On the recognition of rings of upper triangular matrices

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1 Introduction

It is well-known that the elements a of a (unital associative) ring R may be identified with the scalar matrices $aI_n \in \mathbb{M}_n(R)$ and these matrices commute with all the e_{ij}. The converse also holds (see [2] or [5]): *Let R be a ring containing n^2 elements e_{ij} satisfying $e_{ij}e_{rs} = \delta_{jr}e_{is}$* [the δ's equations] *and $\sum_{i=1}^{n} e_{ii} = 1$* [also called a *full set of matrix units*]. *Then $R \cong \mathbb{M}_n(C)$ where the ring C is the centralizer of the e_{ij} in R.*

Sketch of the proof. Define C as in the statement and for $a \in R$ put $a_{ij} = \sum\limits_{r=1}^{n} e_{ri}ae_{jr}$. Then $a_{ij} \in C$ and we have $\sum\limits_{i,j=1}^{n} a_{ij}e_{ij} = a$. Hence the bijection needed is $a \longleftrightarrow [a_{ij}]$, and this is verified to be an isomorphism.

Note that according to the δ's equations, this characterization of (full) matrix rings uses n idempotents and $n^2 - n$ zerosquare elements, that is, n^2 elements satisfying some conditions.

In the last decade of the past century, the "recognition" of (full) matrix rings, has seen a new impetus, mainly by diminishing the number of elements needed for a characterization.

In [7], a characterization was given using only $n+1$ elements (one nilpotent together with all its powers and another n elements) and in [1] (see also [5]), a characterization of matrix rings was given using only 3 elements (only two powers of a nilpotent and another two elements).

More precisely, these characterizations follow.

Robson: *A ring is a complete $n \times n$ matrix ring if and only if it contains elements $a_1, ..., a_n, f$ such that $f^n = 0$ and*

$$1 = a_1f^{n-1} + fa_2f^{n-2} + f^2a_3f^{n-3} + ... + f^{n-1}a_n.$$

Agnarrson et altri: *Let R be a ring and $p, q \geq 1$ be fixed integers. Then $R \cong \mathbb{M}_{p+q}(S)$ for some ring S iff there exist elements $a, b, f \in R$ such that $f^{p+q} = 0$ and $af^p + f^qb = 1$.*

In both characterizations, one nilpotent of index n (or $p+q$) is given together with some other elements which permit the reconstruction of the idempotents.

In the second characterization (see also [5], the "only if" part of **(17.10)**), a (strictly) *lower* triangular matrix (the sum of the elements on the subdiagonal) is taken

$$f = E_{21} + E_{32} + \cdots + E_{p+q,p+q-1},$$

for which $f^{p+q} = 0$, and another a, b two (strictly) *upper* triangular matrices (with identity superdiagonal) are needed.

Notice that here *it was essential* to use both lower and upper triangular matrices.

We proceed with the subject of this exposition, that is, the recognition of the upper triagular matrix rings.

A result of the same genre (the base ring of the ring of upper triangular matrices is the centralizer of the e_{ij}'s) is a special case (Theorem 2.9) in [8]. The characterization of rings of (upper) triangular matrices was given using $\frac{n(n+1)}{2}$ matrix units (i.e. *all* the upper triangular matrix units) together with some conditions.

More precisely, we recall this special case (for which the natural quasi-order on $\{1, 2, ..., n\}$ is considered).

Theorem 1 *Let R be a ring with a set $\{e_{ij} : i \leq j\}$ of matrix units, that is, $e_{11} + ... + e_{nn} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$.*

R is isomorphic with a ring of upper triangular matrices $\mathbb{T}_n(S)$ if and only if R has a subring S with $1 \in S$ and the following properties.

(i) $S \subseteq C(\{e_{ij} : i \leq j\})$;

(ii) $e_{ii}Re_{jj} = Se_{ij}$ for $i \leq j$;

(iii) $e_{jj}Re_{ii} = \{0\}$ for $i < j$;

(iv) If $se_{ij} = 0$ for some $s \in S$ (and $i \leq j$), then $s = 0$.

Moreover, the isomorphism $\phi : \mathbb{T}_n(S) \longrightarrow R$ is given: $\phi([a_{ij}]) = \sum_{i \leq j} a_{ij}e_{ij}$.

The goal of this exposition is to perform the explicit inverse of ϕ, which is not an easy task even for $n = 2$.

