SOLUTIONS OF THEORETICAL EXERCISES

selected from

INTRODUCTORY LINEAR ALGEBRA WITH APPLICATIONS

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Eighth Edition, Prentice Hall, 2005.

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Preface

Back in 1997, somebody asked in the Mathematics Department: "Why are the results in 111 Course (Linear Algebra), so bad?" The solution was to cancel some sections of the 6 chapters selected for this one semester course. The solutions of some of the so-called theoretical Exercises were to be covered in the lectures. But this takes time and less time remains for covering the long material in these 6 chapters. Our collection of solutions is intended to help the students in 111 course, and provides to the lecturers a precious additional time in order to cover carefully all the large number of notions, results, examples and procedures to be taught in the lectures. Moreover, this collection yields all the solutions of the Chapter Tests and as a Bonus, some special Exercises to be solved by the students in their Home Work.

Because often these Exercises are required in Midterms and Final Exam, the students are warmly encouraged to prepare carefully these solutions, and, if some of them are not understood, to use the Office Hours of their teachers for supplementary explanations.

The author

List of symbols

Symbol	**Description**
$\mathbf{N}$	the set of all positive integer numbers
$\mathbf{Z}$	the set of all integer numbers
$\mathbf{Q}$	the set of all rational numbers
$\mathbf{R}$	the set of all real numbers

for R any of the above numerical sets

R^*	the set R, removing zero
R^n	the set of all n-vectors with entries in R
$\mathcal{M}_{m\times n}(R)$	the set of all $m \times n$ matrices with entries in R
$\mathcal{M}_n(R)$	the set of all (square) $n \times n$ matrices
S_n	the set of all permutations of n elements
$\mathcal{P}(M)$	the set of all subsets of M
$R[X]$	the set of all polynomials of indeterminate X with coefficients in R

Second Edition (updated to eighth Edition)

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Chapter 1

Matrices

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T.5. A SQUARE matrix $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0$ for $i > j$. It is called **lower triangular** if $a_{ij} = 0$ for $i < j$.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ 0 & a_{22} & \dots & \dots & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

Upper triangular matrix (The entries below the main diagonal are zero.)

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix}$$

Lower triangular matrix (The entries above the main diagonal are zero.)

(a) Show that the sum and difference of two upper triangular matrices is upper triangular.

(b) Show that the sum and difference of two lower triangular matrices is lower triangular.

(c) Show that if a matrix is upper and lower triangular, then it is a diagonal matrix.

Solution. (a) As A above, let $B = [b_{ij}]$ be also an upper triangular matrix, $S = A+B = [s_{ij}]$ be the sum and $D = A-B = [d_{ij}]$ be the difference of these matrices. Then, for every $i > j$ we have $s_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$ respectively, $d_{ij} = a_{ij} - b_{ij} = 0 - 0 = 0$. Thus the sum and difference of two upper triangular matrices is upper triangular.

Example. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$, sum of two upper triangular matrices, which is also upper triangular;

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, difference of two lower triangular matrices, which is also lower triangular.

(b) Similar.

(c) If a matrix is upper and lower triangular, then the entries above the main diagonal and the entries below the main diagonal are zero. Hence all the entries off the main diagonal are zero and the matrix is diagonal.

T.6. (a) Show that if A is an upper triangular matrix, then A^T is lower triangular.

(b) Show that if A is an lower triangular matrix, then A^T is upper triangular.

Solution. (a) By the definition of the transpose if $A^T = [a^T_{ij}]$, and A is an upper triangular matrix, $a_{ij} = 0$ for every $i > j$ and so $a^T_{ji} = a_{ij} = 0$.

Hence A^T is lower triangular.

(b) Similar.

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T.4. Show that the product of two diagonal matrices is a diagonal matrix.

Solution. Just verify that

$$\begin{bmatrix} a_{11} & 0 & ... & 0 \\ 0 & a_{22} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & ... & 0 \\ 0 & b_{22} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & b_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} & 0 & ... & 0 \\ 0 & a_{22}b_{22} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn}b_{nn} \end{bmatrix}.$$

T.5. Show that the product of two scalar matrices is a scalar matrix.

Solution. Just verify that

$$\begin{bmatrix} a & 0 & ... & 0 \\ 0 & a & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a \end{bmatrix} \begin{bmatrix} b & 0 & ... & 0 \\ 0 & b & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & b \end{bmatrix} = \begin{bmatrix} ab & 0 & ... & 0 \\ 0 & ab & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & ab \end{bmatrix}.$$

Short solution. Notice that any scalar matrix has the form $a.I_n = \begin{bmatrix} a & 0 & ... & 0 \\ 0 & a & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a \end{bmatrix}$. Then, obviously $(a.I_n)(b.I_n) = (ab).I_n$ shows that products of scalar matrices are scalar matrices.

T.6. (a) Show that the product of two upper triangular matrices is an upper triangular matrix.

(b) Show that the product of two lower triangular matrices is a lower triangular matrix.

Sketched solution. (a) A direct computation shows that the product of two upper triangular matrices

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ 0 & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ 0 & 0 & ... & b_{nn} \end{bmatrix}$$

is upper triangular too. Indeed, this is

$$\begin{bmatrix} a_{11}b_{11} & a_{11}b_{11} + a_{12}b_{22} & ... & a_{11}b_{1n} + a_{12}b_{2n} + ... + a_{1n}b_{nn} \\ 0 & a_{22}b_{22} & ... & a_{22}b_{2n} + a_{23}b_{3n} + ... + a_{2n}b_{nn} \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn}b_{nn} \end{bmatrix},$$

an upper triangular matrix.

Complete solution. Let $P = [p_{ij}] = AB$ be the product of two upper triangular matrices. If $i > j$ then $p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{n} a_{ik}b_{kj} = (a_{i1}b_{1j} + ... + a_{i,i-1}b_{i-1,j}) + (a_{ii}b_{ij} + ... + a_{in}b_{nj})$. Since both matrices A and B are upper triangular, in the first sum the a's are zero and in the second sum the b 's are zero. Hence $p_{ij} = 0$ and P is upper triangular too.

(b) Analogous.

T9. (a) Show that the j-th column of the matrix product AB is equal to the matrix product $Acol_j(B)$.

(b) Show that the i-th row of the matrix product AB is equal to the matrix product $row_i(A)B$.

Solution. (a) With usual notations, consider $A = [a_{ij}]$ an $m \times n$ matrix, $B = [b_{ij}]$ an $n \times p$ matrix and $C = AB = [c_{ij}]$ the corresponding matrix product, an $m \times p$ matrix.

As already seen in the Lectures, an arbitrary (i,j)-entry c_{ij} in the product is given by the dot product $row_i(A) \bullet col_j(B) = \sum_{k=1}^{n} a_{ik}b_{kj}$ or $[a_{i1}a_{i2}...a_{in}] \bullet \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$. Hence the j-th column of the product (the matrix C) is the following:

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = \begin{bmatrix} row_1(A) \bullet col_j(B) \\ row_2(A) \bullet col_j(B) \\ \vdots \\ row_n(A) \bullet col_j(B) \end{bmatrix} = Acol_j(B)$$

using the product (just do the computation!) of the initial $m \times n$ matrix A and the $n \times 1$ matrix $col_j(B) = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$.

