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DIRECTED ANGLES

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Abstract. In this paper we give an expository description of *directed angles*, their main properties and investigate some applications to the study of antiparallel pairs of lines. We also give proofs of some classical geometry results by using directed angles.

MSC 2000. 50D05

Key words. Directed angles, antiparallel lines, generalized Simson's theorem, cyclic quadrilaterals.

1. INTRODUCTION

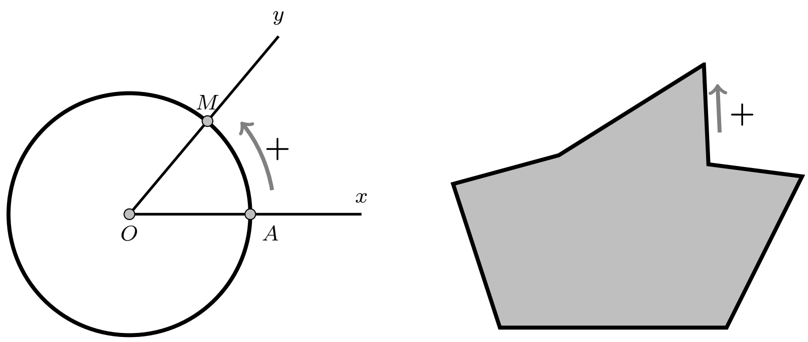
The *directed* or *oriented* angles (alongside with oriented segments) have been introduced in analytical geometry by the great French mathematician Michel Chasles, by the middle of the XIXth century (see his classical textbook *Traité de géométrie supériore*, [2]).

Inspite of its usefulness for the simplifications of many proofs in elementary geometry, by avoiding the necessity to investigate many particular cases, the theory is still not wide-spread. The first exposition in English belongs, to our knowledge, to Roger Johnson (see [3]). The theory was used in the two classical American textbooks of the period between wars, that of R. Johnson (see [4]) and N. Court (see [1]). The directed angles were a constant part of French highschool curriculum starting from the beginning of the XXth century until the 1960th. The current exposition is, largely, based on two such textbooks ([5] and [6]). The examples are adapted mainly from the book of G. Papelier ([7]).

2. DIRECTED ANGLES AND THEIR PROPERTIES

In this paper we will assume that the plane is *oriented*. This, simply, means that we choose a sense of rotation of a line around a point and we define it as *positive*, while the opposite sense will be called *negative*. By convention, we will choose the positive sense to be the *counterclockwise* one (see figure 1(a)). Of course, this will automatically prescribe a sense for the traversation of an arbitrary simple polygon (see figure 1(b)). Thus, in particular, we can speak, for instance, about a *positively oriented triangle*.

What we want, in the end, is to give a definition of the directed angle between two *lines* in the plane. There are several approaches to this definition, the one we shall follow is taken from [5] (see, also, [6]). It starts by defining first the directed angle for rays (which have an in-built orientation), for vectors, oriented lines (axes) and, in the end, for arbitrary (non-oriented) lines.



(a) (b)

Fig. 1

Definition. Let $[OA$ and $[OB$ be two rays with the same origin. We define the *directed angle* between the rays $[OA$ and $[OB$ (in this order!) as being one of the angles through which a ray $[OX$, with the same origin as the first ones must be rotated around O, in a sense or the other (positive or negative) in order that initially it coincides to $[OA$ and, after the rotation, to $[OB$. This angle will be denoted by $\measuredangle\big([OA,[OB\big)$.[1]

We shall assign to the rotation angle a "+" sign if the rotation has been made in the positive sense and a "−" sign if the rotation has been made in the negative sense. We obtain, thus, an *algebraic measure* of the directed angle. This algebraic measure is, also, called a *determination*. On the drawing (figure 2) we marked the rays with an arrow to make clear that they have a built-in orientation ("away from the origin").

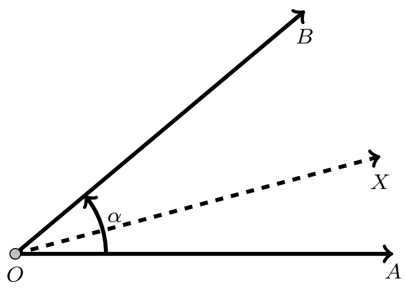


Fig. 2

There are, as it may be seen easily, infinitely many determinations of the same directed angle $\measuredangle\big([OA,[OB\big)$, so it might seem unpractical to work with

[1]The two rays form, actually, two (unoriented angle). The rotation is made on the "shortest path" (smallest unoriented angle).

such a notion, at least when one needs to make computations. Fortunately, it is easy to see that all the determinations have something in common, namely they differ through an integer multiple of 2π[2].