2 The $n = 2$ case

To simplify the writing denote $e := e_{11}$, $t := e_{12}$ and so $\overline{e} = e_{22}$.

2.1 Prerequisites

Lemma 2 *Let $e = e^2$ and $r \in R$. The following conditions are equivalent.*

(i) $ere = re$;

(ii) $\overline{e}re = 0$;

(iii) $\overline{e}r\overline{e} = \overline{e}r$.

Proof. (i) $\Rightarrow$ (ii) $\overline{e}(re) = (\overline{e}e)re = 0$.

(ii) $\Rightarrow$ (iii) $\overline{e}r\overline{e} = \overline{e}r(1-e) = \overline{e}r$.

(iii) $\Rightarrow$ (i) $ere = (1-\overline{e})r(1-\overline{e}) = r - \overline{e}r - r\overline{e} + \overline{e}r\overline{e} = r - r\overline{e} = re$. ■

Definition. An idempotent e is called *left semicentral* if $\overline{e}Re = \{0\}$. Equivalently, $re = ere$ for every $r \in R$ and also equivalently, $eR \lhd R$.

Lemma 3 *If e is left semicentral, and $t \in R$ satisfies $et = t$, $te = 0$ then*

(iv) $trt = 0$,

(v) $tre = 0$, (v') $tr\overline{e} = tr$,

(vi) $\overline{e}rt = 0$; (vi') $ert = rt$,

(vii) $et = t$ equivalent to $\overline{e}t = 0$;

(viii) $t\overline{e} = t$ equivalent to $te = 0$.

Proof. Just notice that in any ring and for any two elements a, b from $ab = b$, $ba = 0$ follows $b^2 = 0$. Then $t^2 = 0$ and the rest is easy. ■

In what follows, (i)-(viii) will be used **only** with respect to this previous lemma.

Remark. Note that $t \neq 0$ implies $e \notin \{0, 1\}$ (from $et = t$ resp $te = 0$). However all equalities hold also for $t = 0$.

We also mention $er\overline{e} = er(1-\overline{e}) \stackrel{(i)}{=} er - re$ (so a left semicentral idempotent may **not** be central).

In the sequel we suppose $e^2 = e$, $et = t$ and $te = 0$ (so $t^2 = 0$). Notice that $eR\overline{e} = e_{11}Re_{22} = e_{12}S = Se_{12}$.

Lemma 4 *Let $S = C(t)$. If $eR\overline{e} = St = tS$ then for every $r \in R$ there exist $s, s', s'' \in S$ such that $rt = ts$, $tr = s't$ and $er\overline{e} = s''t$.*

Proof. The existence follows from the hypothesis and the following equalities $rt \stackrel{(vi)}{=} ert \stackrel{(viii)}{=} e(rt)\overline{e} = ts$, $tr \stackrel{(v')}{=} tr\overline{e} \stackrel{(vii)}{=} etr\overline{e} = s't$ resp. $er\overline{e} = s''t$. ■

2.2 Direct characterization for triangular 2×2 matrix rings

To simplify the wording, we say that t has the *cancellation property* if $s = 0$ whenever $s \in S$ and $st = 0$.

Theorem 5 *Let e be a left semicentral idempotent in a ring R, $t \in R$ is such that $et = t$, $te = 0$ and for $S = C(e, t)$ assume that t has the cancellation property (with respect to S) and $eR\overline{e} = St = tS$. Then the map $\varphi : R \to \mathbb{T}_2(S)$ given by $\varphi(r) = \begin{bmatrix} re + \overline{e}s & s'' \\ 0 & \overline{e}r + s'e \end{bmatrix}$ where $rt = ts$, $tr = s't$ and $er\overline{e} = s''t$ is a ring isomorphism.*

The inverse is given by $\varphi^{-1}\left(\begin{bmatrix} x & y \\ 0 & w \end{bmatrix}\right) = xe + yt + w\overline{e}$.

Proof. We have already mentioned that from $et = t$, $te = 0$ follows $t^2 = 0$. The existence of the elements s, s', s'' follows from Lemma 4. Notice that, by the cancellation property of t, these elements are unique with respect to $rt = ts$, $tr = s't$ and $er\overline{e} = s''t$, respectively.

The entries of $\varphi(r)$ belong to S, the centralizer $C(e,t) = C(t,\overline{e})$.