(b) Analogous.

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T.9. Find a 2×2 matrix $B \neq O_2$ and $B \neq I_2$ such that $AB = BA$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Solution. Obviously $B = A$ satisfies the required conditions (indeed, $AA = AA$, $A \neq O_2$ and $A \neq I_2$).

Solution for a"better" statement: find all the matrices B with this property.

We search for B as an unknown matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and so, using the definition of matrix equality,

$$\begin{cases} a + 2c &= a \\ b + 2d &= 2a + b \\ c &= c \\ d &= 2c + d \end{cases}.$$

These equalities are equivalent to $c = 0$ and $a = d$. Therefore every matrix B of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ with arbitrary (real) numbers a, b verifies $AB = BA$. **Example**: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

T.13. Show that $(-1)A = -A$.

Solution. In the proof of Theorem 1.1 (Properties of the matrix addition) we took $D = (-1)A$ (the scalar multiplication) and we have verified that $A + D = D + A = O$, that is $-A = (-1)A$.

T.23. Let A and B be symmetric matrices.
(a) Show that $A + B$ is symmetric.
(b) Show that AB is symmetric if and only if $AB = BA$.

Solution. (a) This follows at once from $(A + B)^T \overset{\text{Th.1.4(b)}}{=} A^T + B^T = A + B$, A and B being symmetric (above some of the equalities, their justification is given).

(b) First observe that $(AB)^T \overset{\text{Th.1.4(c)}}{=} B^T A^T = BA$ holds for arbitrary symmetric matrices A, B.

Now, if AB is symmetric, $(AB)^T = AB$ and thus (using the previous equality) $AB = BA$.

Conversely, if $AB = BA$ then $(AB)^T = BA = AB$, that is, AB is symmetric.

T.26. If A is an $n \times n$ matrix, show that AA^T and $A^T A$ are symmetric.

Solution. For instance $(AA^T)^T \overset{\text{Th.1.4(c)}}{=} (A^T)^T A^T \overset{\text{Th.1.4(a)}}{=} AA^T$, so that AA^T is symmetric. Likewise $A^T A$ is symmetric.

We recall here a DEFINITION given in Exercise T24: a matrix $A = [a_{ij}]$ is called **skew-symmetric** if $A^T = -A$.

T.27. If A is an $n \times n$ matrix,
(a) Show that $A + A^T$ is symmetric.
(b) Show that $A - A^T$ is skew-symmetric.

Solution. (a) We have only to observe that $(A + A^T)^T \overset{\text{Th.1.4(b)}}{=} A^T + (A^T)^T \overset{\text{Th.1.4(a)}}{=} A^T + A \overset{\text{Th.1.1(a)}}{=} A + A^T$.

(b) Analogously, $(A - A^T)^T \overset{\text{Th.1.4(b)}}{=} A^T - (A^T)^T \overset{\text{Th.1.4(a)}}{=}$
$= A^T - A \overset{\text{Th.1.1(a)}}{=} -(A - A^T)$.

T.28. Show that if A is an $n \times n$ matrix, then A can be written uniquely as $A = S + K$, where S is symmetric and K is skew-symmetric.

Solution. Suppose such a decomposition exists. Then $A^T = S^T + K^T = S - K$ so that $A + A^T = 2S$ and $A - A^T = 2K$.

Now take $S = \frac{1}{2}(A + A^T)$ and $K = \frac{1}{2}(A - A^T)$. One verifies $A = S + K$, $S = S^T$ and $K^T = -K$ similarly to the previous Exercise 26.

T.32. Show that if $A\mathbf{x} = \mathbf{b}$ is a linear system that has more than one solution, then it has infinitely many solutions.

Solution. Suppose $\mathbf{u}_1 \neq \mathbf{u}_2$ are two different solutions of the given linear system. For an arbitrary real number r, such that $0 < r < 1$, consider $\mathbf{w}_r = r\mathbf{u}_1 + (1-r)\mathbf{u}_2$. This is also a solution of the given system: $A\mathbf{w}_r = A(r\mathbf{u}_1 + (1-r)\mathbf{u}_2) = r(A\mathbf{u}_1) + (1-r)(A\mathbf{u}_2) = r\mathbf{b} + (1-r)\mathbf{b} = \mathbf{b}$.

First observe that $\mathbf{w}_r \notin \{\mathbf{u}_1, \mathbf{u}_2\}$. Indeed, $\mathbf{w}_r = \mathbf{u}_1$ implies $\mathbf{u}_1 = r\mathbf{u}_1 + (1-r)\mathbf{u}_2$, $(1-r)(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}$ and hence $\mathbf{u}_1 = \mathbf{u}_2$, a contradiction. Similarly, $\mathbf{w}_r \neq \mathbf{u}_2$.

Next, observe that for $0 < r, s < 1$, $r \neq s$ the corresponding solutions are different: indeed, $\mathbf{w}_r = \mathbf{w}_s$ implies $r\mathbf{u}_1 + (1-r)\mathbf{u}_2 = s\mathbf{u}_1 + (1-s)\mathbf{u}_2$ and so $(r-s)(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}$, a contradiction.

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T.11. Let $\mathbf{u}$ and $\mathbf{v}$ be solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(a) Show that $\mathbf{u}+\mathbf{v}$ is a solution.
(b) For any scalar r, show that $r\mathbf{u}$ is a solution.
(c) Show that $\mathbf{u}-\mathbf{v}$ is a solution.
(d) For any scalars r and s, show that $r\mathbf{u}+s\mathbf{v}$ is a solution.

Remark. We have interchanged (b) and (c) from the book, on purpose.

Solution. We use the properties of matrix operations.

(a) $A(\mathbf{u}+\mathbf{v}) \overset{\text{Th.1.2(b)}}{=} A\mathbf{u}+A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$.

(b) $A(r\mathbf{u}) \overset{\text{Th.1.3(d)}}{=} r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}$.

(c) We can use **our** previous (b) and (a): by (b), for $r=-1$ we have $(-1)\mathbf{v} = -\mathbf{v}$ is a solution; by (a) $\mathbf{u}+(-\mathbf{v}) = \mathbf{u}-\mathbf{v}$ is a solution.

(d) Using twice **our** (b), $r\mathbf{u}$ and $s\mathbf{v}$ are solutions and, by (a), $r\mathbf{u}+s\mathbf{v}$ is a solution.

T.12. Show that if $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x}=\mathbf{b}$, then $\mathbf{u}-\mathbf{v}$ is a solution to the *associated* homogeneous system $A\mathbf{x}=\mathbf{0}$.

Solution. Our hypothesis assures that $A\mathbf{u}=\mathbf{b}$ and $A\mathbf{v}=\mathbf{b}$. Hence $A(\mathbf{u}-\mathbf{v}) \overset{\text{Th.1.2(b)}}{=} A\mathbf{u}-A\mathbf{v} = \mathbf{b}-\mathbf{b} = \mathbf{0}$ and so $\mathbf{u}-\mathbf{v}$ is a solution to the associated homogeneous system $A\mathbf{x}=\mathbf{0}$.