THEOREM 1. *Let α be an arbitrary determination of the directed angle $\measuredangle\big([OA,[OB\big)$. Then all the determinations of the angle are given by*

$$\measuredangle\big([OA,[OB\big) = \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Let's rotate, first, the ray $[OX$ around O in the counterclockwise (positive) sense and let us denote by θ the angle of rotation corresponding to the first superposition between $[OX$ and $[OB$. Clearly, $\theta < 2\pi$. For the next superposition we shall have rotation angles of $\theta + 2\pi, \theta + 4\pi, \ldots$. For the $(n+1)$th superposition, we shall have a rotation angle of $\theta + 2n\pi$. Thus, all the *positive* values of the directed angle $\measuredangle\big([OA,[OB\big)$ will be of the form

$$\theta + 2n\pi, \quad n \in \mathbb{N}.$$

Now, let's rotate the ray $[OX$ in the clockwise (negative) sense, around O, starting again from $[OA$. Clearly, to the first superposition with $[OB$ correspond a rotation angle of $2\pi - \theta$, to the second – a rotation angle of $4\pi - \theta$ and so on. To the $(n+1)$th superposition correspond a rotation angle of $2n\pi - \theta$. To all this angles should be assigned a "$-$" sign, because the rotation is made in the negative sense, therefore all the *negative* values of the directed angle $\measuredangle\big([OA,[OB\big)$ will be of the form

$$\theta - 2n\pi, \quad n \in \mathbb{N}.$$

To sum up, all the determinations of the directed angle $\measuredangle\big([OA,[OB\big)$ will be of the form

$$\measuredangle\big([OA,[OB\big) = \theta + 2\lambda\pi, \quad \lambda \in \mathbb{Z}.$$

If we consider an arbitrary determination, of the form $\alpha = \theta + 2h\pi$, with $h \in \mathbb{Z}$, then we can write

$$\theta + 2\lambda\pi = \theta + 2h\pi + 2(\lambda - h)\pi$$

or

$$\theta + 2\lambda\pi = \alpha + 2k\pi,$$

where we put $k = \lambda - h$.

As λ is arbitrary, so is k and, as such, we can say that any determination of the directed angle $\measuredangle\big([OA,[OB\big)$ is of the form

$$\alpha + 2k\pi,$$

where α is an arbitrary determination of the angle, while k is an arbitrary integer.

Among all the determinations of a given directed angle of rays there is one which is special:

[2]We measure the angles in radians.

Definition. The *principal determination* of the directed angle $\measuredangle\big([OA,[OB\big)$ is the algebraic measure φ of this angle lying in the interval $[-\pi,\pi)$.

While the directed angle $\measuredangle\big([OA,[OB\big)$ is defined only up to an integer multiple of 2π, its principal determination is a well defined *real number*, that can be positive or negative.

Our next step would be to define the directed angle of a pair of *oriented lines* or *axes*.

Definition. An *axis* or an *oriented line* is a pair formed by a line Δ and a non-vanishing vector $\mathbf{v}$, parallel to the line. This actually means that we choose a sense of traversing the line. We define as *positive* the sense of the vector $\mathbf{v}$ and as *negative* the sense opposite to the sense of $\mathbf{v}$. An axis Δ of vector $\mathbf{v}$ will be denoted by $(\Delta,\mathbf{v})$ or, more often, if $\mathbf{v}$ is understood, by $\overrightarrow{\Delta}$.

Definition. Let $\overrightarrow{\Delta_1}\equiv(\Delta_1,\mathbf{v}_1)$ and $\overrightarrow{\Delta_2}\equiv(\Delta_2,\mathbf{v}_2)$ be two axes. If the axes are parallel (or equal), we shall define

$$\measuredangle\left(\overrightarrow{\Delta_1},\overrightarrow{\Delta_2}\right)=\begin{cases}2k\pi,\ \text{with } k\in\mathbb{Z}, & \text{if } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ have the same sense;}\\ (2k+1)\pi,\ \text{with } k\in\mathbb{Z}, & \text{if } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ have opposite senses.}\end{cases}$$

If the two axes are concurrent, let O be their intersection point. We choose $A\in\Delta_1$, such as $\overrightarrow{OA}$ and $\mathbf{v}_1$ have the same sense and $B\in\Delta_2$, such as $\overrightarrow{OB}$ and $\mathbf{v}_2$ have the same sense. We define the directed angle of the two axes as

$$\measuredangle\left(\overrightarrow{\Delta_1},\overrightarrow{\Delta_2}\right)=\measuredangle\big([OA,[OB\big).$$

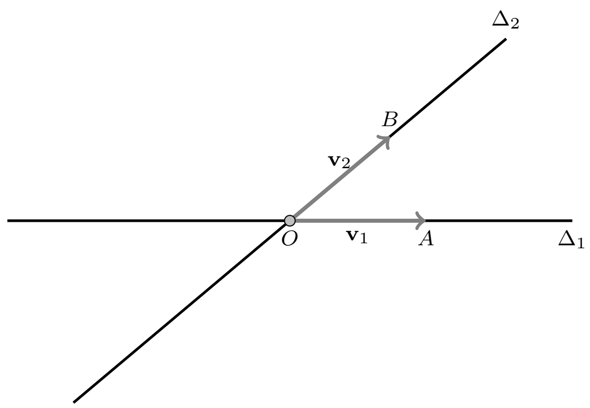


Fig. 3

We shall prove a very simple property of directed angles, credited to Michel Chasles, which has important applications:

Theorem 2 (Chasles). *For any three oriented lines $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$, we have*

$$\measuredangle\left(\overrightarrow{OA},\overrightarrow{OB}\right)+\measuredangle\left(\overrightarrow{OB},\overrightarrow{OC}\right)=\measuredangle\left(\overrightarrow{OA},\overrightarrow{OC}\right)+2k\pi,\quad k\in\mathbb{Z}. \tag{1}$$

