NEGATIVE NIL CLEAN RINGS

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Abstract. A ring is called negative nil clean if the negative of each nil clean element is also nil clean. It turns out that a ring is negative nil-clean iff 2 is nilpotent. Consequently, this is a Morita invariant property. The nil-clean 2×2 matrices whose negative are (not) nil-clean are described over several types of rings.

1. Introduction

The rings we consider are associative and unital (i.e., with identity).

For a ring R, $Id(R)$ denotes the idempotents of R, $N(R)$ the nilpotents of R and $\mathrm{ncn}(R)$ denotes the set of all nil-clean elements of R (i.e., a sum of an idempotent and a nilpotent). In any ring $0, 1$ are the *trivial* idempotents. A nil-clean element $a = e + t$ with $e \in Id(R)$ and $t \in N(R)$ is *nontrivial* if e is not trivial. When we want to emphasize the idempotent we say a is *e-nil-clean*.

We start with the following

Definition. A ring is called *negative nil-clean* if the negative of each nil-clean element is also nil clean. Equivalently, $-\mathrm{ncn}(R) \subseteq \mathrm{ncn}(R)$ and so $\mathrm{ncn}(R) = -\mathrm{ncn}(R)$. Nil-clean rings are negative nil-clean and the converse obviously holds for rings of characteristics 2, e.g., Boolean rings.

Since $\mathrm{ncn}(R) = \{0, 1\}$, any *connected* (i.e., with only trivial idempotents) *reduced* ring R with characteristics $\neq 2$ (e.g., $\mathbb{Z}$ or any field with more than two elements) is *not* negative nil-clean.

Since nil-clean rings are clean, we expect that conditions which refer to nil-clean rings should be simpler to describe than the corresponding ones referring to clean rings.

In [4], the theory of negative clean rings was developed, among others giving examples of images of negative clean rings which are not negative clean.

In this note, a simple characterization is given for negative nil-clean rings. These turn out to be precisely the rings for which 2 is nilpotent.

In the last section, over several types of rings, the nil-clean 2×2 matrices with (not) nil-clean negative are described.

2. Negative nil-clean rings

It is well-known (see [6]) that in any nil-clean ring, 2 is a (central) nilpotent. This remains true also for negative nil-clean rings. Actually this turns out to be the characterization for negative nil-clean rings. We have

Lemma 1. -1 *is nil-clean in a (unital) ring iff* 2 *is nilpotent.*

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Proof. If $-1 = e + t$ with $e^2 = e$ and $t \in N(R)$ then $e = -(1+t) \in U(R)$ so $e = 1$. Hence $2 = -t \in N(R)$. Conversely, if $2 \in N(R)$ then $-1 = 1 - 2$ is a nil-clean decomposition. □

In order to prove the characterization of negative nil-clean rings, we need the following key lemma.

Lemma 2. *If R is a ring, x a nilpotent element of R, and y an element of a nil ideal I of R, then $x + y$ is nilpotent.*

Proof. Say $x^m = 0$ and let I be a nil ideal. Since $y \in I$, the image of $x + y$ in R/I is the same as the image of x, so the image of $(x + y)^m$ in R/I is the same as the image of $x^m = 0$, i.e., $(x+y)^m \in I$. Hence $(x+y)^m$ is nilpotent and so is $x+y$. □

As mentioned in the Introduction, we have

Theorem 3. *A ring R is negative nil-clean iff 2 is a (central) nilpotent.*

Proof. Let R be a negative nil-clean ring. Then since 1 (as idempotent) is nil-clean, -1 must also be nil-clean and the statement follows from the Lemma 1.

Conversely, suppose 2 is nilpotent in R, and consider a nil-clean element $e + t$. We have $-(e + t) = e + (-2e - t)$, where in the second summand, $-2e$ belongs to the nilpotent (and so nil) ideal $2R$ (nilpotent because 2 is nilpotent and central), and $-t$ is nilpotent, so by Lemma 2, $-2e - t$ is nilpotent. Hence $-(e + t)$ is indeed nil-clean. □

As easy examples we mention

Proposition 4. *The following are equivalent.*

(i) $\mathbb{Z}_n$ is nil-clean;

(ii) $\mathbb{Z}_n$ is negative nil-clean;

(iii) $n = 2^k$ for some positive integer k.

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) Assume n is divisible by some odd prime. Then 2 is not nilpotent and so -1 is not nil-clean (by Lemma 1). Hence $\mathbb{Z}_n$ is not negative nil-clean.

