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PHASE PLANE ANALYSIS OF THE MOTION IN THE SCHWARZSCHILD FIELD

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Abstract. The Schwarzschild orbital dynamics are summarised in the setting of phase plane analysis, thus lessening the algebraic calculations whilst emphasising the physics. The standard results are presented. A dimensionless parameter which involves the angular momentum is defined and varied. Based on these variations, the exact phase plane is analysed. Also, there is an emphasis put on the separatrix structure of this phase plane.

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1. INTRODUCTION

In Modern Physics, the problem of dynamics in Schwarzschild field is of fundamental importance since the Schwarzschild solution, introduced in 1916, is considered the exact one to the Einstein field equation due to its direct testability. The solution is believed to be flawless and has successfully passed the test which put General Relativity to the rank of the true relativistic theory of gravitational field. In this paper, we take a look at the general relativistic orbits in Schwarzschild field and analyse them through a phase plane technique. This standard approach of nonlinear analysis applied to Schwarzschild orbital dynamics has also been discussed in Dean's work (see [2]) and also in Anderson and Walsh's work (see [5]). The first section of this paper presents how the general relativistic equations of motions are obtained, while the second one develops the phase plane analysis. Section three presents an exact general relativistic phase plane and a discussion about the Schwarzschild orbital dynamics takes place.

2. GENERAL RELATIVISTIC ORBITS

For a rigorous introduction to General Relativity and an exhibition of how the Schwarzshild solution is derived, the reader is encouraged to see [1]. Consider a large object with mass M and another object with rest mass m_0 which orbits around it. The general relativistic equation of motion arises from the Schwarzschild line element:

$$ds^2 = c^2\Lambda dt^2 - \Lambda^{-1}dr^2 - r^2 d\Omega^2 \tag{1}$$

where $\Lambda = 1 - \frac{r_s}{r}$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and r_s is the Schwarzschild radius:

$$r_s = \frac{2MG}{c^2}$$

The Lagrangian appears as a constant of motion:

$$L = \frac{1}{2}(\frac{ds}{d\tau})^2 = \frac{1}{2}m_0c^2 \tag{2}$$

where τ represents the proper time.
In the particular case when the orbit is bound to the equatorial plane ($\theta = \frac{\pi}{2}$), then:

$$L = \frac{1}{2}m_0c^2\Lambda t_\tau^2 - \frac{1}{2}m_0\Lambda^{-1}r_\tau^2 - \frac{1}{2}m_0r^2\varphi_\tau^2 \tag{3}$$

where $t_\tau = \frac{dt}{d\tau}$, etc.
The Euler-Lagrange equations generate two constraints of motion:

$$\begin{cases} \frac{\partial L}{\partial t} + \frac{d}{d\tau}(\frac{\partial L}{\partial t_\tau}) = 0 \\ \frac{\partial L}{\partial t} = 0 \end{cases} \implies \frac{\partial L}{\partial t_\tau} = E = m_0c^2\Lambda t_\tau \tag{4}$$

$$\begin{cases} \frac{\partial L}{\partial \varphi} + \frac{d}{d\tau}(\frac{\partial L}{\partial \varphi_\tau}) = 0 \\ \frac{\partial L}{\partial \varphi} = 0 \end{cases} \implies \frac{\partial L}{\partial \varphi_\tau} = J = m_0r^2\varphi_\tau \tag{5}$$

where E represents the energy that is necessary for an observer situated at infinity to put m_0 in orbit around M and J is the system's angular momentum. In order do eliminate t_τ and φ_τ from (3), we use (4) and (5) and after rearranging the terms, we obtain:

$$L = \frac{1}{2}(m_0c^2\Lambda\frac{E^2}{m_0^2c^4\Lambda^2} - m_0\Lambda^{-1}r_\tau^2 - m_0r^2\frac{J^2}{m_0^2r^4})$$

