Some known results of Brosowski [1], Hicks and Humpheries [6], Sahab, Khan and Sessa [13] and S.P.Singh [14] are also generalized and improved by increasing the number of mappings and by considering generalized nonexpansive mapping on locally convex spaces. For this purpose, we use the concept given by Köthe [9] and Tarafdar [19]. In this way, we have tried to give a new direction to the line of investigation initiated [1].

2. PRELIMINARIES

In the sequel (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$, $n \in \mathbb{N}$, and $U_{\alpha_i} = \{x \in E : p_{\alpha_i}(x) \leq 1\}$, forms a base of neighbourhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined of E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see page 203 [9]). A subset M of E is τ-bounded if and only if each p_α is bounded on M.

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x \in M : p_\alpha(x) \leq 1\}$. Clearly, $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ- bounded, τ-closed, absolutely convex and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|.\|_B$ on E_B. Thus, $(E_B, \|.\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$. (for detail, see [9, 19]).

Definition 2.1. Let I and T be selfmaps on M. The map T is called
(i) $A^*(\tau)$-nonexpansive if for all $x, y \in M$

$$p_\alpha(Tx - Ty) \leq p_\alpha(x - y),$$

for each $p_\alpha \in A^*(\tau)$.
(ii)$A^*(\tau)$-I-nonexpansive if for all $x, y \in M$

$$p_\alpha(Tx - Ty) \leq p_\alpha(Ix - Iy),$$

for each $p_\alpha \in A^*(\tau)$.

For simplicity, we shall call $A^*(\tau)$-nonexpansive ($A^*(\tau) - I$- nonexpansive) maps to be nonexpansive (I-nonexpansive).

Definition 2.2. Let M be a subset of (E,τ). Let $x_0 \in E$. We denote by $P_M(x_0)$ the set of best M-approximant to x_0, i.e. if $P_M(x_0) = \{y \in M : p_\alpha(y - x_0) = d_{p_\alpha}(x_0, M)$ for all $p_\alpha \in A^*(\tau)\}$, where

$$d_{p_\alpha}(x_0, M) = \inf\{p_\alpha(x_0 - z) : z \in M\}.$$

Definition 2.3. The map $T : M \to E$ is said to be demiclosed at 0 if for every net $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

We give the definition providing the notion of contractive jointly continuous family introduced by Dotson [4] in locally convex space.

Definition 2.4. [4] Let $F = \{f_x\}_{x \in M}$ be a family of function from $[0,1]$ into M such that $f_x(1) = x$ for each $x \in M$, where M is a subset of (E, τ).

The family F is said to be contractive, if there exists a function $\phi : (0,1) \to (0,1)$ such that $p_\alpha(f_x(t) - f_y(t)) \leq \phi(t) p_\alpha(x-y)$, for all $x, y \in M$, all $t \in (0,1)$ and all $p_\alpha \in A^*(\tau)$. The family F is said to be jointly continuous if $t \to t_0$ in [0,1] and $x \to x_0$ in M, then $f_x(t) \to f_{x_0}(t_0)$. Also, if T is a map from M into itself, then for any $x \in X$, $f_{Tx}(t) \subseteq Tx$ for all $t \in [0,1]$. The family F is called jointly weakly continuous in (x,t) provided $f_x(t) \to^w f_{x_0}(t_0)$ whenever $x \to^w x_0$ in M and $t \to t_0$ in [0,1].

Now, we give the definition *Property* Γ on contractive jointly continuous family F.

Definition 2.5. A self mapping A of M is said to satisfy the *Property* Γ, if for any $t \in [0,1]$, for all $x \in M$ and for all $f_x \in F$, we have $A(f_x(t)) = f_{Ax}(t)$, where $\{f_x(t)\}$ is defined as above.

Throughout, this paper $F(T)$ denotes the fixed point set of mapping T.

We also use the following result due to Hadzic [5]:

Theorem 2.6. [5] *Let (X,d) be a complete metric space. Let $S, T : X \to X$ be two continuous maps and $\Re$ a family of self-mappings $A : X \to S(X) \cap T(X)$ such that*

(i) *A commutes with S and T, for each $A \in \Re$;*
(ii) *$d(Ax, By) \leq q d(Sx, Ty)$,*

for any $x, y \in X$ and for $A, B \in \Re$ where $0 \leq q < 1$. Then S, T and A have a unique common fixed point in X for all $A \in \Re$.

3. MAIN RESULT

We use a technique of Tarafdar [19] to obtain the following common fixed point theorem which generalize Theorem 2.6.

Theorem 3.1. *Let M be a nonempty τ-bounded, τ-sequentially complete subset of a Hausdorff locally convex space (E, τ). Let T and S be nonexpansive self maps of M and $\Re$ a family of self-mappings $A : X \to S(X) \cap T(X)$ such that $AS = SA$, $AT = TA$ and*

$$p_\alpha(Ax - By) \leq q p_\alpha(Sx - Ty), \tag{3.1}$$

for any $x, y \in M$, $p_\alpha \in A^*(\tau)$, for every $A, B \in \Re$ and where $0 \leq q < 1$. Then S, T and A have a unique common fixed point in M for all $A \in \Re$.