(iii) $\Rightarrow$ (i) If $n = 2^k$ for some positive integer k, every element of $\mathbb{Z}_n$ can be written either as $0 + 2s$ or as $1 + 2s$. Both are nil-clean. □

Next, we gather some simple properties of negative nil-clean rings. The term *subring* of a ring R will be used including $1 \in R$. *Ring homomorphisms* will be assumed preserving identities.

Proposition 5. *Let R be any ring.*

(i) The negative of a nil-clean element is clean. So though in general $\mathrm{ncn}(R) \subsetneq$ $\mathrm{cn}(R)$ *(the set of clean elements of R), we have* $-\mathrm{ncn}(R) \subseteq \mathrm{cn}(R)$.

(ii) Subrings, overrings and images of negative nil-clean rings are also negative nil-clean.

(iii) If R is not negative nil-clean, and I is an ideal, R/I may be (negative) nil-clean.

(iv) Direct products of rings are negative nil-clean iff all components are negative nil-clean.

(v) Strongly (or uniquely) nil-clean elements need not have nil-clean negative.

(vi) Matrix rings over negative nil-clean rings are also negative nil-clean.

Proof. (i) If $a = e + t$ then $-a = (1 - e) - (1 + t)$ is clean.

(ii) Follows from the characterization theorem.

(iii) Indeed $\mathbb{Z}$ is not negative nil-clean, but $\mathbb{Z}_4$ is (strongly) nil-clean and so negative nil-clean.

(iv) Obvious.

(v) Take $1 \in \mathbb{Z}$ or any other connected reduced ring with characteristics $\neq 2$.

(vi) Indeed, if 2 is nilpotent in R, so is $2I_n$ in $\mathbb{M}_n(R)$. □

In particular, *corners* and *centers* of negative nil-clean rings are negative nil-clean. Combining with (vi) above, it follows that

Proposition 6. *The negative nil-clean property is Morita invariant.*

It is well-known that polynomial rings are not clean and so nor nil-clean. However

Proposition 7. *Polynomial rings over negative nil-clean rings are negative nil-clean.*

Proof. Follows from the characterization theorem. □

Corollary 8. *Negative nil-clean rings need not be clean.*

Moreover, *clean rings need not be negative nil-clean.* As an example, any field with more than two elements is clean but 2 is not nilpotent.

Since polynomial rings over nonzero exchange rings are never exchange rings (see [7]), *negative nil-clean rings need not be exchange.*

Further, since *(nil-)clean rings are both exchange and negative nil-clean* we may wonder about the converse. In order to show that *the converse fails*, we can use [10] (Example 3.1, starting with a field F of characteristic 2). Indeed, there exists an exchange ring of characteristic 2 that is not (nil-)clean, but it is negative nil-clean.

If R is a ring and $\mathbb{T}_n(R)$ denotes the ring of *upper triangular* matrices over R, we provide a direct proof (from definition) for the following

Proposition 9. $\mathbb{T}_n(R)$ *is negative nil-clean iff* R *is negative nil-clean.*

Proof. Let $a = e + t \in \mathrm{ncn}(R)$. Then $aI_n = eI_n + tI_n \in \mathrm{ncn}(\mathbb{T}_n(R))$ and so $-aI_n = E + T \in \mathrm{ncn}(\mathbb{T}_n(R))$, by hypothesis. Then $-a = e_{11} + t_{11} \in \mathrm{ncn}(R)$ (as idempotent or nilpotent upper triangular matrices have idempotent resp. nilpotent entries on the diagonal).

Conversely, suppose $A = E + T$ is nil-clean in $\mathbb{T}_n(R)$. Then $a_{ii} = e_{ii} + t_{ii}$ are all nil-clean in R for $1 \leq i \leq n$. Hence, by hypothesis, $-a_{ii} = f_i + s_i$ (with idempotent f_i and nilpotent s_i) are also nil-clean in R. Then $-A =$

$$\begin{bmatrix} f_1 + s_1 & -a_{12} & \cdots & -a_{1n} \\ 0 & f_2 + s_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & f_n + s_n \end{bmatrix} =$$

$$= \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & f_n \end{bmatrix} + \begin{bmatrix} s_1 & -a_{12} & \cdots & -a_{1n} \\ 0 & s_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix} \in \mathrm{ncn}(\mathbb{T}_n(R)).$$ □

Similarly one proves

Proposition 10. *For any ring R and any $R-R$-bimodule M, the trivial extension* $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ *is negative nil-clean iff R is negative nil-clean.*

Finally for formal power series we have

Proposition 11. *If $R[[X]]$ is negative clean then R is negative clean.*

Proof. As already mentioned, images of negative nil-clean rings are negative nil-clean. In our case, we take the retraction $\varphi : R[[X]] \to R$, $\varphi(a + bx + cx^2 + ...) = a$. □

We mention from [8]: for any ring R, the set of clean elements of $cn(R[[x]]) = cn(R) + R[[x]]$.