Since $L = \frac{1}{2}m_0c^2$, we obtain:

$$(\frac{r_\tau}{c})^2 = (\frac{dr}{ds})^2 = \hat{E}^2 - (1 + \frac{J^2}{m_0^2c^2r^2})\Lambda \tag{6}$$

where $\hat{E} = \frac{E}{m_0c^2}$ is the total energy per unit rest energy.
Since $r = r(\varphi)$, then $r_\tau = \frac{dr}{d\varphi}\varphi_\tau$ which allows the expansion of (6) in terms of J. In order to reduce the degree of equation (6) in r, we make the variable change $u = \frac{r_s}{r}$ and, after simplifications, it becomes:

$$(\frac{du}{d\varphi})^2 = 2\sigma\hat{E}^2 - (2\sigma + u^2)\Lambda \tag{7}$$

where

$$\sigma = \frac{1}{2}(\frac{m_0cr_s}{J})^2 = 2(\frac{GMm_0}{cJ})^2 \tag{8}$$

By differentiating equation (7) with respect to φ:

$$\frac{d^2u}{d\varphi^2} = \sigma + \frac{3}{2}u^2 - u \tag{9}$$

Equation (9) is a nonlinear, second-order, inhomogenous differential equation. In order to perform its phase plane analysis, by defining the new variables $x = u$ and $y = \frac{du}{d\varphi}$, we convert it to a system of two first order equations:

$$\begin{cases} \dot{x} = f_1(x, y) = y \\ \dot{y} = f_2(x, y) = \frac{3}{2}x^2 - x + \sigma \end{cases} \tag{10}$$

where $\dot{x} = \frac{dx}{d\varphi}$ and $\dot{y} = \frac{dy}{d\varphi}$.
The system (10) will be analysed from the viewpoint of the phase plane analysis technique. For an introduction on this procedure, see [3].

3. PHASE PLANE ANALYSIS

We solve $\dot{x} = \dot{y} = 0$ simultaneously for x and y. The solution gives us the equilibrium points of (10):

$$\overrightarrow{x^*} = (\frac{1 + \sqrt{1 - 6\sigma}}{3}, 0), \qquad \overrightarrow{y^*} = (\frac{1 - \sqrt{1 - 6\sigma}}{3}, 0) \tag{11}$$

Alternatively, if y is expressed in terms of x using (7):

$$\dot{x}(= y) = \pm[2\sigma\hat{E}^2 - (2\sigma + x^2)(1 - x)]^{\frac{1}{2}} \tag{12}$$

We solve $\dot{x} = \dot{y} = 0$ simultaneously, but this time for $\hat{E}^2$ and x, we obtain the matching energies at each equilibrium point:

$$\begin{cases} \hat{E}_1^2 - 1 = \frac{2\sigma[1-4\sigma-(1-6\sigma)^{\frac{1}{2}}]}{[(1-6\sigma)^{\frac{1}{2}}-1]^3} \\ \hat{E}_2^2 - 1 = \frac{2\sigma[-1+4\sigma-(1-6\sigma)^{\frac{1}{2}}]}{[(1-6\sigma)^{\frac{1}{2}}-1]^3} \end{cases} \tag{13}$$

In order to find the nature of the equilibrium points (11), a linear stability analysis is performed. This involves performing a series expansion of equation (10), in an arbitrary equilibrium point, in the small parameters: $\delta x = x - x^*$ and $\delta y = y - y^*$. Since the second order terms can be overlooked, we obtain a first order linear system, which yields the matrix form:

$$\delta\dot{x} \approx A\mid_{x=x*} \delta x \tag{14}$$

where $\delta\dot{x} = \begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix}$, $A = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}$ and $\delta x = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$. Then $A = \begin{pmatrix} 0 & 1 \\ 3x - 1 & 0 \end{pmatrix}$.
The solution of equation (14) is an exponential. By classifying the eigenvalues of A, we can analyse its stability at each equilibrium point. We solve the characteristic polynomial

$$\lambda^2 - tr\lambda + \Delta = 0$$

where tr is the trace of A and Δ is the determinant of A. The eigenvalues are $\lambda_{1,2} = \frac{1}{2}(tr \pm \sqrt{tr^2 - 4\Delta})$.

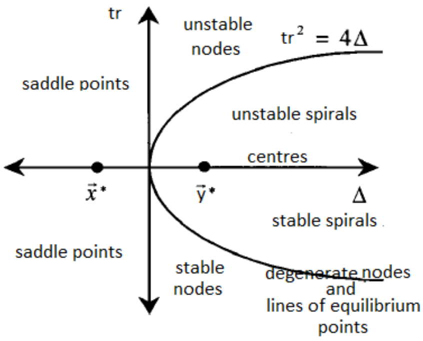

FIG. 3.1 – Eigenvalue classification

When A is evaluated in the equilibrium points (11), the following classification arises:

$$A\,|_{\overrightarrow{x}^*} = \begin{pmatrix} 0 & 1 \\ \sqrt{1-6\sigma} & 0 \end{pmatrix} \implies tr = 0; \Delta = -\sqrt{1-6\sigma} \tag{15}$$

which makes $\overrightarrow{x}^*$ a saddle point.

$$A\,|_{\overrightarrow{y}^*} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-6\sigma} & 0 \end{pmatrix} \implies tr = 0; \Delta = \sqrt{1-6\sigma} \tag{16}$$

which makes $\overrightarrow{y}^*$ a centre.

