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ON δ-PRE-CONVERGENT SEQUENCES

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Abstract. In this paper, we define the concept of δ-pre-convergent sequence. By using this concept we study sequences and separation axioms.

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1. INTRODUCTION

In 1967, Wilansky [12] offered a new separation axiom in general topology called US topological spaces which lay between T_1 and T_2. Recently, Ekici [2, 4] has studied some separation axioms by using δ-preopen sets. The aim of this paper is to investigate and study δ-pre-convergent sequence and δ-pre-separation axioms.

Throughout the present paper, spaces X and Y mean topological spaces. Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

A subset A is said to be α-open [9] (resp. preopen [8]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$).

A subset A of a space X is said to be regular open (respectively regular closed) if $A = int(cl(A))$ (respectively $A = cl(int(A))$) [11].

The δ-interior [13] of a subset A of X is the union of all regular open sets of X contained in A is denoted by δ-$int(A)$. A subset A is called δ-open [13] if $A = \delta$-$int(A)$, i. e., a set is δ-open if it is the union of regular open sets.

The complement of δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [13] if $A = \delta$-$cl(A)$, where δ-$cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$.

A subset S of a topological space X is said to be δ-preopen [10] iff $S \subset int(\delta$-$cl(S))$. The complement of a δ-preopen set is called a δ-preclosed set [10].

Arbitrary union (resp. intersection) of δ-preopen (resp. δ-preclosed) sets in X is δ-preopen (resp. δ-preclosed) [10].

The union (resp. intersection) of all δ-preopen (resp. δ-preclosed) sets, each contained in (resp. containing) a set S in a topological space X is called the δ-preinterior (resp. δ-preclosure) of S and it is denoted by δ-$pint(S)$ (resp. δ-$pcl(S)$) [10].

The family of all δ-preopen sets containing a point $x \in X$ is denoted by $\delta PO(X, x)$. The family of all δ-preopen (resp. δ-preclosed, δ-open) sets of X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$, $\delta O(X)$).

Definition 1. A topological space X is called α-T_0 [7] (resp. pre-T_0 [1, 5]) if for any distinct pair of points in X, there exists an α-open (resp. preopen) set containing one of the points but not the other.

Definition 2. A topological space X is called α-T_1 [7] (resp. pre-T_1 [1, 5]) if for any distinct pair of points x and y in X, there exist an α-open (resp. preopen) U in X containing x but not y and an α-open (resp. preopen) set V in X containing y but not x.

Definition 3. A topological space X is called α-T_2 [6] (resp. pre-T_2 [5]) if for any distinct pair of points x and y in X, there exist α-open (resp. preopen) sets U and V in X containing x and y, respectively, such that $U \cap V = \varnothing$.

Proposition 1. [10] *Let X be a topological space and $S \subset X$. The following properties hold:*
(1) *S is δ-preclosed if and only if $S = \delta$-$pcl(S)$,*
(2) *If $M \subset N$, then δ-$pcl(M) \subset \delta$-$pcl(N)$,*
(3) *δ-$pcl(S)$ is δ-preclosed,*
(4) *δ-$pcl(\delta$-$pcl(S)) = \delta$-$pcl(S)$,*
(5) *$x \in \delta$-$pcl(S)$ if and only if $S \cap V \neq \varnothing$ for every δ-preopen set V containing x.*

2. δ-PRE-CONVERGENCE

In this section, we introduce sequentially δ-precontinuity and δ-pre-US spaces.

Definition 4. [2] A topological space X is called δ-pre-T_0 if for any distinct pair of points in X, there exists a δ-preopen set containing one of the points but not the other.

Definition 5. [3] A space X is said to be δ-pre-T_1 if for each pair of distinct points x and y of X, there exists δ-preopen sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Definition 6. [3] A space X is said to be δ-pre-T_2 if for each pair of distinct points x and y of X , there exists δ-preopen sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Remark 1. [2] The following hold for a topological space X:

$$\alpha\text{-}T_2 \Rightarrow \text{pre-}T_2 \Rightarrow \delta\text{-pre-}T_2,$$
$$\alpha\text{-}T_1 \Rightarrow \text{pre-}T_1 \Rightarrow \delta\text{-pre-}T_1,$$
$$\alpha\text{-}T_0 \Rightarrow \text{pre-}T_0 \Rightarrow \delta\text{-pre-}T_0.$$

Definition 7. [4] A space X is said to be δ-pre-R_1 if and only if for $x, y \in X$ with δ-$pcl(\{x\}) \neq \delta$-$pcl(\{y\})$, there exist disjoint δ-preopen sets U and V such that δ-$pcl(\{x\}) \subset U$ and δ-$pcl(\{y\}) \subset V$.