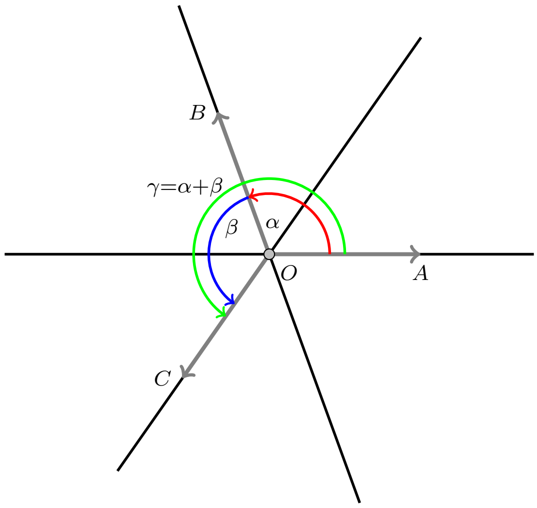


Fig. 4

Proof. We shall take an axis $\overrightarrow{OX}$ and rotate it, in the positive sense, around O, starting from $\overrightarrow{OA}$, such that it coincides, succesively, for the first time, to $\overrightarrow{OB}$ and $\overrightarrow{OC}$ (paying attention, of course, to the fact that at each coincidence, the axis $\overrightarrow{OX}$ has to coincide to the current oriented line both in direction and *sense*). Let us denote the principal determinations of the directed angles $\measuredangle\left(\overrightarrow{OA},\overrightarrow{OB}\right)$, $\measuredangle\left(\overrightarrow{OB},\overrightarrow{OB}\right)$ and $\measuredangle\left(\overrightarrow{OA},\overrightarrow{OC}\right)$ by α,β and γ, respectively. We have to take into account the different relative positions of the three oriented axes. There are, in fact, two cases, which are pictured in the figures 4 and 5. In the first case (when $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, in the counterclockwise order) we have

$$\gamma=\alpha+\beta,$$

while in the second case (when $\overrightarrow{OC}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, in the counterclockwise order) we have

$$\gamma=\alpha+\beta+2\pi,$$

so, in both cases, the equation (1) is verified. $\square$

Remark 1. (i) The equation (1) can be, also, written as

$$\measuredangle\left(\overrightarrow{OA},\overrightarrow{OB}\right)+\measuredangle\left(\overrightarrow{OB},\overrightarrow{OC}\right)+\measuredangle\left(\overrightarrow{OC},\overrightarrow{OA}\right)=2k\pi,\quad k\in\mathbb{Z}. \tag{2}$$

(ii) The Chasles' formula can be extended to an arbitrary number of points:

$$\begin{aligned}&\measuredangle\left(\overrightarrow{OA_1},\overrightarrow{OA_2}\right)+\measuredangle\left(\overrightarrow{OA_2},\overrightarrow{O_3}\right)+\cdots+\measuredangle\left(\overrightarrow{OA_{n-1}},\overrightarrow{OA_n}\right)=\\&=\measuredangle\left(\overrightarrow{OA_1},\overrightarrow{OA_n}\right)+2k\pi,\quad k\in\mathbb{Z},\end{aligned} \tag{3}$$

where n is a natural number, at least equal to three.

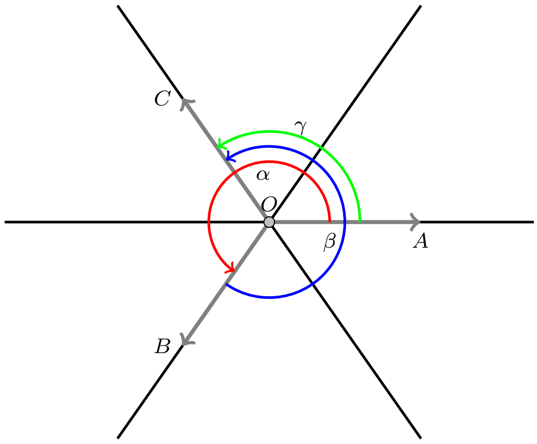


Fig. 5

Indeed, we can prove it by induction. For $n = 3$, this formula is nothing but the formula of Chasles. Let us assume it holds for a certain value of n and prove that it follows, also, for $n + 1$. Thus, we asume that

$$\measuredangle\left(\overrightarrow{OA_1}, \overrightarrow{OA_2}\right) + \measuredangle\left(\overrightarrow{OA_2}, \overrightarrow{O_3}\right) + \cdots + \measuredangle\left(\overrightarrow{OA_{n-2}}, \overrightarrow{OA_{n-1}}\right) = \\ = \measuredangle\left(\overrightarrow{OA_1}, \overrightarrow{OA_{n-1}}\right) + 2k\pi, \quad k \in \mathbb{Z}.$$

Then

$$\measuredangle\left(\overrightarrow{OA_1}, \overrightarrow{OA_2}\right) + \measuredangle\left(\overrightarrow{OA_2}, \overrightarrow{O_3}\right) + \cdots + \measuredangle\left(\overrightarrow{OA_{n-1}}, \overrightarrow{OA_n}\right) = \\ = \measuredangle\left(\overrightarrow{OA_1}, \overrightarrow{OA_{n-1}}\right) + \measuredangle\left(\overrightarrow{OA_{n-1}}, \overrightarrow{OA_n}\right) + 2k\pi = \\ = \measuredangle\left(\overrightarrow{OA_1}, \overrightarrow{OA_n}\right) + 2k\pi, \quad k \in \mathbb{Z}.$$

