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NOTE ON APPLICATION OF FRACTIONAL CALCULUS AND SUBORDINATION TO p-VALENT FUNCTIONS

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Abstract. For p-valent functions of the form $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ that satisfies the condition

$$\frac{z(U_z^{(\lambda,p)} f(z))'}{f_t(z)} \prec \frac{p + (\gamma p + (\alpha - \gamma)(p - \eta)\sin\theta)z}{1 + \gamma z}$$

we will find coefficient inequalities, distortion bounds, radii of starlikeness and convexity, and some properties on this class.

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Key words. p- valent function, subordination, radii of starlikeness and convexity, arithmetic mean.

1. INTRODUCTION

Let $\mathcal{A}_p$ be the class of analytic and p-valent functions in the unit disk $\Delta = \{z : |z| < 1\}$

$$\mathcal{A}_p = \{f(z) | f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k,\ n = 1, 2, \cdots, p \in \mathbb{Z}^+\}.$$

Also T_p denotes the subclass of $\mathcal{A}_p$ and

$$T_p = \{f(z) \in \mathcal{A}_p | f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k,\ a_k \geq 0\}.$$

We define $\Omega_{\lambda,p}(t, \alpha, \gamma, \eta, \theta)$ be the subclass of T_p consisting of all functions in T_p for which

$$\frac{z(U_z^{(\lambda,p)} f(z))'}{f_t(z)} \prec \frac{p + (\gamma p + (\alpha - \gamma)(p - \eta)\sin\theta)z}{1 + \gamma z} \text{ or equivalently} \tag{1}$$

$$\left| \frac{\frac{z(U_z^{(\lambda,p)} f(z))'}{f_t(z)} - p}{\gamma p + (\alpha - \gamma)(p - \eta)\sin\theta - \gamma z \frac{(U_z^{(\lambda,p)} f(z))'}{f_t(z)}} \right| < 1, \tag{2}$$

where $0 \leq \lambda \leq 1, -1 \leq \gamma < \alpha \leq 1, 0 \leq t \leq 1, 0 < \theta \leq \pi, 0 < \eta < p$ and $f_t(z) = (1 - t)z^p + tf(z)$. $(f(z) \in T_p)$ and $U_z^{(\lambda,p)} f(z)$ as a fractional differential operator defined by

$$U_z^{(\lambda,p)} : T_p \to T_p \; U_z^{(\lambda,p)} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k G_p(k,\lambda) z^k, \tag{3}$$

where $G_p(k,\lambda) = \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}$. For $z \neq 0$ we can obtain $U_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z)$ where $D_z^{\lambda} f(z)$ is the fractional derivative of f of order λ. See [2].

For functions f and g, analytic in Δ we say f is subordinate to g denoted by $f \prec g$ if for some analytic function $w(z)$ with $w(0) = 1$ and $|w(z)| < 1$, $f(z) = g(w(z)), z \in \Delta$. By a simple calculation we have $U_z^{(0,p)} f(z) = f(z)$, $U_z^{(1,p)} f(z) = \frac{zf'(z)}{p}$. In special case (i) when $\lambda = 0, \theta = \frac{\pi}{2}$, we obtain the same class $T^*_{p,t}(A,B,\alpha)$ that was introduced and studied by Patel [4]. (ii) If instead of $U_z^{(\lambda,p)} f$ we consider Ruscheweyh derivative of order $n+1$ and put $t = p-1 = 0$, then we obtain the same class $V_n(A,B,\alpha)$ that was investigated by M. K. Aouf. [1].

2. MAIN RESULTS

In this section we obtain sharp coefficient estimates.

THEOREM 1. *Let* $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k (a_k \geq 0)$ *be regular in* Δ. *Then* $f \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ *if and only if*

$$\sum_{k=n+p}^{\infty} [(kG_p(k,\lambda)-pt)(1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta]a_n \leq (\alpha-\gamma)(p-\eta)\sin\theta. \tag{4}$$

Proof. Let $|z| = 1$ and $M = \gamma p + (\alpha-\gamma)(p-\eta)\sin\theta$. Then

$$\begin{aligned}
&\left|z(U_z^{(\lambda,p)} f(z))' - pf_t(z)\right| - \left|Mf_t(z) - \gamma z(U_z^{(\lambda,p)} f(z))'\right| \\
&= \left|z\left(pz^{p-1} - \sum_{k=n+p}^{\infty} a_k G_p(k,\lambda) k z^{k-1}\right) - p\left[(1-t)z^p + tf(z)\right]\right| \\
&\quad - \left|M[(1-t)z^p + tf(z)] - \gamma z\left(pz^{p-1} - \sum_{k=n+p}^{\infty} a_k G_p(k,\lambda) k z^{k-1}\right)\right| \\
&= \left|-\sum_{k=n+p}^{\infty} (kG_p(k,\lambda) - pt) a_k z^k\right| \\
&\quad - \left|(M-\gamma p)z^p - \sum_{k=n+p}^{\infty} [tM - \gamma k G_p(k,\lambda)] a_k z^k\right|.
\end{aligned}$$