Page 86 (Bonus).

Find all values of a for which the resulting linear system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

23. $\left\{\begin{array}{rcl} x+y-z &=& 2 \\ x+2y+z &=& 3 \\ x+y+(a^2-5)z &=& a \end{array}\right.$.

Solution. We use Gauss-Jordan method. The augmented matrix is $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{bmatrix}$. The first elementary operations give

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{bmatrix} \overset{-R_1+R_{2,3}}{\sim} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{bmatrix}.$$

Case 1. $a^2-4=0$, that is $a\in\{\pm 2\}$.

(i) $a=2$. Then the last matrix gives (final Step after Step 8; in what follows we refer to the steps in the Procedure p. 65 - 68) the following reduced row echelon form

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{-R_2+R_1}{\sim} \begin{bmatrix} \mathbf{1} & 0 & -3 & 1 \\ 0 & \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding system is $\begin{cases} x &= 1+3z \\ y &= 1-2z \end{cases}$, which has ($z$ is an arbitrary real number), infinitely many solutions.

(ii) $a=-2$. The last matrix is $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ and so our last equation is $0\times x+0\times y+0\times z=-4$. Hence this is an inconsistent system (it has no solutions).

Case 2. $a^2-4\neq 0$ (i.e., $a\notin\{\pm 2\}$). In the 1×4 submatrix (which remains neglecting the first and second rows), we must only use Step 4 (multiply by $\frac{1}{a^2-4}=\frac{1}{(a-2)(a+2)}$) and after this, the final Step (from REF to RREF):

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{a+2} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2+\frac{1}{a+2} \\ 0 & 1 & 0 & 1-\frac{2}{a+2} \\ 0 & 0 & 1 & \frac{1}{a+2} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 1+\frac{3}{a+2} \\ 0 & 1 & 0 & 1-\frac{2}{a+2} \\ 0 & 0 & 1 & \frac{1}{a+2} \end{bmatrix}.$$

Finally the solution (depending on a) in this case is $x=1+\frac{3}{a+2}$, $y=1-\frac{2}{a+2}$ and $z=\frac{1}{a+2}$.

24. $\begin{cases} x+y+z &= 2 \\ 2x+3y+2z &= 3 \\ 2x+3y+(a^2-1)z &= a+1 \end{cases}.$

Solution. The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 2 & 3 & a^2-1 & a+1 \end{bmatrix}$. The first elementary operations give

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 2 & 3 & a^2-1 & a+1 \end{bmatrix} \overset{-2R_1+R_{2,3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & a^2-3 & a-3 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-3 & a-4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-3 & a-4 \end{bmatrix}.$$

Case 1. $a^2-3 \neq 0$, that is $a \notin \{\pm\sqrt{3}\}$. Then a^2-3 is our third pivot and we continue by Step 4 (multiply the third row by $\frac{1}{a^2-3}$) :

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{a-4}{a^2-3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1-\frac{a-4}{a^2-3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{a-4}{a^2-3} \end{bmatrix}.$$

This is a consistent system with the unique solution $x = 1-\frac{a-4}{a^2-3}$, $y = 1$, $z = \frac{a-4}{a^2-3}$.

Case 2. $a^2-3 = 0$, that is $a \in \{\pm\sqrt{3}\}$. Hence $a-4 \neq 0$ and the last equation is $0 \times x + 0 \times y + 0 \times z = a-4 \neq 0$, an inconsistent system (no solutions).

Pages 105-106 (Bonus).

16. Find all the values of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1} ?

Solution. We use the practical procedure for computing the inverse (see p. 95, textbook):

$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & a & 0 & 0 & 1 \end{array}\right] \overset{-R_1+R_{2,3}}{\sim} \left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & a & -1 & 0 & 1 \end{array}\right] \overset{-R_2}{\sim}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & a & -1 & 0 & 1 \end{array}\right] \overset{-R_2+R_3}{\sim} \left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & a & -2 & 1 & 1 \end{array}\right] \overset{-R_2+R_1}{\sim}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & a & -2 & 1 & 1 \end{array}\right] = \left[C \vdots D\right].$$

CASE 1. If $a = 0$ then C has a zero row ($C \neq I_3$). Hence A is singular (that is, has no inverse).

CASE 2. If $a \neq 0$ we use Step 4 (multiply the third row by $\frac{1}{a}$):

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{array}\right] = \left[C \vdots D\right]$$

so that A is nonsingular and $D = A^{-1} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{array}\right].$

22. If A and B are nonsingular, are $A+B$, $A-B$ and $-A$ nonsingular ? Explain.

Solution. If A, B are nonsingular generally it is **not** true that $A+B$ or $A-B$ are nonsingular. It suffices to give suitable counterexamples:

for $A = I_n$ and $B = -I_n$ we have $A+B = 0_n$, which is not invertible (see the definition p. 91), and,

for $A = B$, the difference $A - B = 0_n$, again a non-invertible matrix.

Finally, if A is nonsingular we easily check that $-A$ is also invertible and $(-A)^{-1} = -A^{-1}$.

Remark. More can be proven (see Exercise 20, (b)): for every $c \neq 0$, if A is nonsingular, cA is also nonsingular and $(cA)^{-1} = \frac{1}{c}A^{-1}$(indeed, one verifies $(cA)(\frac{1}{c}A^{-1}) \overset{\mathrm{T}h.1.3(d)}{=} ((cA)\frac{1}{c})A^{-1} \overset{\mathrm{T}h.1.3.(d)}{=} (c\frac{1}{c}A)A^{-1} = 1AA^{-1} = I_n$ and similarly $(\frac{1}{c}A^{-1})(cA) = I_n$).

Chapter Test 1.

1. Find all solutions to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 - 2x_4 &= 3 \\ 2x_1 + x_2 + 3x_3 + 2x_4 &= 5 \\ -x_2 + x_3 + 6x_4 &= 3. \end{aligned}$$

Solution. Gauss-Jordan method is used:

$$\begin{bmatrix} \mathbf{1} & 1 & 1 & -2 & 3 \\ 2 & 1 & 3 & 2 & 5 \\ 0 & -1 & 1 & 6 & 3 \end{bmatrix} \overset{-2R_1+R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & -1 & 1 & 6 & -1 \\ 0 & -1 & 1 & 6 & 3 \end{bmatrix} \overset{-R_2}{\sim}$$

$$\begin{bmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & \mathbf{1} & -1 & -6 & 1 \\ 0 & -1 & 1 & 6 & 3 \end{bmatrix} \overset{R_2+R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & -1 & -6 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

The corresponding equivalent system has the third equation $0 \times x_1 + 0 \times x_2 + 0 \times x_3 + 0 \times x_4 = 4$, which has no solution.