Theorem 3. *If we change the sense of one of the axes, their directed angle is modified by π.*

Indeed,

$$\measuredangle\left(\overrightarrow{AB}, \overrightarrow{DC}\right) = \measuredangle\left(\overrightarrow{AB}, \overrightarrow{CD}\right) + \measuredangle\left(\overrightarrow{CD}, \overrightarrow{DC}\right) = \measuredangle\left(\overrightarrow{AB}, \overrightarrow{CD}\right) + \pi + 2k\pi.$$

We are, now, ready to define the directed angle of two unoriented lines.

Definition. Let D_1 and D_2 be two lines. The directed angle of the two lines is, by definition, the directed angles between either of the two axis associated to D_1 and either of the two axes associated to D_2. This directed angle will be denoted by $\measuredangle(D_1, D_2)$. If $D_1 = OA$ and $D_2 = OB$, we shall, also, write

$$\measuredangle(OA, OB) = \measuredangle AOB.$$

Now, let's see how well defined is the directed angle of unoriented lines. We have

THEOREM 4. *Let AB and CD be two lines. Then the angle $\measuredangle(AB, CD)$ is defined up to an integer multiple of π.*

Indeed, the directed angle of AB and CD corresponds to two directed angle of axes: $\measuredangle\left(\overrightarrow{AB}, \overrightarrow{CD}\right)$ and $\measuredangle\left(\overrightarrow{AB}, \overrightarrow{DC}\right)$. Let α be an arbitrary determination of $\measuredangle\left(\overrightarrow{AB}, \overrightarrow{CD}\right)$. Thus, we have

$$\measuredangle\left(\overrightarrow{AB}, \overrightarrow{CD}\right) \equiv \measuredangle\left(\overrightarrow{BA}, \overrightarrow{DC}\right) = \alpha + 2m\pi, \quad m \in \mathbb{Z},$$

while

$$\measuredangle\left(\overrightarrow{AB}, \overrightarrow{DC}\right) \equiv \measuredangle\left(\overrightarrow{BA}, \overrightarrow{CD}\right) = \alpha + \pi + 2n\pi, \quad n \in \mathbb{Z}.$$

As such, each angle can be written in the form $\alpha + k\pi$, $k \in \mathbb{Z}$, therefore, we can write

$$\measuredangle(AB, CD) = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

REMARK 2. As it happens with the directed angles between axes, in the case of unoriented lines we also have a special value of the algebraic measure of a directed angle, that lies in the interval $[0, \pi)$. This would, also, be called *the principal determination* of the directed angle.

REMARK 3. As from now on we shall work only with directed angles of unoriented lines (if not specified otherwise) we prefer not to write anymore the arbitrary multiple of π and to consider that all the equalities between directed angles are thought of $\mod \pi$.

We shall prove, now, a series of properties of the directed angles, on which relies, actually, the usefulness of these objects in the plane Euclidean geometry.

PROPERTY 1. *If D_1 and D_2 are two lines in the plane, then*

$$\measuredangle(D_1, D_2) + \measuredangle(D_2, D_1) = 0. \tag{4}$$

Proof. There isn't, actually, anything to proof, only to interpret. The relation (4) simply says that the two angles are supplementary (mod π), which is obvious. $\square$

REMARK 4. The relation (4) can be, also, written under the form

$$\measuredangle(D_1, D_2) = -\measuredangle(D_2, D_1). \tag{5}$$

If we denote by φ the principal determination of the directed angle $\measuredangle(D_1, D_2)$, then

$$\measuredangle(D_1, D_2) = \varphi + k\pi, \quad k \in \mathbb{Z}.$$

We want to find the principal determination of the directed angle $\measuredangle(D_2, D_1)$. Sticking with the rule of rotating the lines in the positive direction, we have that

$$\measuredangle(D_2, D_1) = 2\pi - \varphi - k\pi$$

or, if we want to find the principal determination, between 0 and π,

$$\measuredangle(D_2, D_1) = \pi - \varphi + (1-k)\pi = \pi - \varphi + l\pi, \quad k, l \in \mathbb{Z},$$

i.e. the principal determination of $\measuredangle(D_2, D_1)$ is $\pi - \varphi$. Thus, the principal determinations of the directed angle and its opposite (mod π) are supplementary.

Property 2 (Chasles). *Let D_1, D_2, D_3 be three lines in the plane. Then*

$$\measuredangle(D_1, D_2) + \measuredangle(D_2, D_3) = \measuredangle(D_1, D_3). \tag{6}$$

Proof. If two of the lines are parallel or coincide, the property reduces to the previous one. As such, we assume that the lines are in general position, in other words, they form a triangle.