Theorem 1. [4] *Let X be a topological space. Then X is δ-pre-T_2 if and only if it is δ-pre-R_1 and δ-pre-T_0.*

LEMMA 1. [10] *Let A and Y be subsets of a space X. If $A \in \delta PO(X)$ and Y is δ-open in X, then $A \cap Y \in \delta PO(Y)$.*

DEFINITION 8. A sequence (x_n) is said to be δ-pre-converges to a point x of X, denoted by $(x_n) \xrightarrow{\delta p} x$, if (x_n) is eventually in every δ-preopen set containing x.

DEFINITION 9. A space X is said to be δ-pre-US if every δ-pre-convergent sequence (x_n) in X δ-pre-converges to a unique point.

DEFINITION 10. A set F is said to be sequentially δ-preclosed if every sequence in F δ-pre-converging in X δ-pre-converges to a point in F.

DEFINITION 11. A subset G of a space X is said to be sequentially δ-pre-compact if every sequence in G has a subsequence which δ-pre-converges to a point in G.

THEOREM 2. *A space X is δ-pre-US if and only if the set $A = \{(x, x) : x \in X\}$ is a sequentially δ-preclosed subset of $X \times X$.*

Proof. Let X be δ-pre-US. Let (x_n, x_n) be a sequence in A. Then (x_n) is a sequence in X. As X is δ-pre-US, $(x_n) \xrightarrow{\delta p} x$ for a unique $x \in X$. i.e., (x_n) δ-pre-converge to x and y. Thus, $x = y$. Hence, A is sequentially δ-preclosed set.

Conversely, let A be sequentially δ-preclosed. Let a sequence (x_n) δ-pre-converge to x and y. Hence, sequence (x_n, x_n) δ-pre-converges to (x, y). Since A is sequentially δ-preclosed, $(x, y) \in A$ which means that $x = y$ implies space X is δ-pre-US. □

THEOREM 3. *Every δ-pre-T_2 space is δ-pre-US.*

Proof. Let X be δ-pre-T_2 space and (x_n) be a sequence in X. Suppose that (x_n) δ-pre-converge to two distinct points x and y. That is, (x_n) is eventually in every δ-preopen set containing x and also in every δ-preopen set containing y. This is contradiction since X is δ-pre-T_2 space. Hence, the space X is δ-pre-US. □

THEOREM 4. *Every δ-pre-US space is δ-pre-T_1.*

Proof. Let X be δ-pre-US space. Let x and y be two distinct points of X. Consider the sequence (x_n) where $x_n = x$ for every n. Clearly, (x_n) δ-pre-converges to x. Also, since $x \neq y$ and X is δ-pre-US, (x_n) cannot δ-pre-converge to y, i.e, there exists a δ-preopen set V containing y but not x. Similarly, if we consider the sequence (y_n) where $y_n = y$ for all n, and proceeding as above we get a δ-preopen set U containing x but not y. Thus, the space X is δ-pre-T_1. □

THEOREM 5. *In a δ-pre-US space every sequentially δ-pre-compact set is sequentially δ-preclosed.*

Proof. Let X be δ-pre-US space. Let Y be a sequentially δ-pre-compact subset of X. Let (x_n) be a sequence in Y. Suppose that (x_n) δ-pre-converges to a point in $X \backslash Y$. Let (x_{n_k}) be subsequence of (x_n) which δ-pre-converges to a point $y \in Y$ since Y is sequentially δ-pre-compact.

Also, a subsequence (x_{n_k}) of (x_n) δ-pre-converges to $x \in X \backslash Y$. Since (x_{n_k}) is a sequence in the δ-pre-US space X, $x = y$. Thus, Y is sequentially δ-preclosed set. □

THEOREM 6. *Every δ-open set of a δ-pre-US space is δ-pre-US.*

Proof. Let X be a δ-pre-US space and $Y \subset X$ be an δ-open set. Let (x_n) be a sequence in Y. Suppose that (x_n) δ-pre-converge to x and y in Y. We shall prove that (x_n) δ-pre-converges to x and y in X. Let U be any δ-preopen subset of X containing x and V be any δ-preopen set of X containing y. Then, $U \cap Y$ and $V \cap Y$ are δ-preopen sets in Y. Therefore, (x_n) is eventually in $U \cap Y$ and $V \cap Y$ and so in U and V. Since X is δ-pre-US, this implies that $x = y$. Hence the subspace Y is δ-pre-US. □

THEOREM 7. *A space X is δ-pre-T_2 if and only if it is both δ-pre-R_1 and δ-pre-US.*

Proof. Let X be a δ-pre-T_2 space. Then X is δ-pre-R_1 by Theorem 2 and δ-pre-US by Theorem 4.