Indeed, $e(re + \overline{e}s) = ere \overset{(i)}{=} re \overset{(ii)}{=} (re + \overline{e}s)e$, and $t(re + \overline{e}s) \overset{\{v),(viii)}{=} ts$, $(re + \overline{e}s)t \overset{(vi)}{=} rt$, and

$\overline{e}(\overline{e}r + s'e) = \overline{e}r + \overline{e}s'e \overset{(ii)}{=} \overline{e}r \overset{(iii)}{=} \overline{e}r\overline{e} = (\overline{e}r + s'e)\overline{e}$, and $t(\overline{e}r + s'e)t \overset{(viii),(v)}{=} tr \overset{(v')}{=} tr\overline{e} \overset{(vii)}{=} etr\overline{e} = s't \overset{(vi)}{=} (\overline{e}r + s'e)t$

φ is **unital**, i.e., $\varphi(1) = I_2$: indeed, for $r = 1$ we get $s = s' = 1$ and $s'' = 0$ [using the cancellation property of elements of S with respect to t and $e + \overline{e} = 1$].

φ is **additive**: we have $r_1t = ts_1$, $r_2t = ts_2$, $tr_1 = s_1't$, $tr_2 = s_2't$ and $er_1\overline{e} = s_1''t$, $er_2\overline{e} = s_2''t$. ■

Then $\varphi(r_1) = \begin{bmatrix} r_1e + \overline{e}s_1 & s_1'' \\ 0 & \overline{e}r_1 + s_1'e \end{bmatrix}$, $\varphi(r_2) = \begin{bmatrix} r_2e + \overline{e}s_2 & s_2'' \\ 0 & \overline{e}r_2 + s_2'e \end{bmatrix}$ and so $\varphi(r_1) + \varphi(r_2) = \begin{bmatrix} (r_1 + r_2)e + \overline{e}(s_1 + s_2) & s_1'' + s_2'' \\ 0 & \overline{e}(r_1 + r_2) + (s_1' + s_2')e \end{bmatrix} = \varphi(r_1 + r_2)$. Indeed, $(r_1 + r_2)t = ts_1 + ts_2 = t(s_1 + s_2)$, $t(r_1 + r_2) = s_1't + s_2't = (s_1' + s_2')t$ and $e(r_1 + r_2)\overline{e} = s_1''t + s_2''t = (s_1'' + s_2'')t$.

φ is **multiplicative** amounts to [comparing the corresponding entries of both upper triangular matrices]:

(1,1): $(r_1e + \overline{e}s_1)(r_2e + \overline{e}s_2) = r_1r_2e + \overline{e}s_1s_2$ [indeed $\overline{e}(s_1r_2)e \overset{(ii)}{=} 0$ and $r_1(er_2e) \overset{(i)}{=} r_1r_2e$, $(\overline{e}s_1\overline{e})s_2 \overset{(iii)}{=} \overline{e}s_1s_2$]

(1,2): $(r_1e + \overline{e}s_1)s_2'' + s_1''(\overline{e}r_2 + s_2'e) = s_{1,2}''$ where $e(r_1r_2)\overline{e} = s_{1,2}''t$ [here we have to check $[(r_1e + \overline{e}s_1)s_2'' + s_1''(\overline{e}r_2 + s_2'e)]t = e(r_1r_2)\overline{e} = er_1r_2 - r_1r_2e$;

indeed $[(r_1e + \overline{e}s_1)s_2'' + s_1''(\overline{e}r_2 + s_2'e)]t = (r_1e + \overline{e}s_1)er_2\overline{e} + s_1''\overline{e}r_2t + s_1''tr_2 \overset{(ii)}{=}$
$= r_1er_2\overline{e} + er_1\overline{e}r_2 = r_1er_2 - r_1r_2e + er_1r_2 - r_1er_2 \overset{(i)}{=} -r_1r_2e + er_1r_2]$

(2,2): $(\overline{e}r_1 + s_1'e)(\overline{e}r_2 + s_2'e) = \overline{e}r_1r_2 + s_1's_2'e$ [indeed $\overline{e}(r_1s_2')e \overset{(ii)}{=} 0$ and $\overline{e}r_1\overline{e}r_2 \overset{(iii)}{=} \overline{e}r_1r_2$, $s_1'es_2'e \overset{(i)}{=} s_1's_2'e$].

