On products of stable range one elements

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Abstract

1 Introduction

All the rings we consider are associative and have an identity $1 \neq 0$. For a ring R, $U(R)$ denotes the set of all the units of R and $N(R)$ denotes the set of all the nilpotents of R.

An element $a \in R$ has *left stable range 1* (*lsr1*, for short) if whenever $Ra + Rb = R$ for some $b \in R$, there exists a $y \in R$ such that $a + yb$ is a unit. Equivalently, if $xa + b = 1$ implies the existence of an element $y \in R$ such that $a + yb$ is a unit. Removing b, equivalently again, if for every $x \in R$ there exists $y \in R$ such that $a + y(1 - xa) \in U(R)$. In this case we also write $lsr(a) = 1$. A symmetric definition can be given on the right (*rsr1*, for short).

In the sequel, y will be called a *unitizer* for a with respect to x, denoted $y_x^{(a)}$.

It is now well-known (see [1], Lemma 17) that *any finite product of left (or right) stable range 1 elements has left (or right) stable range 1.*

In this short note we give a formula for the unitizer of a product of left stable range 1 elements and make some applications of it.

2 The formula

The proof goes along the lines of the proof of Lemma 17 in [1].

Theorem 1 *If $lsr(a') = 1$, for every $x \in R$ there exists a unitizer $y_{xa}^{(a')}$ such that $u := a' + y_{xa}^{(a')}(1 - xaa') \in U(R)$. If also $lsr(a) = 1$, a unitizer for the product aa' is given by the formula*

$$y_x^{(aa')} = ay_{xa}^{(a')} + y_{ux}^{(a)}u(1 - xay_{xa}^{(a')}).$$

Proof. Since $lsr(a') = 1$, a unitizer $y_{xa}^{(a')}$ and a unit $u =: a' + y_{xa}^{(a')}(1 - xaa')$, both exist. We start with $x(aa') + b = 1$ in R, that is, $b = 1 - xaa'$. Then

$1 = xa(u - y_{xa}^{(a')}b) + b = xau - cb$ with $c = xay_{xa}^{(a')} - 1$. Conjugation by u gives $1 = uxa - ucbu^{-1}$, while $a - y_{ux}^{(a)}ucbu^{-1} =: v \in U(R)$. Hence $au - y_{ux}^{(a)}ucb = vu \in U(R)$ and finally, since $u = a' + y_{xa}^{(a')}b$, $aa' + (ay_{xa}^{(a')} - y_{ux}^{(a)}uc)b = aa' + [ay_{xa}^{(a')} + y_{ux}^{(a)}u(1 - xay_{xa}^{(a')})] \in U(R)$, as stated. ■

Notice that if a is a unit, a unitizer (independent of x) is $y_x^{(a)} = 0$. Therefore

Corollary 2 *Let $u \in U(R)$. Then $y_x^{(au)} = y_{ux}^{(a)}u$ and $y_x^{(ua)} = uy_{xu}^{(a)}$.*

Corollary 3 *If $lsr(a) = 1$ then $lsr(-a) = 1$.*

Proof. Just take $u = -1$ in the previous corollary: $y_x^{(-a)} = -y_{-x}^{(a)}$. ■

Corollary 4 *Left sr1 elements are invariant to equivalences. If $u, v \in U(R)$ then*

$$y_x^{(uav)} = uy_{xu}^{(av)} = uy_{vx}^{(a)}v.$$

Proof. As above, this follows choosing zero the unitizers of units. ■

Recall that unit-regular elements (in particular, idempotents) have sr1. With respect to unitizers we provide the following

Proposition 5 *If $a = aua$ for $a \in R$ and $u \in U(R)$ then a unitizer for a (independent of x) is $u^{-1} - a$. If $e = e^2$, a unitizer for e (independent of x) is the complementary idempotent $1 - e$.*

Proof. Suppose $a = aua$ with $u \in U(R)$ and x arbitrary in R. We show that $a + (u^{-1} - a)(1 - xa) \in U(R)$, that is, a unitizer for a (independent of x) is $y_x^{(a)} = u^{-1} - a$. It suffices to replace a with auu^{-1}:

$a + (u^{-1} - a)(1 - xa) = auu^{-1} + (u^{-1} - auu^{-1})(1 - xauu^{-1}) =$

$= auu^{-1} + (1 - au)u^{-1}(1 - xauu^{-1}) = u^{-1} - (1 - au)u^{-1}xauu^{-1} =$

$= [1 - (1 - au)u^{-1}x(au)]u^{-1} \in U(R)$, since au is idempotent and $(1 - au)u^{-1}x(au)$ is square-zero.

Just taking $u = 1$ in the previous proof shows that $1 - e$ is a unitizer for an idempotent e. ■

References

[1] Huanyin Chen,W.K.Nicholson *Stable modules and a theorem of Camillo and Yu*. J. of Pure and Applied Algebra **218** (2014), 1431-1442.