Coversely, let X be both δ-pre-R_1 and δ-pre-US space. By Theorem 5 we obtain that every δ-pre-US space is δ-pre-T_1 and X is both δ-pre-T_1 and δ-pre-R_1 and, it follows from Theorem 2 that space X is δ-pre-T_2. □

Next, we prove the product theorem for δ-pre-US spaces.

THEOREM 8. *If X_1 and X_2 are δ-pre-US spaces, then $X_1 \times X_2$ is δ-pre-US.*

Proof. Let $X = X_1 \times X_2$ where X_i is δ-pre-US and $I = \{1, 2\}$. Let a sequence (x_n) in X δ-pre-converges to $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$. Then the sequence (x_{n_i}) δ-pre-converges to x_i and y_i for all $i \in I$. Suppose that for $2 \in I$, (x_{n_2}) does not δ-pre-converges to x_2. Then there exists δ-preopen set U_2 containing x_2 such that (x_{n_2}) is not eventually in U_2. Consider the set, $U = X_1 \times U_2$. Then U is a δ-preopen subset of X and $x \in U$. Also, (x_n) is not eventually in U which contradicts the fact that (x_n) δ-pre-converges to x. Thus we get (x_{n_i}) δ-pre-converges to x_i and y_i for all $i \in I$. Since X_i is δ-pre-US for each $i \in I$, we obtain $x = y$. Hence, X is δ-pre-US. □

Now we define the notion of sequentially δ-precontinuous functions in the following:

DEFINITION 12. A function $f : X \to Y$ is said to be

(1) *sequentially δ-precontinuous at* $x \in X$ if $f(x_n)$ δ-pre-converges to $f(x)$ whenever (x_n) is a sequence δ-pre-converging to x,

(2) *sequentially δ-precontinuous* if f is sequentially δ-precontinuous at all $x \in X$.

DEFINITION 13. A function $f : X \to Y$ is said to be sequentially nearly δ-precontinuous if for each point $x \in X$ and each sequence (x_n) in X δ-pre-converging to x, there exists a subsequence (x_{n_k}) of (x_n) such that $f(x_{n_k}) \xrightarrow{\delta p} f(x)$.

DEFINITION 14. A function $f : X \to Y$ is said to be *sequentially sub-δ-precontinuous* if for each point $x \in X$ and each sequence (x_n) in X δ-pre-converging to x, there exists a subsequence (x_{n_k}) of (x_n) and a point $y \in Y$ such that $f(x_{n_k}) \xrightarrow{\delta p} y$.

DEFINITION 15. A function $f : X \to Y$ is said to be *sequentially δ-pre-compact preserving* if the image $f(K)$ of every sequentially δ-pre-compact set K of X is sequentially δ-pre-compact in Y.

THEOREM 9. *Every function $f : X \to Y$ is sequentially sub-δ-precontinuous if Y is a sequentially δ-pre-compact.*

Proof. Let (x_n) be a sequence in X δ-pre-converging to a point x of X. Then $(f(x_n))$ is a sequence in Y and as Y is sequentially δ-pre-compact, there exists a subsequence $(f(x_{n_k}))$ of $(f(x_n))$ δ-pre-converging to a point $y \in Y$. Hence, $f : X \to Y$ is sequentially sub-δ-precontinuous.

□

THEOREM 10. *A function $f : X \to Y$ is sequentially δ-pre-compact preserving if and only if $f \mid M : M \to f(M)$ is sequentially sub-δ-precontinuous for each sequentially δ-pre-compact subset M of X.*

Proof. Suppose that $f : X \to Y$ is a sequentially δ-pre-compact preserving function. Then $f(M)$ is sequentially δ-pre-compact set in Y for each sequentially δ-pre-compact set M of X. Therefore, by Theorem 10, $f \mid M : M \to f(M)$ is sequentially sub-δ-precontinuous function.

Conversely, let M be any sequentially δ-pre-compact set of X. We shall show that $f(M)$ is sequentially δ-pre-compact set in Y. Let (y_n) be any sequence in $f(M)$. Then for each positive integer n, there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in the sequentially δ-pre-compact set M, there exists a subsequence (x_{n_k}) of (x_n) δ-pre-converging to a point $x \in M$. Since $f \mid M : M \to f(M)$ is sequentially sub-δ-precontinuous, then there exists a subsequence (y_{n_k}) of (y_n) δ-pre-converging to a point $y \in f(M)$. This implies that $f(M)$ is sequentially δ-pre-compact set in Y. Thus, $f : X \to Y$ is sequentially δ-pre-compact preserving function. □

The following theorem gives a sufficient condition for a sequentially sub-δ-precontinuous function to be sequentially δ-pre-compact preserving.