2. Find all values of a for which the resulting linear system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

$$\begin{aligned} x + z &= 4 \\ 2x + y + 3z &= 5 \\ -3x - 3y + (a^2 - 5a)z &= a - 8. \end{aligned}$$

Solution. Gauss-Jordan method is used:

$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 4 \\ 2 & 1 & 3 & 5 \\ -3 & -3 & a^2 - 5a & a - 8 \end{bmatrix} \overset{(-2)R_1+R_2, 3R_1+R_3}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & \mathbf{1} & 1 & -3 \\ 0 & -3 & a^2 - 5a + 3 & a + 4 \end{bmatrix} \overset{3R_2+R_3}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & a^2-5a+6 & a-5 \end{bmatrix}.$$

As usually we distinguish two cases (notice that
$a^2-5a+6=(a-2)(a-3)=0 \Leftrightarrow a \in \{2,3\}$):

CASE 1. $a=2$ or $a=3$. In both cases $a+1 \neq 0$ and so the third equation of the corresponding system is $0 \times x + 0 \times y + 0 \times z = a+1$, with no solution.

CASE 2. If $a \notin \{2,3\}$ then $a^2-5a+6 \neq 0$ and the procedure continues with Step 4:

$$\overset{\frac{1}{a^2-5a+6}R_3}{\sim} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & \frac{a-5}{a^2-5a+6} \end{bmatrix} \overset{-R_3+R_{2,1}}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4-\frac{a-5}{a^2-5a+6} \\ 0 & 1 & 0 & -3-\frac{a-5}{a^2-5a+6} \\ 0 & 0 & 1 & \frac{a-5}{a^2-5a+6} \end{bmatrix},$$

with the corresponding equivalent system (unique solution)
$x = 4-\frac{a-5}{a^2-5a+6}$, $y=-3-\frac{a-5}{a^2-5a+6}$, $z=\frac{a-5}{a^2-5a+6}$.

3. If possible, find the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution. We use the Practical Procedure (see p. 95, textbook):

$$\begin{bmatrix} \mathbf{1} & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \overset{-R_1+R_3}{\sim} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{bmatrix} \overset{2R_2+R_3}{\sim}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 2 & 1 \end{bmatrix} \overset{\frac{1}{2}R_3}{\sim} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \overset{-R_3+R_2,R_3+R_1}{\sim}$$

$$\left[\begin{array}{cccccc} 1 & 2 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right] \overset{(-2)R_2+R_1}{\sim} \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right] =$$

$$= \left[C \vdots D\right].$$

Since C has no zero rows, A^{-1} exists (that is, A is nonsingular) and $A^{-1} = D = \left[\begin{array}{ccc} -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$.

4. If $A = \left[\begin{array}{cc} -1 & -2 \\ -2 & 2 \end{array}\right]$, find all values of λ for which the homogeneous system $(\lambda I_2 - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Solution. The homogeneous system has a nontrivial solution if and only if $\lambda I_2 - A = \left[\begin{array}{cc} \lambda+1 & 2 \\ 2 & \lambda-2 \end{array}\right]$ is singular (see Theorem 1.13, p. 99). Use the practical procedure for finding the inverse:

Case 1. $\lambda + 1 \neq 0$ then this is the first pivot in

$$\left[\begin{array}{cccc} \lambda+1 & 2 & 1 & 0 \\ 2 & \lambda-2 & 0 & 1 \end{array}\right] \overset{\frac{1}{\lambda+1}R_1}{\sim} \left[\begin{array}{cccc} 1 & \frac{2}{\lambda+1} & \frac{1}{\lambda+1} & 0 \\ 2 & \lambda-2 & 0 & 1 \end{array}\right] \overset{-2R_1+R_2}{\sim}$$

$$\left[\begin{array}{cccc} 1 & \frac{2}{\lambda+1} & \frac{1}{\lambda+1} & 0 \\ 2 & \lambda-2-\frac{4}{\lambda+1} & -\frac{2}{\lambda+1} & 1 \end{array}\right] = \left[\begin{array}{cccc} 1 & \frac{2}{\lambda+1} & \frac{1}{\lambda+1} & 0 \\ 2 & \frac{\lambda^2-\lambda-6}{\lambda+1} & -\frac{2}{\lambda+1} & 1 \end{array}\right] = \left[C \vdots D\right].$$

Now, if $\lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3) = 0 \Leftrightarrow \lambda \in \{-2, 3\}$ then C has a zero row and $\lambda I_2 - A$ is singular (as required).

Case 2. If $\lambda + 1 = 0$ then the initial matrix is $\left[\begin{array}{cccc} 0 & 2 & 1 & 0 \\ \mathbf{2} & -3 & 0 & 1 \end{array}\right]$ so that using Step 3 we obtain

$$\left[\begin{array}{cccc} 2 & -3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Hence the coefficient matrix of the system is a nonsingular matrix with inverse $\begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ and therefore the homogeneous system has a unique (trivial) solution.

5. (a) If $A^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$, compute $(AB)^{-1}$.

(b) Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ if $A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Solution. (a) $(AB)^{-1} \overset{\text{Th.1.10(b)}}{=} B^{-1}A^{-1} = \begin{bmatrix} 3 & 6 & 8 \\ -2 & 2 & -8 \\ 0 & 5 & -3 \end{bmatrix}$.

(b) If A^{-1} exists (that is, A is nonsingular) then (see p.98) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -4 \\ 14 \\ 25 \end{bmatrix}$.

7. Answer each of the following as true or false.

(a) If A and B are $n \times n$ matrices, then

$$(A + B)(A + B) = A^2 + 2AB + B^2.$$

(b) If $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{w} = \frac{1}{4}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

(c) If A is a nonsingular matrix, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(d) A homogeneous system of three equations in four unknowns has a nontrivial solution.

(e) If A, B and C are $n \times n$ nonsingular matrices, then $(ABC)^{-1} = C^{-1}A^{-1}B^{-1}$.

Solution. (a) False, since generally $AB = BA$ fails. Indeed, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. As a matter of fact $(A+B)^2 = (A+B)(A+B) \overset{\mathrm{T}h.1.2(b)}{=} (A+B)A + (A+B)B =\overset{\mathrm{T}h.1.2(c)}{=} A^2 + BA + AB + B^2$.

(b) True, indeed if $A\mathbf{u}_1 = \mathbf{b}$ and $A\mathbf{u}_2 = \mathbf{b}$ then $A\mathbf{w} = A(\frac{1}{4}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2) \overset{\text{Th.1.2(b),Th.1.3(d)}}{=} \frac{1}{4}A\mathbf{u}_1 + \frac{3}{4}A\mathbf{u}_2 = \frac{1}{4}\mathbf{b} + \frac{3}{4}\mathbf{b} = \mathbf{b}$.

(c) False: see Theorem 1.13 (p. 99).

(d) True: special case of Theorem 1.8 (p.77) for $m = 3 < 4 = n$.

(e) False: a special case of Corollary 1.2 (p. 94) gives $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. It suffices to give an example for $B^{-1}A^{-1} \neq A^{-1}B^{-1}$.

Chapter 3

Determinants

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T3. Show that if c is a real number and A is an $n \times n$ matrix then $\det(cA) = c^n \det(A)$.

Solution. You only have to repeatedly use (n times) Theorem 3.5 (p. 187): the scalar multiplication of a matrix by c consists of the multiplication of the first row, of the second row, ..., and the multiplication of the n-th row, by the same real number c. Hence $\det(cA) = c(c(...cA)...)) = c^n \det(A)$.

T5. Show that if $\det(AB) = 0$ then $\det(A) = 0$ or $\det(B) = 0$.