For a converse of the previous proposition, that is, $R[[X]]$ is negative nil-clean whenever R is negative nil-clean, we need $\text{ncn}(R[[X]]) = \text{ncn}(R) + XR[[X]]$.

While the inclusion $\text{ncn}(R) + XR[[X]] \subseteq \text{ncn}(R[[X]])$ holds, the converse inclusion fails: If R is a domain, $1 + X \in \text{ncn}(R) + XR[[X]]$ but is not in $\text{ncn}(R[[x]])$.

It seems difficult in general to compute $\text{ncn}(R[[x]])$. In fact, the nilpotent elements of a power series ring have (so far) no clear description. There exists a commutative ring and a power series $s = \sum a_i x^i$ such that the coefficients a_i are nilpotent of bounded degree and yet s is not nilpotent (see [2]).

2.1. **Comparison with weakly nil-clean.** In [1] the concept of a *weakly nil clean* ring was introduced as follows: a unital ring in which every element can be expressed as sum or difference of a nilpotent and an idempotent (already introduced by Danchev, McGovern [5] in the commutative case).

Clearly, a ring is weakly nil-clean iff $\text{ncn}(R) \cup (-\text{ncn}(R)) = R$.

Proposition 12. *The negative nil-clean and the weakly nil-clean properties are independent.*

Proof. Indeed, by Proposition 7, there are negative nil-clean polynomial rings over commutative rings, but (see [1]), if R is commutative then $R[X]$ is not weak nil clean. Conversely, $\mathbb{Z}_3$ or $\mathbb{Z}_6$ are weakly nil-clean but not negative nil-clean (as 2 is not nilpotent). □

Obviously

Proposition 13. *Any negative and weakly nil-clean ring is nil-clean.*

3. Nil-clean 2×2 matrices whose negative is (not) nil-clean

First, we deal with the trivial nil-clean matrices, that is, the nilpotents (0_2-nil-clean) and the unipotents (I_2-nil-clean) matrices. Since negative of nilpotents are also nilpotent, these are also nil-clean.

Since nontrivial nil-clean 2×2 matrices over commutative domains are characterized by systems of equations, we have to deal separately with unipotents and with nontrivial nil-clean matrices. In this section R denotes a commutative domain.

More precisely, we have to answer four questions:

(A) which are the unipotents whose negative is not unipotent;

(B) which are the unipotents whose negative is not nontrivial nil-clean;

(C) which are the nontrivial nil-clean matrices whose negative is not unipotent;

(D) which are the nontrivial nil-clean matrices whose negative is not nontrivial nil-clean.

(A) This case is simple.

Proposition 14. *Over commutative rings, negatives of unipotent matrices are unipotent iff the characteristics equals 4. Negatives of unipotent* 2×2 *matrices over commutative domains are not unipotent unless of characteristics 2.*

Proof. Given a nilpotent $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 0$, we are looking for a nilpotent $S = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$ such that $-I_2 - T = I_2 + S$. The equality amounts to a linear system including $1 + x = -1 - a$ and $1 - x = -1 + a$. Such an element x exists only if $2 = -2$, that is, $\text{char}(R) = 4$, a necessary condition, possible if R is a commutative domain only if $2 = 0$. Conversely, if $4 = 0$, $-I_2 - T = \begin{bmatrix} 3-a & -b \\ -c & 3+a \end{bmatrix} = I_2 + \begin{bmatrix} 2-a & -b \\ -c & 2+a \end{bmatrix}$, as desired.

In this case, $x = 2 - a$, $y = -b$, $z = -c$ and $x^2 = a^2$. □

Example. For $R = \mathbb{Z}_4$ take the nilpotent $T = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. Then $-(I_2 + T) = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = I_2 + \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$.

(B) Recall (e.g., see [3]) that nontrivial nil-clean matrices are characterized by

Theorem 15. *A* 2×2 *matrix* A *over a commutative domain* D *is nontrivial nil-clean iff* A *has the form* $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ *for some* $a, b, c \in D$ *such that* $\det(A) \neq 0$ *and the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (2a+1)x + cy + bz = a^2 + bc & (2) \end{cases}$$

with unknowns x, y, z*, has at least one solution over* D*. We can suppose* $b \neq 0$ *and if (2) holds, (1) is equivalent to*

$$bx^2 - (2a+1)xy - cy^2 + bx + (a^2 + bc)y = 0 \qquad (3).$$

Recall that $A = E + N$ with nontrivial idempotent $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ i.e., $-\det(E) = x^2 + x + yz = 0$, that is (1), and nilpotent N. Since the condition $Tr(N) = 0$ is already fulfilled, using (1), the condition $\det(N) = 0$ amounts to $(2a+1)x + cy + bz = a^2 + bc$, that is (2).