THEOREM 11. *If a function $f : X \to Y$ is sequentially sub-δ-precontinuous and $f(M)$ is sequentially δ-preclosed set in Y for each sequentially δ-pre-compact set M of X, then f is sequentially δ-pre-compact preserving function.*

Proof. We use the previous theorem. It suffices to prove that $f \mid M : M \to f(M)$ is sequentially sub-δ-precontinuous for each sequentially δ-pre-compact subset M of X. Let (x_n) be any sequence in M δ-pre-converging to a point $x \in M$. Then since f is sequentially sub-δ-precontinuous, there exist a subsequence (x_{n_k}) of (x_n) and a point $y \in Y$ such that $f(x_{n_k})$ δ-pre-converges to y. Since $f(x_{n_k})$ is a sequence in the sequentially δ-preclosed set $f(M)$ of Y, we obtain $y \in f(M)$. This implies that $f \mid M : M \to f(M)$ is sequentially sub-δ-precontinuous. □

THEOREM 12. *Every sequentially nearly δ-precontinuous function is sequentially δ-pre-compact preserving.*

Proof. Suppose that $f : X \to Y$ is a sequentially nearly δ-precontinuous function and let M be any sequentially δ-pre-compact subset of X. We shall show that $f(M)$ is a sequentially δ-pre-compact set of Y. Let (y_n) be any sequence in $f(M)$. Then for each positive integer n, there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in the sequentially δ-pre-compact set M, there exists a subsequence (x_{n_k}) of (x_n) δ-pre-converging to a point $x \in M$. Since f is sequentially nearly δ-precontinuous, then there exists a subsequence (x_j) of (x_{n_k}) such that $f\left(x_j\right) \overset{\delta p}{\to} f\left(x\right)$. Thus, there exists a subsequence (y_j) of (y_n) δ-pre-converging to $f(x) \in f(M)$. This shows that $f(M)$ is sequentially δ-pre-compact set in Y. □

THEOREM 13. *Every sequentially δ-pre-compact preserving function is sequentially sub-δ-precontinuous.*

Proof. Suppose $f : X \to Y$ is a sequentially δ-pre-compact preserving function. Let x be any point of X and (x_n) any sequence in X δ-pre-coverging to x. We shall denote the set $\{x_n : n = 1, 2, 3, \ldots\}$ by N and $M = N \cup \{x\}$. Then M is sequentially δ-pre-compact since $x_n \overset{\delta p}{\to} x$. Since f is sequentially δ-pre-compact preserving, it follows that $f(M)$ is a sequentially δ-pre-compact set of Y. Since $(f(x_n))$ is a sequence in $f(M)$, there exists a subsequence $(f\left(x_{n_k}\right))$ of $(f(x_n))$ δ-pre-converging to a point $y \in f(M)$. This implies that f is sequentially sub-δ-precontinuous. □

THEOREM 14. *Let $f : X \to Y$ and $g : X \to Y$ be two sequentially δ-precontinuous functions. If Y is δ-pre-US, then the set $A = \{x : f(x) = g(x)\}$ is sequentially δ-preclosed.*

Proof. Let Y be δ-pre-US and suppose that there exists a sequence (x_n) in A δ-pre-converging to $x \in X$. Since f and g are sequentially δ-precontinuous functions, $f\left(x_n\right) \overset{\delta p}{\to} f\left(x\right)$ and $g\left(x_n\right) \overset{\delta p}{\to} g\left(x\right)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, A is sequentially δ-preclosed. □

THEOREM 15. *Let $f : X \to Y$ be a sequentially δ-precontinuous function. If Y is δ-pre-US, then the set $E = \{(x, y) \in X \times X : f(x) = f(y)\}$ is sequentially δ-preclosed in $X \times X$.*

Proof. Suppose that there exists a sequence (x_n, y_n) in E δ-pre-converging to $(x, y) \in X \times X$. Since f is sequentially δ-precontinuous functions, $f(x_n) \overset{\delta p}{\to} f(x)$ and $f(y_n) \overset{\delta p}{\to} f(y)$. Hence $f(x) = f(y)$ and $(x, y) \in E$. Hence, E is sequentially δ-preclosed. □

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