Solution. Indeed, by Theorem 3.8 (p. 191)
$\det(AB) = \det(A)\det(B) = 0$ and so $\det(A) = 0$ or $\det(B) = 0$ (as a zero product of two real numbers).

T6. Is $\det(AB) = \det(BA)$? Justify your answer.

Solution. Yes. Generally $AB \neq BA$ but because of Theorem 3.8
$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$ (determinants are real numbers).

T8. Show that if $AB = I_n$ then $\det(A) \neq 0$ and $\det(B) \neq 0$.

Solution. Using Theorem 3.8, $1 = \det(I_n) = \det(AB) = \det(A)\det(B)$ and so $\det(A) \neq 0$ and $\det(B) \neq 0$ (as a nonzero product of two real numbers).

T9. (a) Show that if $A = A^{-1}$, then $\det(A) = \pm 1$.

(b) Show that if $A^T = A^{-1}$, then $\det(A) = \pm 1$.

Solution. (a) We use Corollary 3.2 (p. 191): from $A = A^{-1}$ (if the inverse exists $\det(A) \neq 0$, by Exercise T8 above) we derive $\det(A) = \det(A^{-1}) = \frac{1}{\det(A)}$ and so $(\det(A))^2 = 1$. Hence $\det(A) = \pm 1$.

(b) By Theorem 3.1 (p. 185) $\det(A^T) = \det(A)$. One uses now (a).

T10. Show that if A is a nonsingular matrix such that $A^2 = A$, then $\det(A) = 1$.

Solution. Using again Theorem 3.8, $\det(A) = \det(A^2) = \det(AA) =$
$= \det(A)\det(A)$ and so $\det(A)(\det(A) - 1) = 0$. If A is nonsingular, by Exercise T8 above, $\det(A) \neq 0$ and so $\det(A) - 1 = 0$ and finally $\det(A) = 1$.

T16. Show that if A is $n \times n$, with A skew symmetric ($A^T = -A$, see Section 1.4, Exercise T.24) and n is odd, then $\det(A) = 0$.

Solution. By Theorem 3.1, $\det(A^T) = \det(A)$. By the Exercise T3 above, $\det(-A) = \det((-1)A) = (-1)^n \det(A) = -\det(A)$ because n is odd. Hence $\det(A) = -\det(A)$ and so $2\det(A) = 0$ and $\det(A) = 0$.

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T7. Show that if A is singular, then $\mathrm{adj}A$ is singular.

Solution. If A is singular, then $\det(A) = 0$. Since $A(\mathrm{adj}A) = \det(A)I_n$ (this is Theorem 3.11), $A(\mathrm{adj}A) = O$.

First of all, if $A = O$ then $\mathrm{adj}A = O$ by the definition of the adjoint matrix: all cofactors are zero.

In the remaining case, if $A \neq O$ then $\mathrm{adj}A$ cannot be nonsingular because, otherwise, multiplying to the right with $(\mathrm{adj}A)^{-1}$ the equality $A(\mathrm{adj}A) = O$ we obtain $A = O$. So $adjA$ is singular.

T8. Show that if A is an $n \times n$ matrix, then $\det(\mathrm{adj}A) = \det(A)^{n-1}$.

Solution. Use the equality given in Theorem 3.11: $A(\mathrm{adj}A) = \det(A)I_n$. Taking the determinants of both members of the equality one obtains: $\det(A)\det(\mathrm{adj}A) = (\det(A))^n$ (notice that $\det(A)I_n$ is a scalar matrix having n copies of the real number $\det(A)$ on the diagonal; Theorem 3.7, p. 188, is used).

If $\det(A) = 0$, according to the previous Exercise, together with A, $\mathrm{adj}A$ is also singular, and so $\det(\mathrm{adj}A) = 0$ and the formula holds.

If $\det(A) \neq 0$ we can divide both members in the last equality by $\det(A)$ and thus $\det(\mathrm{adj}A) = (\det(A))^n \div \det(A) = [\det(A)]^{n-1}$.

T10. Let $AB = AC$. Show that if $\det(A) \neq 0$, then $B = C$.

Solution. Using Theorem 3.12 (p. 203), A is nonsingular and so, A^{-1} exists. Multiplying $AB = AC$ to the left with A^{-1} we obtain $A^{-1}(AB) = A^{-1}(AC)$ and finally $B = C$.

T12. Show that if A is nonsingular, then $\mathrm{adj}A$ is nonsingular and

$$(\mathrm{adj}A)^{-1} = \frac{1}{\det(A)}A = \mathrm{adj}(A^{-1}).$$

Solution. First use Exercise T8 previously solved: if A is nonsingular then $\det(A) \neq 0$ and so $\det(\mathrm{adj}A) = [\det(A)]^{n-1} \neq 0$. Thus $\mathrm{adj}A$ is also nonsingular.

Further, the formulas in Theorem 3.11, $A(\mathrm{adj}A) = (\mathrm{adj}A)A = \det(A)I_n$, give $(\mathrm{adj}A)(\frac{1}{\det(A)}A) = (\frac{1}{\det(A)}A)(\mathrm{adj}A) = I_n$ and so $(\mathrm{adj}A)^{-1} = \frac{1}{\det(A)}A$, by definition.

Finally, one can write the equalities given in Theorem 3.11 for A^{-1}: $A^{-1}(\mathrm{adj}(A^{-1})) = \det(A^{-1})I_n = \frac{1}{\det(A)}I_n$ (by Corollary 3.2, p. 191). Hence, by left multiplication with A, one finds $\mathrm{adj}(A^{-1}) = \frac{1}{\det(A)}A$.

Supplementary Exercises (Bonus):

1) THE PROOF OF THEOREM 1.11, SECTION 1.7

Suppose that A and B are $n \times n$ matrices.
(a) If $AB = I_n$ then $BA = I_n$
(b) If $BA = I_n$ then $AB = I_n$.

PROOF. (a) If $AB = I_n$, taking determinants and using suitable properties, $\det(A)\det(B) = \det(AB) = \det(I_n) = 1$ shows that $\det(A) \neq 0$ and $\det(B) \neq 0$ and so A and B are nonsingular. By left multiplication with

A^{-1} (which exists, A being nonsingular), we obtain $A^{-1}(AB) = A^{-1}I_n$ and $B = A^{-1}$. Hence $BA = A^{-1}A = I_n$.

(b) Similarly.

2) EXERCISE T3. Show that if A is symmetric, then adj$A$ is also symmetric.

Solution. Take an arbitrary cofactor $A_{ij} = (-1)^{i+j}\det(M_{ij})$ where M_{ij} is the submatrix of A obtained by deleting the i-th row and j-th column. Observe that the following procedure gives also M_{ij}:

(1) consider the transpose A^T;

(2) in A^T delete the j-th row and i-th column;

(3) transpose the resulting submatrix.

Further, if A is symmetric (i.e., $A = A^T$) the above procedure shows that $M_{ij} = (M_{ji})^T$. Therefore $A_{ji} = (-1)^{j+i}\det(M_{ji}) = (-1)^{i+j}\det((M_{ji})^T) = (-1)^{i+j}\det(M_{ij}) = A_{ij}$ and so adj$A$ is symmetric.