Proof. Since the norm topology on E_B has a base of neighborhoods of zero consisting of τ-closed sets and M is τ-sequentially complete, therefore, M is a $\|.\|_B$-sequentially complete subset of $(E_B, \|.\|_B)$ (Theorem 1.2, [19]). From (3.1) we obtain for $x, y \in M$, $\sup_\alpha p_\alpha(\frac{Ax-By}{\lambda_\alpha}) \leq q \sup_\alpha p_\alpha(\frac{Sx-Ty}{\lambda_\alpha})\}$. Thus

$$\|Ax - By\|_B \leq q\|Sx - Ty\|_B. \tag{3.2}$$

Note that, if S and T are nonexpansive on a τ-bounded, τ- sequentially complete subset M of E, then S and T are also nonexpansive with respect to $\|.\|_B$ and hence $\|.\|_B$-continuous . A comparison of our hypothesis with that of Theorem 2.6 tells that we can apply Theorem 2.6 to M as a subset of $(E_B, \|.\|_B)$ to conclude that there exists a unique $v \in M$ such that $v = Tv = Sv = Av$ for each $A \in \Re$. □

Theorem 3.2. *Let M be a nonempty τ-bounded, τ-sequentially complete subset of a Hausdorff locally convex space (E, τ). Let T and S be nonexpansive self maps of M and $\Re$ a family of self-mappings $A : M \to M$ such that $AS = SA$, $AT = TA$. If M is nonempty and has a contractive jointly continuous family $F = \{f_x\}_{x \in M}$ such that S and T satisfy the Property Γ for all $x \in M$ and $t \in [0, 1]$ and $S(M) = M = T(M)$. If $A, B \in \Re$, T and S satisfy the following:*

$$p_\alpha(Ax - By) \leq p_\alpha(Sx - Ty), \tag{3.3}$$

for any $x, y \in M$, $p_\alpha \in A^*(\tau)$. Then $A \in \Re$, S and T have a unique common fixed point provided one of the following conditions holds:

(i) *M is τ-sequentially compact;*
(ii) *A is a compact map;*
(iii) *M is weakly compact in (E, τ), S and T are weakly continuous and $S - A$ and $T - A$ are demiclosed at 0.*

Proof. Choose a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 1$. For each $n \in \mathbb{N}$, define $A_n : M \to M$ as follows:

$$A_n(x) = f_{Ax}(k_n). \tag{3.4}$$

Obviously, for each n, A_n maps M into itself . As S commutes with A and satisfies the $Property\ \Gamma$, we have

$$A_n(Sx) = f_{A(Sx)}(k_n) = f_{S(Ax)}(k_n) = Sf_{Ax}(k_n) = SA_nx. \tag{3.5}$$

Thus $A_nS = SA_n$, for all $n \in N$ and for all $x \in M$, i.e. A_n and S commute for each n and $A_n(M) \subseteq M = S(M)$. Similarly we can show that A_n and T commute for each n and $A_n(M) \subseteq M = T(M)$. Hence $A_n(M) \subseteq S(M) \cap T(M)$, i.e. $A_n : M \to S(M) \cap T(M)$.

Also, from (3.3), (3.4) and contractiveness of F, it follows

$$\begin{aligned} p_\alpha(A_nx - B_ny) =& p_\alpha(f_{Ax}(k_n) - f_{By}(k_n)) \leq \phi(k_n)p_\alpha(Ax - By) \\ \leq& \phi(k_n)p_\alpha(Sx - Ty), \text{ i.e.} \end{aligned}$$

$$p_\alpha(A_nx - B_ny) \leq \phi(k_n)p_\alpha(Sx - Ty) \tag{3.6}$$

for all $x, y \in M$.

Moreover, S and T are nonexpansive on M, implies that S and T are $\|.\|_B$-nonexpansive and, hence, $\|.\|_B$-continuous. Since the norm topology on E_B has a base of neighborhoods of zero consisting of τ-closed sets and M is τ-sequentially complete, therefore, M is a $\|.\|_B$-sequentially complete subset of $(E_B, \|.\|_B)$ (see proof of Theorem 1.2 in [19]). Thus from Theorem 3.1, for every $n \in N$, A_n, S and T have unique common fixed point x_n in M, i.e.

$$x_n = A_n x_n = S x_n = T x_n, \tag{3.7}$$

for each $n \in N$.

(i) As M is τ-sequentially compact and $\{x_n\}$ is a sequence in M, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in M$. As A, S and T are continuous and

$$x_m = S x_m = T x_m = A_m x_m = f_{Ax_m}(k_m),$$

Proceeding to the limit as $m \to \infty$, we have from the joint continuity of F

$$A_m x_m = f_{Ax_m}(k_m) \to f_{Ay}(1) = Ay.$$

Thus,

$$Ay = y.$$

so it follows that $y = Ay = Ty = Sy$.