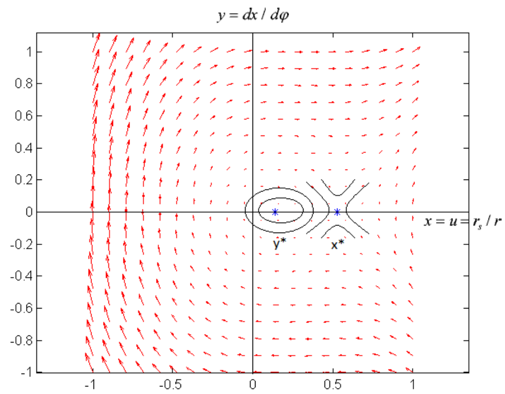

FIG. 3.2 – The phase portrait

The above figure shows the phase plane trajectories of the exponential solution of equation (14) about each equilibrium point (11).

REMARK 1. Newtonian theory does not foresee the saddle node equilibrium point which is present in the phase portrait. This point appears because of an unstable orbital radius which originates from the r^{-3} term of the effective potential. To learn more on this subject see [4].
The effective potential is derived from relation (7):

$$\hat{V}_{eff}^2 = (1 + \frac{x^2}{2\sigma})(1 - x) \tag{17}$$

It is due to this, that there exist orbital effects which are not part of Newtonian theory.

4. AN EXACT PHASE PLANE

The previous section only offers local information about the general relativistic orbits and neglects the hyperbolic, the parabolic or the orbits in the vicinity of the event horizon. However, this shortcoming can be overcome if an exact phase plane is constructed. This will offer global features of the Schwarzschild orbital dynamics. To see more on this, consult [2].
Consider the level curves obtained by taking the ratio $\frac{dy}{dx}$ from (10):

$$\frac{dy}{dx} = \frac{(\frac{3}{2}x^2 - x + \sigma)}{y}. \tag{18}$$

By integrating, a conserved quantity is obtained:

$$y^2 = \beta + x^3 - x^2 + 2\sigma x$$

where β is a constant that can be obtained by comparison with (7):

$$\beta = 2\sigma(\hat{E}^2 - 1)$$

Thus, equation (18) is equivalent to:

$$\hat{E}^2 - 1 = \frac{(y^2 + x^2 - x^3)}{2\sigma} - x \tag{19}$$

Consider the effective potential V_{eff}^2 given in (17) and let $\sigma = \frac{1}{9}$. For various values of the potential energy per unit rest energy, $\hat{E}$, the following phase plane portrait arises:

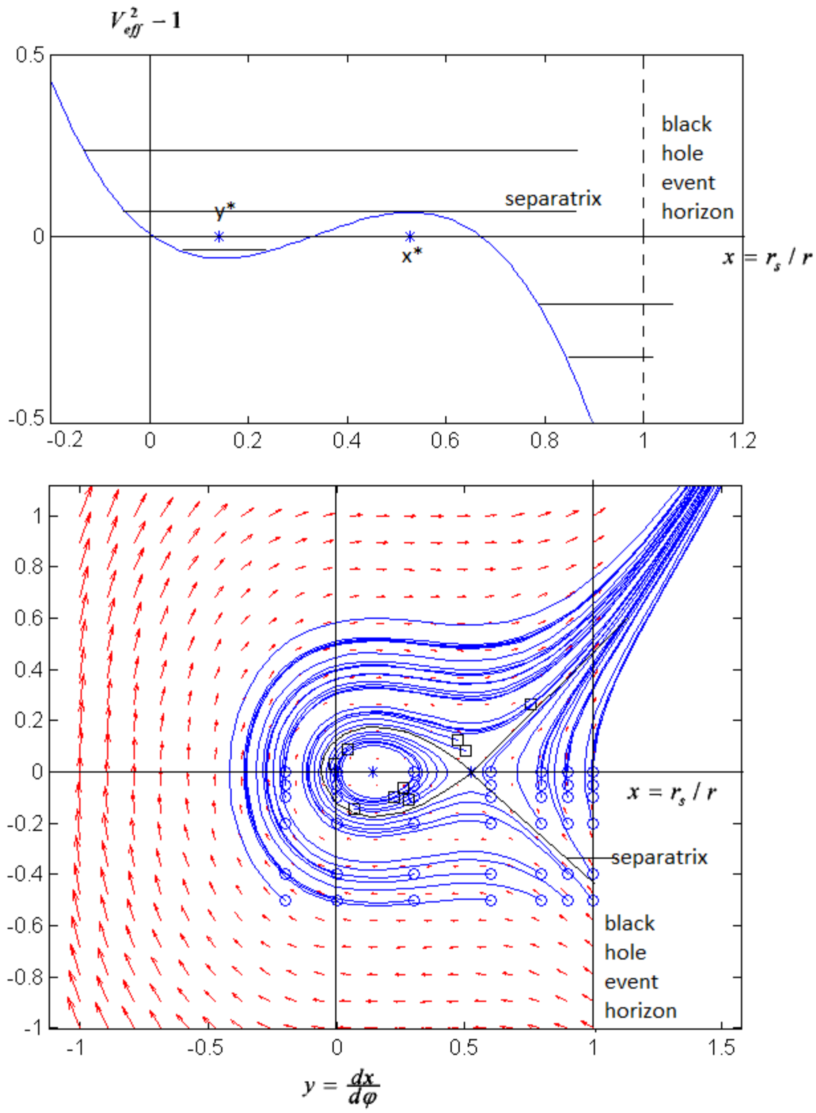

FIG. 4.3 – An exact phase plane for $\sigma = \frac{1}{9}$

The level curves, marked with blue, represent exact solutions for the system (10) for various initial conditions and energies.
For σ the random value $\frac{1}{9}$ has been assigned. It is useful then to see the exact interval in which σ can move and how values in this interval can affect qualitatively the solutions.
Two critical values for σ occur. One when considering the two equilibrium points given in (11). In this case, $\sigma \leq \frac{1}{6}$, because, otherwise, no real equilibrium points would exist. The second one appears when solving $\hat{E}^2 - 1 = 0$.

Here, $\sigma = \frac{1}{8}$.
Based on the following values for σ:

$$0 < \sigma < \frac{1}{8}, \quad \sigma = \frac{1}{8}, \quad \frac{1}{8} < \sigma < \frac{1}{6}, \quad \sigma = \frac{1}{6}, \quad \sigma > \frac{1}{6} \tag{20}$$

qualitative distinct orbits exist.
By placing an upper bound on σ we can deduce the existence of stable or unstable orbits for either equilibrium points in the phase plane. Thus, when $\sigma > \frac{1}{6}$, as seen from (11), no real equilibria exists for any given energy or angular momentum because the angular momentum isn't sufficient for m_0 to preserve an orbit and it just falls into M. Accordingly, $\hat{V}^2_{eff}$ has no extrema. Thus $\frac{1}{6}$ represents an upper bound for σ.

Remark 2. Solving the equation $\partial_x \hat{V}^2_{eff} = 0$ for x gives the localization of the stable and unstable orbits.
We have that $\hat{V}^2_{eff} = (1 + \frac{x^2}{2\sigma})(1 - x)$, thus

$$\partial_x \hat{V}^2_{eff} = -\frac{3x^2}{2\sigma} + \frac{x}{\sigma} - 1$$

Making this equation equal zero yields

$$3x^2 - 2x + 2\sigma = 0$$

for which the discriminant is $\Delta = 4 - 24\sigma$.
Since $\sigma \leq \frac{1}{6}$, implies $\Delta \geq 0$.
Suppose that $\Delta > 0$ thus, the equation yields only real roots,

$$x_{1,2} = \frac{1 \pm \sqrt{1 - 6\sigma}}{3}$$

which will give the minima and the maxima of the effective potential. Since $x = \frac{r_s}{r}$, we obtain that there exist two radii for circular orbits instead of only the one of the Newtonian case:

$$r_{1,2} = \frac{3r_s}{1 \pm \sqrt{1 - 6\sigma}}$$

When these values are substituted in the second derivative of the effective potential

$$\partial_x^2 \hat{V}^2_{eff} = \frac{1}{\sigma}(1 - 3x)$$

we conclude that the smaller radius of the two represent the stable one, while the larger, the unstable one.
From a physical standpoint, when a small perturbation occurs at the unstable circular radius, the particle will immediately fall into or out of the black hole, while, if this perturbation were to occur at the stable circular radius, it will cause only small radial oscillations.