φ is **injective**: assume $\varphi(r) = \begin{bmatrix} re + \overline{e}s & s'' \\ 0 & \overline{e}r + s'e \end{bmatrix} = 0_2$, that is, $re + \overline{e}s = \overline{e}r + s'e = s'' = 0$ with $rt = ts$, $tr = s't$ and $er\overline{e} = s''t$.

As $s'' = 0$ we have $er\overline{e} = 0$ and (using (ii)) $er = ere = re$. Right multiplication by e of $re + \overline{e}s = 0$ gives (by (iii)) $re = 0$. Left multiplication by $\overline{e}$ of $\overline{e}r + s'e = 0$ gives (also by (iii)) $\overline{e}r = 0$, and so $0 = r - er = r - re = r$, as desired.

φ is **surjective**: $xe + yt + w\overline{e} = (re + \overline{e}s)e + er\overline{e} + (\overline{e}r + s'e)\overline{e} = re + er(1 - e) + (1 - e)r(1 - e) = r$, as desired.

3 The 3×3 case

Here we start with the δ's relations for three orthogonal idempotents e_{11}, e_{22}, e_{33}, two zerosquare e_{12}, e_{23} and we define $e_{13} := e_{12}e_{23}$.

The additional δ's relations hold: $e_{13}^2 = e_{12}(e_{23}e_{12})e_{23} = 0$, $e_{11}e_{13} = e_{11}e_{12}e_{23} = e_{12}e_{23} = e_{13}$, $e_{13}e_{11} = e_{12}(e_{23}e_{11}) = 0$, $e_{13}e_{22} = e_{12}(e_{23}e_{22}) = 0$, $e_{22}e_{13} = (e_{22}e_{12})e_{23} = 0$.

Notice that $e_{11} + e_{22} + e_{33} = 1$ implies $\overline{e_{11}} = e_{22} + e_{33}$ and other two permutations.

We gather here some obvious equivalences, consequences of the assumed δ's relations.

Lemma 6 *(vii)* $e_{11}e_{12} = e_{12}$ *equivalent to* $\overline{e_{11}}e_{12} = 0$*;*

(viii) $e_{12}\overline{e_{11}} = e_{12}$ *equivalent to* $e_{12}e_{11} = 0$.

(xii) $e_{11}e_{13} = e_{13}$ *equivalent to* $\overline{e_{11}}e_{13} = 0$.

(xiii) $e_{13}\overline{e_{11}} = e_{13}$ *equivalent to* $e_{13}e_{11} = 0$.

$e_{12}e_{22} = e_{12}$ *equivalent to* $e_{12}\overline{e_{22}} = 0$.

$\overline{e_{22}}e_{12} = e_{12}$ *equivalent to* $e_{22}e_{12} = 0$.

$e_{22}e_{23} = e_{23}$ *equivalent to* $\overline{e_{22}}e_{23} = 0$.

$e_{23}\overline{e_{22}} = e_{23}$ *equivalent to* $e_{23}e_{22} = 0$.

(xxi) $e_{23}e_{33} = e_{23}$ *equivalent to* $e_{23}\overline{e_{33}} = 0$.

(xxii) $\overline{e_{33}}e_{23} = e_{23}$ *equivalent to* $e_{33}e_{23} = 0$.

(xxv) $e_{13}e_{33} = e_{13}$ *equivalent to* $e_{13}\overline{e_{33}} = 0$.

$\overline{e_{33}}e_{13} = e_{13}$ *equivalent to* $e_{33}e_{13} = 0$.

Summarizing, for any idempotent e and element a,
$ea = a$ is equivalent to $\overline{e}a = 0$
$ae = a$ is equivalent to $a\overline{e} = 0$
$ae = 0$ is equivalent to $a\overline{e} = a$, and
$ea = 0$ is equivalent to $\overline{e}a = a$.

As $\overline{e_{33}} = e_{11} + e_{22}$, we also have $\overline{e_{33}}e_{22} = e_{22}\overline{e_{33}} = e_{22}$ and similarly $\overline{e_{33}}e_{11} = e_{11}\overline{e_{33}} = e_{11}$, $\overline{e_{22}}e_{11} = e_{11}\overline{e_{22}} = e_{11}$.