Chapter Test 3.

1. Evaluate

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ -1 & 2 & -3 & 4 \\ 0 & 5 & 0 & -2 \end{bmatrix}.$$

Solution. Use the expansion of $\det(A)$ along the second row (fourth row, first column and third column are also good choices, having also two zero entries).

$\det(A) = 0 \times A_{21} + 1 \times A_{22} + 0 \times A_{23} + 3 \times A_{24} = A_{22} + 3A_{24} =$

$$= \begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & 4 \\ 0 & 0 & -2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & -3 \\ 0 & 5 & 0 \end{vmatrix} = 2 + 15 = 17.$$

2. Let A be 3×3 and suppose that $|A| = 2$. Compute

(a) $|3A|$. (b) $|3A^{-1}|$. (c) $|(3A)^{-1}|$.

Solution. Notice that $|A|$ is here (and above) only an alternative notation for $\det(A)$. Thus:
(a) $|3A| = 3^3\,|A| = 54$
(b) $|3A^{-1}| = 3^3\,|A^{-1}| = \frac{27}{2}$
(c) $|(3A)^{-1}| = \frac{1}{|3A|} = \frac{1}{54}$.

3. For what value of a is

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & a \end{vmatrix} + \begin{vmatrix} 0 & a & 1 \\ 1 & 3a & 0 \\ -2 & a & 2 \end{vmatrix} = 14\ ?$$

Solution. Evaluating the sum of the (two) determinants, we obtain $2(-a-3)+a+6a-2a = 14$, a simple equation with the solution $a = \frac{20}{3}$.

4. Find all values of a for which the matrix

$$\begin{bmatrix} a^2 & 0 & 3 \\ 5 & a & 2 \\ 3 & 0 & 1 \end{bmatrix}$$

is singular.

Solution. The determinant $\begin{vmatrix} a^2 & 0 & 3 \\ 5 & a & 2 \\ 3 & 0 & 1 \end{vmatrix} = a^3 - 9a = a(a-3)(a+3) = 0$ if and only if $a \in \{-3, 0, 3\}$. By Theorem 3.12 (p. 203) these are the required values.

5. Not requested (Cramer's rule).

6. Answer each of the following as true or false.
(a) $\det(AA^T) = \det(A^2)$.
(b) $\det(-A) = -\det(A)$.
(c) If $A^T = A^{-1}$, then $\det(A) = 1$.
(d) If $\det(A) = 0$ then $A = 0$.
(e) If $\det(A) = 7$ then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(f) The sign of the term $a_{15}a_{23}a_{31}a_{42}a_{54}$ in the expansion of the determinant of a 5×5 matrix is $+$.

(g) If $\det(A) = 0$, then $\det(\mathrm{adj}A) = 0$.

(h) If $B = PAP^{-1}$, and P is nonsingular, then $\det(B) = \det(A)$.

(i) If $A^4 = I_n$ then $\det(A) = 1$.

(j) If $A^2 = A$ and $A \neq I_n$, then $\det(A) = 0$.

Solution. (a) True: $\det(AA^T) \overset{\text{Th.3.8}}{=} \det(A)\det(A^T) \overset{\text{Th.3.1}}{=}$ $= \det(A)\det(A) = \det(A^2)$.

(b) False: $\det(-A) = \det((-1)A) \overset{\text{Th.3.5}}{=} (-1)^n \det(A) = -\det(A)$ only if n is odd. For instance, if $A = I_2$ then $\det(-A) = \det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \neq -1 = -\det(A)$.

(c) False: indeed, $A^T = A^{-1} \Rightarrow \det(A^T) = \det(A^{-1}) \Rightarrow \det(A) = \frac{1}{\det(A)}$ which implies $\det(A) \in \{\pm 1\}$, and not necessarily $\det(A) = 1$. For instance, for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^T = A^{-1}$ holds, but $\det(A) = -1$.

(d) False: obviously $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$ but $\det(A) = 0$.

(e) True: indeed, $7 \neq 0$ and one applies Corollary 3.4 (p. 203).

(f) True: indeed, the permutation 53124 has $4 + 2 = 6$ inversions and so is even.

(g) True: use Exercise T.8 (p. 210) which tells us that $\det(\mathrm{adj}A) = [\det(A)]^{n-1}$.

(h) True: using Theorems 3.8 and Corollary 3.2 (notice that P is nonsingular) we have
$\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) =$
$= \det(P)\frac{1}{\det(P)}\det(A) = \det(A)$.

(i) False: for instance, if $n = 2$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the equality $A^4 = I_2$ holds and $\det(A) = -1$.

(j) True: by the way of contradiction. If $\det(A) \neq 0$ then A is nonsingular (see Theorem 3.12, p. 203) and thus the inverse A^{-1} exists. By left multiplication of $A^2 = A$ with this inverse A^{-1} we obtain at once: $A^{-1}(A^2) = A^{-1}A$ and $A = I_n$.

Chapter 4

n-Vectors

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In the following Exercises we denote the n-vectors considered by $\mathbf{u} = (u_1, u_2, ..., u_n)$, $\mathbf{v} = (v_1, v_2, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, ..., w_n)$.

T7. Show that $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$.

Solution. Easy computation: $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) =$
$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + ... + u_n(v_n + w_n) =$
$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + ... + u_nv_n + u_nw_n =$
$= u_1v_1 + u_2v_2 + ... + u_nv_n + u_1w_1 + u_2w_2 + ... + u_nw_n =$
$= \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$.

T8. Show that if $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w}$ for all $\mathbf{u}$, then $\mathbf{v} = \mathbf{w}$.

Solution. First take $\mathbf{u} = \mathbf{e}_1 = (1, 0, ..., 0)$. Then $\mathbf{e}_1 \bullet \mathbf{v} = \mathbf{e}_1 \bullet \mathbf{w}$ gives $v_1 = w_1$.

Secondly take $\mathbf{u} = \mathbf{e}_2 = (0, 1, 0, ..., 0)$. Then $\mathbf{e}_2 \bullet \mathbf{v} = \mathbf{e}_2 \bullet \mathbf{w}$ implies $v_2 = w_2$, and so on.

Finally, take $\mathbf{u} = \mathbf{e}_n = (0, ..., 0, 1)$. Then $\mathbf{e}_n \bullet \mathbf{v} = \mathbf{e}_n \bullet \mathbf{w}$ gives $v_n = w_n$. Hence $\mathbf{v} = (v_1, v_2, ..., v_n) = \mathbf{w} = (w_1, w_2, ..., w_n)$.

T9. Show that if c is a scalar, then $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$, where $|c|$ is the absolute value of c.

Solution. Indeed, by the norm (length) definition,
$\|c\mathbf{u}\| = \sqrt{(cu_1)^2 + (cu_2)^2 + ... + (cu_n)^2} =$
$= \sqrt{c^2 u_1{}^2 + c^2 u_2{}^2 + ... + c^2 u_n{}^2} = \sqrt{c^2(u_1{}^2 + u_2{}^2 + ... + u_n{}^2)} =$
$= |c| \sqrt{u_1{}^2 + u_2{}^2 + ... + u_n{}^2} = = |c| \, \|\mathbf{u}\|.$

T10. (**Pythagorean Theorem in $\mathbf{R}^n$**) Show that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$.