(ii) As A is compact and $\{x_n\}$ is bounded, so $\{Ax_n\}$ has a subsequence $\{Ax_m\}$ such that $\{Ax_m\} \to z \in M$. Now we have

$$x_m = A_m x_m = f_{Ax_m}(k_m).$$

Proceeding to the limit as $m \to \infty$, we have from the joint continuity of F

$$A_m x_m = f_{Ax_m}(k_m) \to f_{Az}(1) = Az.$$

Thus,

$$Az = z.$$

and using the continuity of S and T, we have $Az = z = Tz = Sz$.

(iii) The sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converges to $u \in M$. Since S is weakly continuous and so as in (i), we have $Su = u$. Now,

$$y = Sx_m - Ax_m = x_m - Ax_m = A_m x_m - Ax_m = f_{Ax_m}(k_m) - Ax_m. \tag{3.8}$$

We have from the joint weakly continuity of F, as $m \to \infty$

$$y = Sx_m - Ax_m = f_{Au}(1) - Au = Au - Au = 0. \tag{3.9}$$

Now, $S - A$ is demiclosed at 0 and the sequence $\{x_m\}$ converges weakly to u. Also, from (3.9), $y \to 0$ where $y = Sx_m - Ax_m$. Thus, $0 = (S - A)u$ implies that $Su = Au$. Hence $Au = u = Su$. Similarly we show that $Tu = u = Au$. □

An application of Theorem 3.2, we prove the following more general result in best approximation theory.

Theorem 3.3. *Let* $T, S : M \to M$, $\Re$ *a family of self-mappings* $A : M \to M$ *and* M *be a subset of* E *such that* $A(\partial M) \subseteq M$, *where* ∂M *stands for the boundary of* M *and* $x_0 \in F(T) \cap F(S) \cap F(A)$ *for each* $A \in \Re$. *Suppose that* T *and* S *are nonexpansive, commute with* A *on* $D = P_M(x_0)$ *for each* $A \in \Re$. *Further, suppose* T, S *and* $A, B \in \Re$ *satisfy* (3.3) *for each* $x, y \in D \cup \{x_0\}, p_\alpha \in A^*(\tau)$. *If* D *is nonempty and has a contractive jointly continuous family* $F = \{f_x\}_{x \in D}$ *such that* S *and* T *satisfy the Property* Γ *for all* $x \in D$ *and* $t \in [0,1]$ *and* $S(D) = D = T(D)$, *then* T, S *and* $A \in \Re$ *have a common fixed point in* D *provided one of the following conditions holds:*

(i) D *is* τ*-sequentially compact;*
(ii) A *is a compact map;*
(iii) D *is weakly compact in* (E, τ), S *and* T *are weakly continuous and* $S - A$ *and* $T - A$ *are demiclosed at* 0.

Proof. First, we show that A is a self map on D, i.e. $A : D \mapsto D$. Let $y \in D$, then $Sy, Ty \in D$, since $S(D) = D = T(D)$. Also, if $y \in \partial M$, then $Ay \in M$, since $A(\partial M) \subseteq M$. Now since $Bx_0 = x_0 = Tx_0$, so for each $p_\alpha \in A^*(\tau)$ and for each $A \in \Re$, we have from (3.3)

$$p_\alpha(Ay - Bx_0) \le p_\alpha(Sy - Tx_0),$$

yielding thereby $Ay \in D$; consequently A, S and T are self-maps on D. The conditions of Theorem 3.2 ((i)–(iii)) are satisfied and, hence, there exists a $w \in D$ such that $Aw = w = Sw = Tw$. This completes the proof. □

Remark 3.4. Theorem 3.1 and Theorem 3.2 generalize and improve the Theorem 2.6 due to Hadzic [5] to locally convex space.

Remark 3.5. In the light of the comment given by Dotson [4] that if $M \subseteq X$ is p-starshaped and $f_\alpha(t) = (1-t)p + tx$, $(x \in M, t \in [0,1])$ then $\{f_\alpha\}_{\alpha \in M}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and jointly continuity contains the class of convex set.

Remark 3.6. With the remark 3.5, Theorem 3.2 and Theorem 3.3 generalize the results of Jungck and Sessa [8] by increasing the number of mappings, by taking generalized form of nonexpansive mapping for non-starshaped condition of domain and without affineness of mapping to locally convex space.

Remark 3.7. Theorem 3.3 also generalizes and improves the result of Mukherjee and Som [11] by increasing the number of mappings and generalized form of nonexpansive mapping to locally convex space.

Remark 3.8. With the remark 3.5, Theorem 3.3 also generalizes the results of Brosowski [1], Hicks and Humpheries [6], Sahab, Khan and Sessa [13] and Singh [14] by increasing the number of mappings and by considering the generalized form of nonexpansive mapping to locally convex space.

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