We still suppose e_{11} is *left semicentral* (i.e., $\overline{e_{11}}Re_{11} = \{0\}$). Hence all properties in Lemma 3 hold but we can add many new ones (related to e_{11}) corresponding to the zerosquares e_{23}, e_{13}. However, *here* $\overline{e_{11}} = e_{22}$ *fails !* We now have $\overline{e_{11}} = e_{22} + e_{33}$ (from $e_{11} + e_{22} + e_{33} = 1$).

Lemma 7 *(i)* $e_{11}re_{11} = re_{11}$*;*

(ii) $\overline{e_{11}}re_{11} = 0$*;*

(iii) $\overline{e_{11}}r\overline{e_{11}} = \overline{e_{11}}r$.

(iv) $e_{12}re_{12} = 0$,

(v) $e_{12}re_{11} = 0$, *(v')* $e_{12}r\overline{e_{11}} = e_{12}r$,

(vi) $\overline{e_{11}}re_{12} = 0$*; (vi')* $e_{11}re_{12} = re_{12}$,

(ix) $e_{13}re_{13} = 0$,

(x) $e_{13}re_{11} = 0$, *(x')* $e_{13}r\overline{e_{11}} = e_{13}r$,

(xi) $\overline{e_{11}}re_{13} = 0$*; (xi')* $e_{11}re_{13} = re_{13}$,

However e_{22} may not be left nor right semicentral but e_{33} *is right semicentral*.

Assuming $e_{33}r\overline{e_{33}} = 0$ leads to another (similar and symmetric properties):

Lemma 8 *(xiv)* $e_{33}re_{33} = e_{33}r$,
(xv) $e_{33}r\overline{e_{33}} = 0$,
(xvi) $\overline{e_{33}}r\overline{e_{33}} = r\overline{e_{33}}$,
(xvii) $e_{33}re_{23} = 0$, *(xvii')* $\overline{e_{33}}re_{23} = re_{23}$,
(xviii) $e_{23}re_{23} = 0$
(xx) $e_{23}r\overline{e_{33}} = 0$; *(xx')* $e_{23}re_{33} = e_{23}r$,
(ix) $e_{13}re_{13} = 0$,
(xxiii) $e_{33}re_{13} = 0$, *(xxiii')* $\overline{e_{33}}re_{13} = re_{13}$,
(xxiv) $e_{13}r\overline{e_{33}} = 0$, *(xxiv')* $e_{13}re_{33} = e_{13}r$.

We start with

$$r = x_{11}e_{11} + x_{12}e_{12} + x_{13}e_{13} + x_{22}e_{22} + x_{23}e_{23} + x_{33}e_{33} \quad (2)$$

and are looking to $x_{ij} \in C(e_{11}, e_{22}, e_{12}, e_{23}) = C(e_{12}, e_{23}, e_{22}, e_{33}) =$
$= C(e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33})$.

The 12 relations obtained by left and right multiplication of (2) with the six e_{ij}. The x_{ij}'s are supposed to belong to the centralizer.

$e_{11}r = x_{11}e_{11} + \underline{x_{12}}e_{12} + x_{13}e_{13}$, $re_{11} = x_{11}e_{11}$
$e_{22}r = x_{22}e_{22} + x_{23}e_{23}$, $re_{22} = \underline{x_{12}}e_{12} + x_{22}e_{22}$
$e_{33}r = x_{33}e_{33}$, $re_{33} = x_{13}e_{13} + x_{23}e_{23} + x_{33}e_{33}$
$e_{12}r = x_{22}e_{12} + x_{23}e_{13}$, $re_{12} = x_{11}e_{12}$
$e_{13}r = x_{33}e_{13}$, $re_{13} = x_{11}e_{13}$
$e_{23}r = x_{33}e_{23}$, $re_{23} = \underline{x_{12}}e_{13} + x_{22}e_{23}$.

11. For x_{11}, by left and right multiplication with e_{11} we get $(x_{11} - r)e_{11} = 0 = (x_{11} - r)e_{12}$ [actually also $(x_{11} - r)e_{13} = 0$] and the continuation is (almost !) identical to the $n = 2$ case.

Since $(x_{11} - r)e_{11} = (x_{11} - r)e_{12} = 0$ we are searching for $x_{11} = r + \alpha$, commuting with e_{11} and e_{12}, where $\alpha e_{11} = \alpha e_{12} = 0$.