For the proof, we choose, first, an arbitrary point O and take three lines d_1, d_2, d_3 through O, such that $d_1 \parallel D_1$, $d_2 \parallel D_2$ and $d_3 \parallel D_3$. Then, obviously,

$$\measuredangle(d_1, d_2) = \measuredangle(D_1, D_2),\ \measuredangle(d_2, d_3) = \measuredangle(D_2, D_3),\ \measuredangle(d_1, d_3) = \measuredangle(D_1, D_3).$$

Thus, we are left with the proof of the relation

$$\measuredangle(d_1, d_2) + \measuredangle(d_2, d_3) = \measuredangle(d_1, d_3). \tag{7}$$

We rotate d_1 around O, in the counterclockwise sense, until it coincide for the first time with d_2. Let θ_1 be the rotation angle. We continue the rotation until it coincide for the first time with d_3 and we denote by θ_2 this second rotation angle. Now we rotate d_1, still in the counterclockwise sense, until it first coincides to d_3 and we denote by θ_3. Now, clearly, $\theta_1 + \theta_2 = \theta_3$. $\square$

As an application of the second property, let us consider a triangle ABC, in which the vertices are listed in the counterclockwise manner. Then, we have

$$\measuredangle BAC = \widehat{A},\ \measuredangle CBA = \widehat{B},\ \measuredangle ACB = \widehat{C}.$$

We apply the Property 2 for the lines $D_1 = BC$, $D_2 = AB$ and $D_3 = CA$. Then the formula

$$\measuredangle(D_1, D_2) + \measuredangle(D_2, D_3) = \measuredangle(D_1, D_3)$$

becomes

$$\measuredangle(BC, AB) + \measuredangle(AB, AC) = \measuredangle(BC, AC)$$

or

$$\measuredangle CBA + \measuredangle BAC = \measuredangle BCA$$

or, in the language of classical Euclidean geometry,

$$\widehat{B} + \widehat{A} = \pi - \widehat{C},$$

which is nothing but the *Theorem of the exterior angle* in the triangle ABC.

The Property 2 can be easily extended to an arbitrary number of lines:

Property 3. *Let $D_1, D_2, \ldots, D_n$ be n lines in the plane. Then we have*

$$\measuredangle(D_1, D_2) + \measuredangle(D_2, D_3) + \cdots + \measuredangle(D_{n-1}, D_n) = \measuredangle(D_1, D_n). \tag{8}$$

Proof. We prove by induction. For $n = 3$, the property reduces to Property 2. Let us assume it is true for $n - 1$, i.e.

$$\measuredangle(D_1, D_2) + \measuredangle(D_2, D_3) + \cdots + \measuredangle(D_{n-2}, D_{n-1}) = \measuredangle(D_1, D_{n-1}).$$

Then

$$\begin{aligned}\measuredangle(D_1, D_2) + \cdots + \measuredangle(D_{n-2}, D_{n-1}) + \measuredangle(D_{n-1}, D_n) = \\ = \measuredangle(D_1, D_{n-1}) + \measuredangle(D_{n-1}, D_n) = \measuredangle(D_1, D_n).\end{aligned}$$ □

PROPERTY 4. *Three points A, B, C are colinear iff for any other point D from the plane we have*

$$\measuredangle ABD = \measuredangle CBD.$$

Proof. There is not much to prove, actually. Clearly, the three points are colinear iff the lines AB and CB coincide. The property simply says that the two lines coincide iff they are equally inclined to BD, which is obvious. □

PROPERTY 5. *Let D_1 and D_2 be two lines in the plane. Then D_1 is parallel to D_2 iff for any secant D we have*

$$\measuredangle(D_1, D) = \measuredangle(D_2, D), \tag{9}$$

or, which is the same,

$$\measuredangle(D, D_1) = \measuredangle(D, D_2). \tag{10}$$

Proof. Let us assume, first, that $D_1 \parallel D_2$. This means, as we saw before, that

$$\measuredangle(D_1, D_2) = 0.$$

But, from Property 2 it follows that

$$0 = \measuredangle(D_1, D_2) = \measuredangle(D_1, D) + \measuredangle(D, D_2) = \measuredangle(D_1, D) - \measuredangle(D_2, D),$$

hence

$$\measuredangle(D_1, D) = \measuredangle(D_2, D).$$

Assume, now, conversely, that the lined D_1, D_2, D verify the relation (9). This means that

$$\measuredangle(D_1, D) = -\measuredangle(D, D_2),$$

hence

$$\measuredangle(D_1, D) + \measuredangle(D, D_2) = 0.$$

But

$$\measuredangle(D_1, D) + \measuredangle(D, D_2) = \measuredangle(D_1, D_2),$$

whence $\measuredangle(D_1, D_2) = 0$, i.e. $D_1 \parallel D_2$. □

After establishing a parallelism criterion in terms of directed angles, we shall find a criterion of perpendicularity.

PROPERTY 6. *Two concurent lines D_1 and D_2 are perpendicular iff*

$$\measuredangle(D_1, D_2) = \measuredangle(D_2, D_1). \tag{11}$$

Proof. Clearly, if the two lines are perpendicular, the two angles are equal. Conversely, if

$$\measuredangle(D_1, D_2) = \measuredangle(D_2, D_1),$$

then we have the system of equations

$$\begin{cases} \measuredangle(D_1, D_2) - \measuredangle(D_2, D_1) = 0, \\ \measuredangle(D_1, D_2) + \measuredangle(D_2, D_1) = 0. \end{cases}$$

If we add up the two equations, we get

$$2\measuredangle(D_1, D_2) = 0.$$

We need to remind, again, that all the equalities should be understood mod π. As such, $2\measuredangle(D_1, D_2) = 0$ doesn't necessarily mean that $\measuredangle(D_1, D_2) = 0$. In fact, in this particular case, the angle cannot be zero, because the lines are concurrent. As the double of the angle is zero (or π) it follows that $\measuredangle(D_1, D_2) = \pi/2$. □

Finally, we shall give a condition of concyclicity of four points in terms of directed angles.