When $\sigma = \frac{1}{6}$, then $\Delta = 0$ and $x_{1,2} = \frac{1}{3}$, thus the stable and unstable radii will correspond at $r_{1,2} = 3r_s$. As a consequence, a bifurcation point appears in the graph of $\hat{V}_{eff}^2 - 1$.

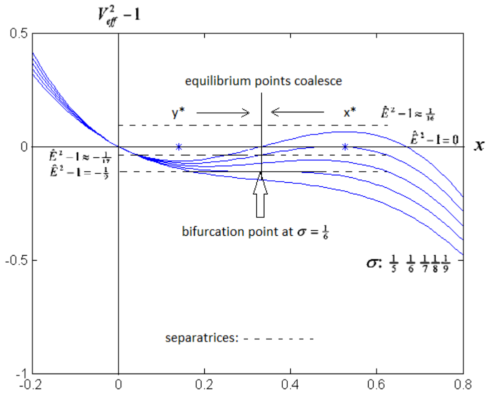

FIG. 4.4 – The Schwarzschild effective potential

In order to understand the significance of the other values which appear in (20) the separatrix structure of (19) has to be analysed.

DEFINITION 1. The separatrix is the boundary separating two modes of behaviour in a differential equation.

The primary use of the separatrix is to represent graphic the critical relationship between the angular momentum and the energy of the system at the unstable orbital radius.
Let σ be given. The critical energy corresponding to the unstable orbit is obtained from $\hat{E}_1^2$ of (11):

$$\hat{E}_1^2 - 1 = \frac{2\sigma[1 - 4\sigma - (1 - 6\sigma)^{\frac{1}{2}}]}{[(1 - 6\sigma)^{\frac{1}{2}} - 1]^3}$$

In Figure **??**, for $\sigma \in \{\frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$, the energies are computed and marked with horizontal lines. Substituting the obtained values into (19), we plot the separatrices of (20).

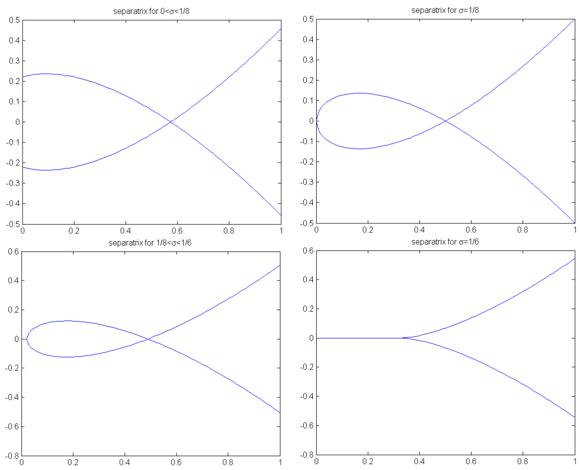

FIG. 4.5 – Separatrices for selected values of σ

We observe that, when $0 < \sigma < \frac{1}{6}$ these separatrices have divided the phase plane into four regions of motion.

As the above figure illustrates, depending on the values of angular momentum, separatrices are summarised as distinct unstable orbits:

(1) $0 < \sigma < \frac{1}{8} \implies$ unstable hyperbolic
(2) $\sigma = \frac{1}{8} \implies$ unstable parabolic
(3) $\frac{1}{8} < \sigma < \frac{1}{6} \implies$ unstable elliptic

It is also noted that, for $0 < \sigma < \frac{1}{8}$, hyperbolic, parabolic and elliptic orbits are possible before reaching the unstable hyperbolic orbit, whilst for $\frac{1}{8} \leq \sigma < \frac{1}{6}$ only elliptic orbits can occur before the unstable one.

When $\sigma = \frac{1}{6}$ a bifurcation occurs. This means that the phase plane's topological structure morphs as the two equilibrium points travel together, coalesce into one equilibrium point which will disappear from the phase plane as σ takes values greater than $\frac{1}{6}$. It is due to this fact that the Schwarzschild orbital dynamics can be approached as a conservative two dimensional bifurcation phenomena, namely a saddle-centre bifurcation phenomena.

If $\sigma \leq 0$, then, on one hand, no physical interpretation can be attributed to σ, since (5) asks for positive values of σ. On the other hand, the two equilibrium points change stability at $\sigma = 0$ which yields another bifurcation before the saddle-centre one.

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