$e_{11}x_{11} = x_{11}e_{11}$ is equivalent to $e_{11}\alpha = re_{11} - e_{11}r \overset{(i)}{=} e_{11}re_{11} - e_{11}r = -e_{11}r\overline{e_{11}}$ whence $e_{11}(\alpha + r\overline{e_{11}}) = 0$, $\alpha + r\overline{e_{11}} \in r(e_{11}) \ni \overline{e_{11}}$

and so we choose $\alpha + r\overline{e_{11}} = \overline{e_{11}}\beta$. This way $e_{11}x_{11} = e_{11}re_{11} \overset{(i)}{=} re_{11} = x_{11}e_{11}$ and we have to choose β for $e_{12}x_{11} = x_{11}e_{12}$.

We have $e_{12}x_{11} = e_{12}re_{11} + e_{12}\overline{e_{11}}\beta \overset{(v),(viii)}{=} e_{12}\beta$ and $x_{11}e_{12} = re_{11}e_{12} + \overline{e_{11}}\beta e_{12} \overset{(vi)}{=} re_{12}$, so we need β such that $re_{12} = e_{12}\beta$.

As $re_{12} \overset{(vi')}{=} e_{11}re_{12} \overset{(viii)}{=} e_{11}re_{12}\overline{e_{11}} \in e_{11}R\overline{e_{11}} = e_{11}R(e_{22}+e_{33}) \subseteq e_{11}Re_{22} + e_{11}Re_{33} = e_{12}S + e_{13}S = e_{12}S + e_{12}e_{23}S = e_{12}S$.

For $n = 3$, $\overline{e_{11}} \neq e_{22}$. It is now $\overline{e_{11}} = e_{22} + e_{33}$.

Hence

$$x_{11} = re_{11} + \overline{e_{11}}s$$

with $re_{12} = e_{12}s$.

The commutation with e_{13}: $x_{11}e_{13} \overset{(xi)}{=} re_{13} = re_{12}e_{23} = e_{12}se_{23}$ and $e_{13}x_{11} \overset{(x),(xiii)}{=} e_{13}s = e_{12}e_{23}s = e_{12}se_{23}$ since e_{ij} commute with elements in S.

The commutation with e_{22}: $x_{11}e_{22} = \overline{e_{11}}se_{22} = \overline{e_{11}}e_{22}s = e_{22}s$ and $e_{22}x_{11} = e_{22}re_{11} + e_{22}\overline{e_{11}}s = e_{22}s$ since $e_{22}Re_{11} = \{0\}$[was already checked]

The commutation with e_{23}: $x_{11}e_{23} = \overline{e_{11}}se_{23} = \overline{e_{11}}e_{23}s = e_{23}s$ and $e_{23}x_{11} = e_{23}re_{11} + e_{23}\overline{e_{11}}s = e_{23}s$ since $e_{23}re_{11} = e_{22}\,(e_{23}r)\,e_{11} = 0$ (as above).

Checking $x_{11} = re_{11} + \overline{e_{11}}s$ with $re_{12} = e_{12}s$.

$re_{11} = (re_{11} + \overline{e_{11}}s)e_{11} \overset{(ii)}{=} re_{11}$, ok, $re_{13} = (re_{11} + \overline{e_{11}}s)e_{13} \overset{(xi)}{=} re_{13}$, ok

$re_{12} = (re_{11} + \overline{e_{11}}s)e_{12} \overset{(vi)}{=} re_{12}$ ok, and

$e_{11}r = (re_{11} + \overline{e_{11}}s)e_{11} + x_{12}e_{12} + x_{13}e_{13} \overset{(ii)}{=} re_{11} + x_{12}e_{12} + x_{13}e_{13}$, that is,

$$x_{12}e_{12} + x_{13}e_{13} = e_{11}r - re_{11} \qquad (3)$$

remains to be checked. That x_{11} belongs to the centralizer was checked above.

33. Now x_{33} will be similar to x_{11}. We start with $[e_{13}(x_{33} - r) =]e_{23}(x_{33} - r) = e_{33}(x_{33} - r) = 0$ that is $x_{33} = r + \alpha$ with $[e_{13}\alpha =]e_{23}\alpha = e_{33}\alpha = 0$.