PROPERTY 7. *Four points A, B, C, D from the plane are concyclic iff*

$$\measuredangle ACB = \measuredangle ADB. \tag{12}$$

Proof. Is left to the reader. □

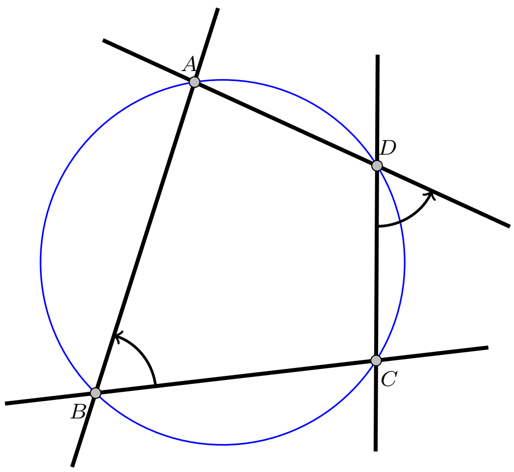


Fig. 6

3. ANTIPARALLEL LINES

DEFINITION. Two pairs of lines, D, D' and Δ, Δ' are called *antiparallel* if we have

$$\measuredangle(D, \Delta) = \measuredangle(\Delta', D').$$

More specifically, we shall say that the line Δ' (for instance) is antiparallel to the line Δ with respect to the pair D, D'.

THEOREM 5. *The two pairs of opposite sides of the fourpoint $ABCD$ are antiparallel iff the quadrangle is cyclic (in other words, iff the points A, B, C, D are concyclic).*

Indeed, let $D = AB$, $D' = CD$, $\Delta = AD$ and $\Delta' = BC$. According to the definition, the two pairs are antiparallel iff

$$\measuredangle(AB, AD) = \measuredangle(BC, CD)$$

or

$$\measuredangle BAD = \measuredangle BCD.$$

But, as we saw before, this is nothing but the necessary and sufficient condition of concyclicity of the points A, B, C, D.

THEOREM 6. *Let $\Delta, \Delta', \Delta'', D, D'$ be five lines in the plane. If Δ' and Δ'' are both antiparallel to Δ with respect to the pair D, D', then Δ' is parallel to Δ'.*

Indeed, Δ' is antiparallel to Δ with respect to D, D' iff

(*) $$\measuredangle(D, \Delta) = \measuredangle(\Delta', D'),$$

while Δ'' is antiparallel to Δ with respect to D, D' iff

(**) $$\measuredangle(D, \Delta) = \measuredangle(\Delta'', D').$$

Let us compute the directed angle $\measuredangle(\Delta', \Delta'')$. We have

$$\measuredangle(\Delta', \Delta'') = \measuredangle(\Delta', D') + \measuredangle(D', \Delta'') = \measuredangle(D, \Delta) - \measuredangle(D, \Delta) = 0,$$

which means that the two lines are parallel.

THEOREM 7. *If the lines D, D', Δ, Δ' are concurrent at a point O, then the pairs of lines D, D' and Δ, Δ' are antiparallel iff the bisectors of the angles formed by D and D' coincide to the bisectors of the angles formed by the lines Δ and Δ'.*

Let us assume, first, that the two pairs of lines have the same bisectors and let H be one of them. Then, we have the equalities

$$\begin{cases} \measuredangle(D, H) = \measuredangle(H, D'), \\ \measuredangle(\Delta, H) = \measuredangle(H, \Delta'). \end{cases}$$

Subtracting the previous equations side by side, we get

$$\measuredangle(D, H) - \measuredangle(\Delta, H) = \measuredangle(H, D') - \measuredangle(H, \Delta')$$

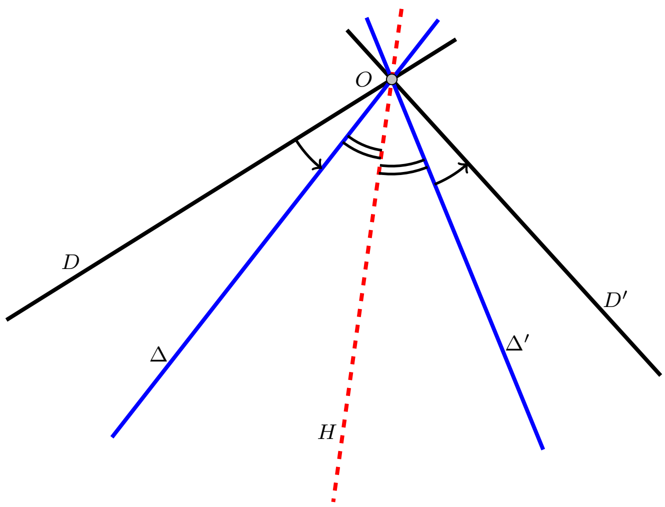


Fig. 7

or

$$\measuredangle(D,H)+\measuredangle(H,\Delta)=\measuredangle(\Delta,H)+\measuredangle(H,D'),$$

whence

$$\measuredangle(D,\Delta)=\measuredangle(\Delta',D'),$$

i.e. the two pairs of lines are antiparallel.