Thus, this case is settled by the following

Proposition 16. *Negatives of unipotent* 2×2 *matrices over commutative domains are not nontrivial nil-clean unless the characteristics of the domain is 3. If* $\text{char}(D) = 3$*, the negative of an unipotent is nontrivial nil-clean iff the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (1+a)x - cy - bz = 1 + a & (2) \end{cases}$$

with

$$-bx^2 + axy + cy^2 - bx + (1+a)y = 0 \qquad (3)$$

is solvable over D.

Proof. With the notation in **(A)**, according to the previous theorem, the trace $Tr(-I_2 - T) = 1$ is a necessary condition. This amounts to $(-1-a)+(-1+a) = -2 = 1$, possible only if $3 = 0$. If the characteristics is 3, $-I_2 - T = \left[\begin{array}{cc} 2-a & -b \\ -c & 2+a \end{array}\right]$ has trace and determinant equal 1 and the system becomes

$$\left\{\begin{array}{cc} x^2+x+yz=0 & (1) \\ (1+a)x-cy-bz=1+a & (2) \end{array}\right.$$

with

$$-bx^2+axy+cy^2-bx+(1+a)y=0 \quad (3).$$

□

Example. For $D = \mathbb{F}_3$ take $T = \left[\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\right]$. Then $-I_2 - T = \left[\begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} 1 & 2 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right]$ is (nontrivial) nil-clean but not unipotent.
(For $-b = 1+a = 2$, a solution is $x = y = 0$, $z = 1$).

(C) Analogous with **(B)**.

Proposition 17. *Negatives of nontrivial nil-clean* 2×2 *matrices over commutative domains are not unipotent unless of the characteristics of the domain is 3. If* $\mathrm{char}(D) = 3$ *then the negative of* $\left[\begin{array}{cc} a+1 & b \\ c & -a \end{array}\right]$ *is unipotent iff* $a^2+bc = 2a+1$.

Proof. We start with a nontrivial nil-clean matrix $A = \left[\begin{array}{cc} a+1 & b \\ c & -a \end{array}\right]$ and looking for a nilpotent N such that $-A = I_2 + N$. Again $Tr(-A) = -1$, $Tr(I_2+N) = 2$ so characteristics 3 of the domain is necessary for this equality. If the characteristics is 3, then $-A = \left[\begin{array}{cc} 2-a & -b \\ -c & a \end{array}\right] = I_2 + \left[\begin{array}{cc} 1-a & -b \\ -c & -1+a \end{array}\right]$ is indeed (Trace $=0$, determinant $= 0$) unipotent iff $a^2+bc = 2a+1$. □

Example. We can reverse the example of **(B)**. Over $\mathbb{F}_3$ take the nontrivial nil-clean matrix $A = \left[\begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}\right]$ and $-A = \left[\begin{array}{cc} 2 & 1 \\ 2 & 0 \end{array}\right] = I_2 + \left[\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\right]$ is indeed unipotent.

(D) Since the general case is hard to handle (including two successive applications of Theorem 15), we add a restriction following Steger (see [9]): we suppose R is a *ID* ring, that is, idempotent matrices are similar to diagonal matrices. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

In this case, every nontrivial idempotent 2×2 matrix is similar to E_{11}, so, up to similarity, it suffices to characterize the E_{11}-nil-clean 2×2 matrices whose negative is (not) nontrivial nil-clean.

We obtain

Proposition 18. *Negatives of nontrivial nil-clean* 2×2 *matrices over commutative ID domains are not nontrivial nil-clean unless the characteristics of the domain is 2.*

Proof. We start with a nilpotent $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ (with $a^2 + bc = 0$) and consider $A = E_{11} + T$ respectively $-A = -E_{11} - T = \begin{bmatrix} -a-1 & -b \\ -c & a \end{bmatrix}$, matrix with trace -1. If the characteristics is 2, we apply Theorem 15 (for matrices with trace 1). As $-1 = 1$ this amounts just to change the signs of all a, b, c in the equations of Theorem 15. □

Example. Over $\mathbb{F}_2$ consider the nontrivial nil-clean matrix $A = E_{11} + E_{12}$. Since $-1 = 1$, $-A = A$ is again nontrivial nil-clean.

As already mentioned in the Introduction, for characteristics 2, negative of elements (incl. idempotents) coincide with these, so the example is just a special idempotent.

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