$e_{33}x_{33} = x_{33}e_{33}$ is equivalent to $\alpha e_{33} = e_{33}r - re_{33} \overset{(xiv)}{=} e_{33}re_{33} - re_{33} = -\overline{e_{33}}re_{33}$ whence $(\alpha + \overline{e_{33}}r)e_{33} = 0$, $\alpha + \overline{e_{33}}r \in l(e_{33}) \ni \overline{e_{33}}$ and so we choose $\alpha + \overline{e_{33}}r = \beta\overline{e_{33}}$. Thus $x_{33} = e_{33}r + \beta\overline{e_{33}}$ and we choose β for $e_{23}x_{33} = x_{33}e_{23}$. This (using (xx) resp. (xxii), (xvii)) gives $e_{23}r = \beta e_{23}$. Since $e_{23}r \overset{(xx')}{=} e_{23}re_{33} = e_{22}e_{23}re_{33} \in e_{22}Re_{33} = e_{23}S$ so there is s_3 such that $e_{23}r = s_3e_{23}$ and

$$x_{33} = e_{33}r + s_3\overline{e_{33}}.$$

The commutation with e_{13}: $x_{33}e_{13} = e_{33}re_{13} + s_3\overline{e_{33}}e_{13} \overset{(xviii'),(xviii)}{=} s_3e_{13}$ and $e_{13}(e_{33}r + s_3\overline{e_{33}}) \overset{(xiv)}{=} e_{13}r$, which holds, left multiplying $e_{23}r = s_3e_{23}$ by e_{12}, since e_{ij} commute with elements in S.

The commutation with e_{22}: $x_{33}e_{22} = (e_{33}r + s_3\overline{e_{33}})e_{22} = s_3e_{22}$ and $e_{22}x_{33} = e_{22}(e_{33}r + s_3\overline{e_{33}}) = s_3e_{22}$ since $e_{22}\overline{e_{33}} = \overline{e_{33}}e_{22} = e_{22}$.

The commutation with e_{12}: $x_{33}e_{12} = (e_{33}r + s_3\overline{e_{33}})e_{12} = s_3e_{12}$ and $e_{12}x_{11} = e_{12}(e_{33}r + s_3\overline{e_{33}}) = s_3e_{12}$ since $e_{12}\overline{e_{33}} = \overline{e_{33}}e_{12} = e_{12}$.

It remains to verify $re_{33} = x_{13}e_{13} + x_{23}e_{23} + x_{33}e_{33}$, which amounts to $x_{13}e_{13} + x_{23}e_{23} = re_{33} - (e_{33}r + s_3\overline{e_{33}})e_{33} = re_{33} - e_{33}r$.

12+22. The relations are $e_{11}r = x_{11}e_{11} + \underline{x_{12}}e_{12} + x_{13}e_{13}$, $re_{22} = \underline{x_{12}}e_{12} + x_{22}e_{22}$ and $re_{23} = \underline{x_{12}}e_{13} + x_{22}e_{23}$.

Replacement in the first $e_{11}r = (re_{11} + \overline{e_{11}}s)e_{11} + \underline{x_{12}}e_{12} + x_{13}e_{13}$ gives (again) (3): $x_{12}e_{12} + x_{13}e_{13} = e_{11}r - re_{11}$.

We can write the other two $(r - x_{22})e_{22} = x_{12}e_{12}$ and $(r - x_{22})e_{23} = x_{12}e_{13} = x_{12}e_{12}e_{23}$ [the later follows from previous by right multiplication with e_{23}).

So $r - x_{22} - x_{12}e_{12} \in l(e_{23}) = \{e_{11}, e_{13}, e_{23}, e_{33}\}$.

We write also the other x_{22} relations: $e_{22}r = x_{22}e_{22} + x_{23}e_{23}$, $e_{12}r = x_{22}e_{12} + x_{23}e_{13}$. Hence

$e_{22}(r - x_{22}) = x_{23}e_{23}$ and $e_{12}(r - x_{22}) = x_{23}e_{13}$ [the x's are in the centralizer]. Both include x_{23}.

The later can be written $r - x_{22} - x_{23}e_{23} \in r(e_{12}) = \{e_{11}, e_{12}, e_{13}, e_{33}\}$ and so $(1 - e_{22})(r - x_{22}) \in r(e_{12}) = \{e_{11}, e_{12}, e_{13}, e_{33}\}$.

Here $1 - e_{22} = e_{11} + e_{33}$.

Too complicated, so I give up !

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