Conversely, let us assume that the pairs of lines are antiparallel and le H be one of the bisectors of the lines D and D'. We intend to prove that H is also a bisector for the lines Δ and Δ'.

As the lines are antiparallel, we have

$$\measuredangle(D,\Delta)=\measuredangle(\Delta',D'),$$

which, because of the formula of Chasles, can be written as

$$\measuredangle(D,H)+\measuredangle(H,\Delta)=\measuredangle(\Delta',H)+\measuredangle(H,D').$$

As H is one of the bisectors of the angle made by D and D', we have $\measuredangle(D,H)=\measuredangle(H,D')$, so the previous relation reduces to

$$\measuredangle(H,\Delta)=\measuredangle(\Delta',H)$$

or

$$\measuredangle(\Delta,H)=\measuredangle(H,\Delta'),$$

which means that H is, also, a bisector of the angle made by Δ and Δ'.

It is easy to check, also, the more general result.

Theorem 8. *Two pairs of lines D, D' and Δ,Δ' are antiparallel iff the bisectors of the angles formed by D and D' are parallel to the bisectors of the angles formed by the lines Δ and Δ'.*

A final property of antiparallel lines is the following:

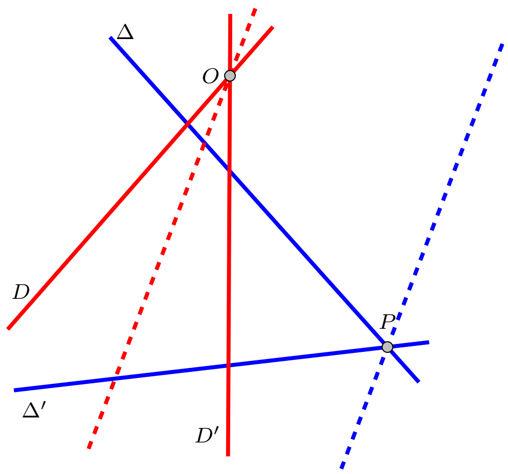


Fig. 8

THEOREM 9. *If two pairs of lines, D, D' and D_1, D_1' are, separately, antiparallel to the same pair of lines, Δ, Δ', then they are antiparallel to each other.*

Proof. From the antiparallelism of the first two pairs of lines to the third pair, we get the equations:

$$\begin{cases} \measuredangle(D,\Delta) = \measuredangle(\Delta', D'), \\ \measuredangle(D_1,\Delta) = \measuredangle(\Delta', D_1'). \end{cases}$$

Subtracting these equations side by side, we get

$$\measuredangle(D,\Delta) - \measuredangle(D_1,\Delta) = \measuredangle(\Delta', D') - \measuredangle(\Delta', D_1')$$

or

$$\measuredangle(D,\Delta) + \measuredangle(\Delta, D_1) = \measuredangle(D_1', \Delta') + \measuredangle(\Delta', D'),$$

whence, using the formula of Chasles,

$$\measuredangle(D, D_1) = \measuredangle(D_1', D'),$$

i.e. the pair D, D' is antiparallel to the pair D_1, D_1'. □

4. OTHER APPLICATIONS OF DIRECTED ANGLES

We can provide now proofs for some interesting results, using, in an essential manner, the notion of directed angle of lines.

THEOREM 10. *Let ABC a triangle, which we assume positively oriented. On the sides BC, CA, AB, respectively, we choose the points P, Q, R, different from the vertices of the triangle ABC. If U is the second intersection point of the circumcircles of the triangles PRB and PCQ, then the quadrilateral $ARUQ$ is cyclic.*

Proof. We notice, first of all, that the quadrilaterals $PUQC$ and $PURB$ are cyclic. From the first quadrilater, we deduce, then, that

$$\measuredangle PUQ = \measuredangle PCQ$$

or

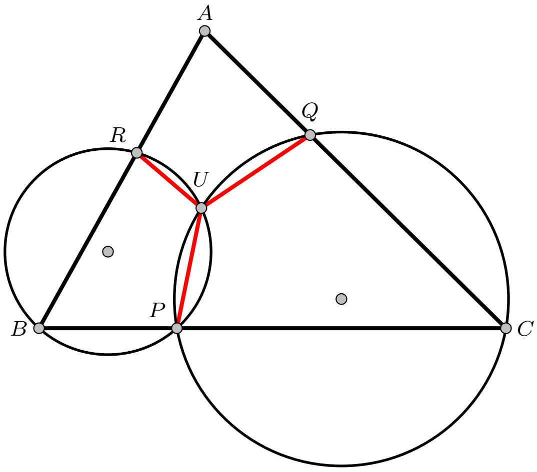


Fig. 9

$$\measuredangle PUQ + \measuredangle QCP = 0. \tag{*}$$

Similarly, from the cyclicity of $PURB$ we get

$$\measuredangle RUP + \measuredangle PBR = 0. \tag{**}$$

If we add up the equations $(*)$ and $(**)$, we get

$$\measuredangle PUQ + \measuredangle QCP + \measuredangle RUP + \measuredangle PBR = 0$$

or

$$\measuredangle PUQ + \measuredangle RUP + \measuredangle QCP + \measuredangle PBR = 0. \tag{***}$$