By putting $tM-\gamma kG_p(k,\lambda)=t(M-\gamma p)-(kG_p(k,\lambda)-pt)\gamma$ and using (4) the above expression reduces to

$$\sum_{n=k+1}^{\infty}[(kG_p(k,\lambda)-pt)(1-\gamma)+t(M-\gamma p)]a_k-(M-\gamma p)\leq 0. \tag{5}$$

To prove the converse, let $f(z)\in\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ thus

$$\left|\frac{\frac{z(U^{(\lambda,p)}f(z))'}{f_t(z)}-p}{M-\gamma\frac{z(U_z^{(\lambda,p)}f(z))'}{f_t(z)}}\right|$$

$$=\frac{\left|z(pz^{p-1}-\sum\limits_{k=n+p}^{\infty}ka_kG_p(k,\lambda)z^{k-1})-p[(1-t)z^p+tf(z)]\right|}{\left|M[(1-t)z^p+tf(z)]-\gamma z\left(pz^{p-1}-\sum\limits_{k=n+p}^{\infty}ka_kG_p(k,\lambda)z^{k-1}\right)\right|}<1$$

for all $z\in\Delta$. By Re $(z)\leq|z|$ for all z, we have

$$Re\left\{\frac{\sum\limits_{k=n+p}^{\infty}[kG_p(k,\lambda)-pt]a_kz^k}{(M-\gamma p)z^p-\sum_{k=n+p}^{\infty}[tM-\gamma kG_p(k,\lambda)]a_kz^k}\right\}<1.$$

By letting $z\to 1$ through positive values and choose the values of z such that $\frac{z(U_z^{(\lambda,p)}f(z))'}{f_t(z)}$ is real we have

$$\sum_{k=n+p}^{\infty}(kG_p(k,\lambda)-pt)a_k\leq(M-\gamma p)-\sum_{k=n+p}^{\infty}[tM-\gamma kG_p(k,\lambda)]a_k.$$

By (5) we obtain $\sum_{k=n+p}^{\infty}[(kG_p(k,\lambda)-pt)(1-\gamma)+t(M-\gamma p)]a_k\leq M-\gamma p$ and this completes the proof. □

The function

$$f(z)=z^p-\sum_{k=n+p}^{\infty}\frac{(\alpha-\gamma)(p-\eta)\sin\theta}{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}z^k \tag{6}$$

shows that the inequality (4) is sharp.

Corollary 1. *If $f\in\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ then*

$$a_k\leq\frac{(\alpha-\gamma)(p-\eta)\sin\theta}{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}\quad k\geq n+p. \tag{7}$$

Next we find distortion bounds for functions in $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$.

3. DISTORTION BOUNDS FOR $U_z^{(\lambda,p)}f(z)$ AND RADII OF STARLIKENESS AND CONVEXITY

THEOREM 2. *Let $f(z) \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$, then for $|z| = r < 1$*

$$r^p - \frac{(\alpha-\gamma)(p-\eta)\sin\theta G_p(n+p,\lambda)}{[(n+p)G_p(n+p,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta} r^{n+p}$$
$$\leq |U_z^{(\lambda,p)}f(z)|$$
$$\leq r^p + \frac{(\alpha-\gamma)(p-\eta)\sin\theta G_p(n+p,\lambda)}{[(n+p)G_p(n+p,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)p-\eta)\sin\theta} r^{n+p} \quad (8)$$

Proof. Let f is in $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$. By (3), (4) we have

$$|U_z^{(\lambda,p)}f(z)| = z^p - \sum_{k=n+p}^{\infty} a_k G_p(k,\lambda) z^k| \leq |z|^p + \sum_{k=n+p}^{\infty} a_k G_p(k,\lambda)|z|^k$$
$$\leq r^p + (\sum_{k=n+p}^{\infty} a_k G_p(k,\lambda)) r^{n+p}$$
$$\leq r^p + \frac{(\alpha-\gamma)(p-\eta)\sin\theta G_p(n+p,\lambda)}{[(n+p)G_p(n+p,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta} r^{n+p}$$

and

$$|U_z^{(\lambda,p)}f(z)| \geq r^p - \frac{(\alpha-\beta)(p-\eta)\sin\theta G_p(n+p,\lambda) r^{n+p}}{[(n+p)G_p(n+p,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}.$$

□

REMARK 1. In the theorem 3.1 if we put $\lambda = 0$, we obtain growth theorem for $f(z)$.

Next we introduce the radii of starlikeness and convexity for $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$.

THEOREM 3. *If $f(z) \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$, then $f(z)$ is p-valently starlike of order $\delta (0 \leq \delta < p)$ in $|z| < R_1$ where*

$$R_1 = \inf_{k \geq n+p} \left[\frac{(p-\delta)[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(k-\delta)(\alpha-\gamma)(p-\eta)\sin\theta} \right]^{\frac{1}{k-p}}. \quad (9)$$

The bound for $|z|$ is sharp for each k with function of the form (6).