Solution. First notice that for arbitrary n-vectors we have
$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \bullet \mathbf{v} + \|\mathbf{v}\|^2$ (using Theorem 4.3, p. 236 and the previous Exercise T7).

Finally, if the equality in the statement holds, then $2\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$. Conversely, $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$ implies $2\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$ and $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Chapter Test 4.

1. Find the cosine of the angle between the vectors $(1, 2, -1, 4)$ and $(3, -2, 4, 1)$.

Solution. Use the formula of the angle (p. 237) $\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{3-4-4+4}{\sqrt{1+4+1+16}\sqrt{9+4+16+1}} = -\frac{1}{\sqrt{22}\sqrt{30}}$

2. Find the unit vector in the direction of $(2, -1, 1, 3)$.

Solution. Use the remark after the definition (p. 239): if $\mathbf{x}$ is a nonzero vector, then $\mathbf{u} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$, is a unit vector in the direction of $\mathbf{x}$. Therefore, $\|(2, -1, 1, 3)\| = \sqrt{4+1+1+9} = \sqrt{15}$ and the required vector is
$\frac{1}{\sqrt{15}}(2, -1, 1, 3)$.

3. Is the vector $(1, 2, 3)$ a linear combination of the vectors $(1, 3, 2)$, $(2, 2, -1)$ and $(3, 7, 0)$?

Solution. Yes: searching for a linear combination $x(1, 3, 2) + y(2, 2, -1) +$

$z(3,7,0) = (1,2,3)$, yields a linear system

$$\begin{array}{rcl} x + 2y + 3z & = & 1 \\ 3x + 2y + 7z & = & 2 \\ 2x - y & = & 3 \end{array}.$$

The coefficient matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 7 \\ 2 & -1 & 0 \end{bmatrix}$ is nonsingular because (compute!) its determinant is $14 \neq 0$. Hence the system has a (unique) solution.

4. Not requested (linear transformations).
5. Not requested (linear transformations).

6. Answer each of the following as true or false.
(a) In $\mathbf{R}^n$, if $\mathbf{u} \bullet \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
(b) In $\mathbf{R}^n$, if $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
(c) In $\mathbf{R}^n$, if $c\mathbf{u} = \mathbf{0}$ then $c = 0$ or $\mathbf{u} = \mathbf{0}$.
(d) In $\mathbf{R}^n$, $\|c\mathbf{u}\| = c\,\|\mathbf{u}\|$.
(e) In $\mathbf{R}^n$, $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.
(f) **Not requested** (linear transformations).
(g) The vectors $(1,0,1)$ and $(-1,1,0)$ are orthogonal.
(h) In $\mathbf{R}^n$, if $\|\mathbf{u}\| = 0$, then $\mathbf{u} = \mathbf{0}$.
(i) In $\mathbf{R}^n$, if $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $2\mathbf{v}+3\mathbf{w}$.
(j) **Not requested** (linear transformations).

Solution. (a) False: for instance $\mathbf{u} = \mathbf{e}_1 = (1,0,...,0) \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{e}_2 = (0,1,0...,0) \neq \mathbf{0}$ but $\mathbf{u} \bullet \mathbf{v} = \mathbf{e}_1 \bullet \mathbf{e}_2 = 0$.

(b) False: for $\mathbf{u}, \mathbf{v}$ as above: take $\mathbf{w} = \mathbf{0}$; then $\mathbf{e}_1 \bullet \mathbf{e}_2 = \mathbf{e}_1 \bullet \mathbf{0}$ but $\mathbf{e}_2 \neq \mathbf{0}$.

(c) True: indeed, if $c\mathbf{u} = \mathbf{0}$ and $c \neq 0$ then $(cu_1, cu_2, ..., cu_n) = (0,0,...,0)$ or $cu_1 = cu_2 = ... = cu_n = 0$, together with $c \neq 0$, imply $u_1 = u_2 = ... = u_n = 0$, that is, $\mathbf{u} = \mathbf{0}$.

(d) False: compare with the correct formula on p. 246, Theoretical Exercise T9; for instance, if $c = -1$, and $\mathbf{u} \neq \mathbf{0}$ then $\|-\mathbf{u}\| \neq -\|\mathbf{u}\|$.

(e) False: compare with the correct Theorem 4.5 (p.238); if $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} = -\mathbf{u}$ then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{0}\| = 0 \neq 2\|\mathbf{u}\| = \|\mathbf{u}\| + \|-\mathbf{u}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

(g) False: the dot product $\mathbf{u} \bullet \mathbf{v} = -1 + 0 + 0 \neq 0$, so that the vectors are not orthogonal (see Definition p.238).

(h) True: indeed, $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + ... + u_n^2} = 0$ implies $u_1^2 + u_2^2 + ... + u_n^2 = 0$ and (for real numbers) $u_1 = u_2 = ... = u_n = 0$, that is, $\mathbf{u} = \mathbf{0}$.

(i) True: indeed, compute (once again by the definition p. 238) $\mathbf{u} \bullet (2\mathbf{v} + 3\mathbf{w}) \overset{\text{T7,p.246}}{=} 2(\mathbf{u} \bullet \mathbf{v}) + 3(\mathbf{u} \bullet \mathbf{w}) = 2 \times 0 + 3 \times 0 = 0$.

Chapter 5

Lines and Planes

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T2. Show that $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$.

Solution. Indeed,
$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \bullet (w_1, w_2, w_3) =$
$= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 =$
$= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) =$
$= (u_1, u_2, u_3) \bullet (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) =$
$= \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$.

T4. Show that

$$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Solution. In the previous Exercise we already obtained
$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$. But this computation can be continued as follows:
$= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) =$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

using the expansion of the determinant along the first row.

T5. Show that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Solution. In Section 4.2 (p. 238), the following definition is given: two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if $|\mathbf{u} \bullet \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\|$ (that is, $\cos\theta = \pm 1$, or, equivalently $\sin\theta = 0$, θ denoting the angle of $\mathbf{u}$ and $\mathbf{v}$).

Notice that $\|\mathbf{u}\| \neq 0 \neq \|\mathbf{v}\|$ for nonzero vectors and $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Leftrightarrow$
$\|\mathbf{u} \times \mathbf{v}\| = 0$.

Using the length formula $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$ we obtain
$\sin\theta = 0$ if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the required result.

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T5. Show that an equation of the plane through the noncollinear points $P_1(x_1, y_1, z_1), P_1(x_2, y_2, z_2)$ and $P_1(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Solution. Any three noncollinear points $P_1(x_1, y_1, z_1), P_1(x_2, y_2, z_2)$ and $P_1(x_3, y_3, z_3)$ determine a plane whose equation has the form

$$ax + by + cz + d = 0,$$

where a, b, c and d are real numbers, and a, b, c are not all zero. Since $P_1(x_1, y_1, z_1)$, $P_1(x_2, y_2, z_2)$ and $P_1(x_3, y_3, z_3)$ lie on the plane, their coordinates satisfy the previous Equation:

$$\begin{array}{c} ax_1 + by_1 + cz_1 + d = 0 \\ ax_2 + by_2 + cz_2 + d = 0 \\ ax_3 + by_3 + cz_3 + d = 0 \end{array} .$$

We can write these relations as a homogeneous linear system in the unknowns a, b, c and d

$$\begin{aligned} ax + by + cz + d &= 0 \\ ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0 \end{aligned}$$

which must have a nontrivial solution. This happens if and only if (see Corollary 3.4 p. 203) the determinant of the coefficient matrix is zero, that is, if and only if

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Chapter Test 5.