We notice, immediately, that, as lines, we have $QC{=}AC$, $CP{=}PB{=}BC$ and $BR{=}AB$. Therefore,

$$\begin{aligned} &\measuredangle QCP \equiv \measuredangle(QC, CP) = \measuredangle(AC, BC) \equiv \measuredangle ACB \quad \text{and} \\ &\measuredangle PBR \equiv \measuredangle(PB, BR)\measuredangle(BC, AB) \equiv \measuredangle CBA, \end{aligned}$$

therefore

$$\begin{aligned} \measuredangle QCP + \measuredangle PBR = \measuredangle ACB + \measuredangle CBA \equiv \measuredangle(AC, BC) + \measuredangle(BC, AB) = \\ = \measuredangle(AC, AB) \equiv \measuredangle CAB = \measuredangle QAR. \end{aligned} \tag{****}$$

On the other hand,

$$\begin{aligned} \measuredangle RUP + \measuredangle PUQ \equiv \measuredangle(RU, PU) + \measuredangle(PU, UQ) = \\ = \measuredangle(RU, UQ) \equiv \measuredangle RUQ. \end{aligned} \tag{*****}$$

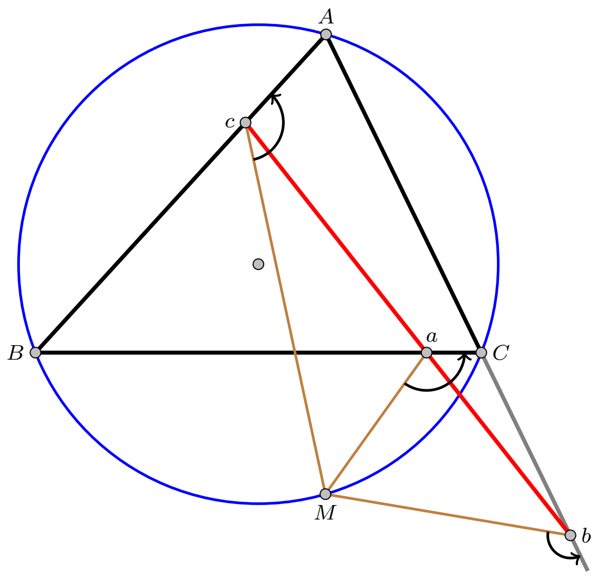


Fig. 10

Substituting $(****)$ and $(*****)$ in $(***)$, we get

$$\measuredangle QAR + \measuredangle RUQ = 0$$

or

$$\measuredangle QAR = \measuredangle QUR,$$

which shows that the quadrilateral $ARUQ$ is cyclic. □

THEOREM 11 (Generalized Simson's theorem.). *Let ABC be a given triangle and M – a point in the plane. The points P, Q, R are taken on BC, CA, AB, respectively, so that , for angles measured in the counterclockwise sense, the angles between BC and MP; between CA and MQ; between AB and MR are equal. Than M lies on the circumcircle of the triangle ABC iff the points P, Q, R are collinear.*

Proof. We are given the following directed angle equalities

$$\measuredangle(BC, MP) = \measuredangle(CA, MQ) = \measuredangle(AB, MR). \tag{13}$$

But, since $BC = CP$ and $CA = CQ$ (as lines) the first equality of (13) can be written as

$$\measuredangle(CP, MP) = \measuredangle(CQ, MQ)$$

or

$$\measuredangle CPM = \measuredangle CQM \tag{14}$$

i.e. the points P, Q, M, C are concyclic.

Consider, now, the second equality from (13). This time, we have $R \in AB$, hence $AB = BR = AR$. As such, this equality is equivalent to

$$\measuredangle(AQ, MQ) = \measuredangle(AR, MR)$$

or

$$\measuredangle AQM = \measuredangle ARM, \tag{15}$$

which is the same with saying that the points R, Q, A, M are concyclic.

As we saw previously, the points P, Q, R are collinear iff

$$\measuredangle PQM = \measuredangle RQM, \tag{16}$$

for any point in the plane. Now, from the cyclic quadrilateral $PQCM$ we get

$$\measuredangle PQM = \measuredangle PCM = \measuredangle(CP, CM) = \measuredangle(CB, CM) \equiv \measuredangle BCM. \tag{17}$$

On the other hand, from the cyclic quadrilater $QRAM$, we get

$$\measuredangle RQM = \measuredangle RAM = \measuredangle(RA, AM) = \measuredangle(AB, AM) \equiv \measuredangle BAM. \tag{18}$$

Thus, from the relations (16), (17) and (18) it follows that

$$\measuredangle PQM = \measuredangle RQM \quad \text{iff} \quad \measuredangle BCM = \measuredangle BAM \tag{19}$$

or, in other words, the points P, Q, R are collinear iff the points A, B, C, M are concyclic or, to put it another way, iff M lies on the circumcircle of ABC. $\square$

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