Proof. It is sufficient to show that $\left|\frac{zf'}{f} - p\right| \leq p - \delta$ for $|z| < R_1$. But

$$\left|\frac{zf'}{f} - p\right| = \left| \frac{\sum\limits_{k=n+p}^{\infty} (p-k) a_k z^k}{z^p - \sum\limits_{k=n+p}^{\infty} a_k z^k} \right| \leq \frac{\sum\limits_{k=n+p}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum\limits_{k=n+p}^{\infty} a_k |z|^{k-p}} \leq p - \delta$$

or $\sum_{k=n+p}^{\infty} \left(\frac{k-\delta}{p-\delta}\right) a_k|z|^{k-p} \le 1$ or $|z|^{k-p} \le \frac{(p-\delta)[kG_p(k,\lambda)-pt](1-\gamma)+(\alpha-\gamma)(p-\eta)\sin\theta}{(k-\delta)(\alpha-\gamma)(p-\eta)\sin\theta}$
and this gives the result. □

THEOREM 4. *If $f(z) \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$, then $f(z)$ is p-valently convex of order $\delta(0 \le \delta < p)$ in $|z| < R_2$ where*

$$R_2 = \inf_{k \ge n+p} \left[\frac{p(p-\delta)[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{k(k-\delta)(\alpha-\gamma)(p-\eta)\sin\theta}\right]^{\frac{1}{k-p}}. \quad (10)$$

The bound for $|z|$ is sharp for each k with function of the form (6).

Proof. By using the fact that "f is convex if and only if zf' is starlike" the proof is trivial. □

4. SOME PROPERTIES OF $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$

First we introduce an integral operator due to Bernardi [3]

$$L_c[f] = \frac{p+c}{z^c}\int_0^z f(t)t^{c-1}dt \ (c > -p). \quad (11)$$

THEOREM 5. *If $f \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ then $L_c[f]$ is also in $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$.*

Proof. If $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ then

$$\begin{aligned} L_c[f] &= \frac{p+c}{z^c}\int_0^z \left(t^p - \sum_{k=n+p}^{\infty} a_k t^k\right) t^{c-1}dt \\ &= \frac{p+c}{z^c}\left[\frac{1}{p+c}t^{p+c} - \sum_{k=n+p}^{\infty}\frac{1}{k+c}a_k t^{k+c}\Bigg|_0^z\right] = z^p - \sum_{k=n+p}^{\infty}\frac{p+c}{k+c}a_k z^k. \end{aligned}$$

Since $c \ge 1, k \ge n+p > p$ then $\frac{p+c}{k+c} \le 1$ so we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(\alpha-\gamma)(p-\eta)\sin\theta}\left[\frac{p+c}{k+c}\right] a_k \\ &\le \sum_{k=n+p}^{\infty} \frac{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(\alpha-\gamma)(p-\eta)\sin\theta} a_k < 1. \end{aligned}$$

Thus $L_c[f] \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$. □

Next we shall prove that the class $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ is closed under arithmetic mean.

THEOREM 6. *Let*

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k (j = 1, 2, \cdots, m) \in \Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta).$$

Then the function $F(z) = z^p - \sum\limits_{k=n+p}^{\infty} b_k z^k$ *is also in* $\Omega_{\lambda,p}(t,\alpha,\gamma,\eta,\theta)$ *where* $b_k = \frac{1}{m}\sum\limits_{j=1}^{m} a_{k,j}$.

Proof. Since $f_j(z) \in \Omega_{\lambda,p}(t,\alpha,\beta,\gamma,\theta)$, then by (4) we have

$$\sum_{k=n+p}^{\infty} \frac{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(\alpha-\gamma)(p-\eta)\sin\theta} a_{k,j} \leq 1,\ \ j=1,2,\ldots,m. \tag{12}$$

Therefore

$$\begin{aligned}&\sum_{k=n+p}^{\infty} \frac{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(\alpha-\gamma)(p-\eta)\sin\theta} b_k \\ &= \sum_{k=n+p}^{\infty} \frac{[kG_p(k,\lambda)-pt](1-\gamma)+t(\alpha-\gamma)(p-\eta)\sin\theta}{(\alpha-\gamma)(p-\eta)\sin\theta} \left(\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\right) \\ &\leq \sum_{j=1}^{m} \frac{1}{m} = 1 \text{ by (12)}\end{aligned}$$

and this completes the proof. □

REFERENCES

[1] Aouf, M. K. *On a new criterion for univalent functions of order* α, Rendiconti di Matematica, Serie VII, **11** (1991), 47–59.

[2] Bajerji, P. K. and Shenan, G. M., *Certain generalized subclass of analytic and multivalent functions in terms of fractional calculus*, Proc. Nat. Acad. Sci. India, **71(A)** (2001), 321–338.

[3] Bernardi, S. D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), 429–446.

[4] Patel, J., *On a class of p-valent functions with negative and missing coefficients*, Kyungpook Math. J., **36** (1996), 29–40.

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