3. Find parametric equations of the line through the point $(5, -2, 1)$ that is parallel to the vector $\mathbf{u} = (3, -2, 5)$.

Solution. $x = 5+3t$, $y = -2-2t$, $z = 1+5t$, $-\infty < t < \infty$ (see p.266).

4. Find an equation of the plane passing through the points $(1, 2, -1)$, $(3, 4, 5), (0, 1, 1)$.

Solution. $\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & -1 & 1 \\ 3 & 4 & 5 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 0$ (see the previous T. Exercise 5, p. 271).

Thus

$$x \begin{vmatrix} 2 & -1 & 1 \\ 4 & 5 & 1 \\ 1 & 1 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \\ 0 & 1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

or $x - y + 1 = 0$.

5. Answer each of the following as true or false.

(b) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \times \mathbf{w} = \mathbf{0}$ then $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = 0$.

(c) If $\mathbf{v} = -3\mathbf{u}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(d) The point $(2, 3, 4)$ lies in the plane $2x - 3y + z = 5$.

(e) The planes $2x - 3y + 3z = 2$ and $2x + y - z = 4$ are perpendicular.

Solution. (b) True, using Theorem 5.1 (p.260) properties (c) and (a): $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

(c) True, using Theoretical Exercise **T.5** (p.263), or directly: if $\mathbf{u} = (u_1, u_2, u_3)$ then for $\mathbf{v} = (-3u_1, -3u_2, -3u_3)$ the cross product

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ -3u_1 & -3u_2 & -3u_3 \end{vmatrix} = 0$$

because (factoring out -3) it has two equal rows.

(d) False. Verification: $2 \times 2 - 3 \times 3 + 1 \times 4 = -1 \neq 0$.

(e) False. The planes are perpendicular if and only if the corresponding normal vectors are perpendicular, or, if and only if these vectors have the dot product zero. $(2, -3, 3) \bullet (2, 1, -1) = 4 - 3 - 3 = -2 \neq 0$.

Chapter 6

Vector Spaces

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T2. Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$ and let S_1 be a subset of S_2. Show that:

(a) If S_1 is linearly dependent, so is S_2.

(b) If S_2 is linearly independent, so is S_1.

Solution. Since S_1 is a subset of S_2, denote the vectors in the finite sets as follows: $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$.

(a) If S_1 is linearly dependent, there are scalars c_1, c_2, ..., c_k not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$. Hence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + ... + 0\mathbf{v}_m = \mathbf{0}$, where not all the scalars are zero, and so S_2 is linearly dependent too.

(b) If S_1 is not linearly independent, it is (by Definition) linearly dependent. By (a), S_2 is also linearly dependent, a contradiction. Hence S_1 is linearly independent.

T4. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is also linearly independent, where $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3$.

Solution. Take $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$, that is, $c_1(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) +$

$c_2(\mathbf{v}_2+\mathbf{v}_3)+c_3\mathbf{v}_3=\mathbf{0}$. Hence $c_1\mathbf{v}_1+(c_1+c_2)\mathbf{v}_2+(c_1+c_2+c_3)\mathbf{v}_3=\mathbf{0}$ and by linearly independence of S, $c_1=c_1+c_2=c_1+c_2+c_3=0$. But this homogeneous system has obviously only the zero solution. Thus T is also linearly independent.

T6. Suppose that $S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Is $T=\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$, where $\mathbf{w}_1=\mathbf{v}_1$, $\mathbf{w}_1=\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{w}_1=\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3$, linearly dependent or linearly independent ? Justify your answer.

Solution. T is linearly independent. We prove this in a similar way to the previous Exercise T4.

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28 (Bonus). Find all values of a for which $\{(a^2,0,1),(0,a,2),(1,0,1)\}$ is a basis for $\mathbf{R}^3$.

Solution 1. Use the procedure given in Section 6.3, p. 294, in order to determine the values of a for which the vectors are linearly independent; this amounts to find the reduced row echelon form for the matrix (we have reversed the order of the vectors to simplify computation of RREF)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 2 \\ a^2 & 0 & 1 \end{bmatrix} \overset{-a^2R_1+R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 2 \\ 0 & 0 & 1-a^2 \end{bmatrix}.$$

If $a=0$ or $a\in\{\pm1\}$ the reduced row echelon form has a zero row. For $a\notin\{-1,0,1\}$ the three vectors are linearly independent, and so form a basis in $\mathbf{R}^3$ (by Theorem 6.9 (a), p.312).

Solution 2. Use the Corollary 6.4 from Section 6.6, p.335: it is sufficient to find the values of a for which the determinant

$$\begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 2 \\ 1 & 0 & 1 \end{vmatrix} = a(a^2-1)\neq 0.$$

These are $a \in \mathbf{R} - \{-1, 0, 1\}$.

T9. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a vector space V and $c \neq 0$ then $\{c\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is also a basis for V.

Solution. By definition: if $k_1(c\mathbf{v}_1) + k_2\mathbf{v}_2 + ... + k_n\mathbf{v}_n = 0$, then (the system $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ being linearly independent) $k_1 c = k_2 = ... = k_n = 0$ and $(c \neq 0)$, $k_1 = k_2 = ... = k_n = 0$. The rest is covered by Theorem 6.9, (a).

T10. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for vector space V. Then show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3$, is also a basis for V.

Solution. Using Theorem 6.9, (a), it is sufficient to show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

If $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = 0$ then $c_1(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + c_2(\mathbf{v}_2 + \mathbf{v}_3) + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_1 + c_2 + c_3)\mathbf{v}_3 = 0$, and, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ being linearly independent, $c_1 = c_1 + c_2 = c_1 + c_2 + c_3 = 0$. Hence (by elementary computations) $c_1 = c_2 = c_3 = 0$.

T12. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for $\mathbf{R}^n$. Show that if A is an $n \times n$ nonsingular matrix, then $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ is also a basis for $\mathbf{R}^n$.

Solution. First we give a solution for

Exercise T10, Section 6.3: Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Show that if A is an $n \times n$ nonsingular matrix, then $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ is linearly independent.

Solution using section 6.6. Let $M = [\mathbf{v}_1\mathbf{v}_2...\mathbf{v}_n]$ be the matrix which has the given vectors as columns ($\mathrm{col}_i(M) = \mathbf{v}_i$, $1 \leq i \leq n$). Thus the product $AM = [A\mathbf{v}_1 A\mathbf{v}_2 ... A\mathbf{v}_n]$, that is $\mathrm{col}_i(AM) = A\mathbf{v}_i$, $1 \leq i \leq n$.

The given vectors being linearly independent, $\det(M) \neq 0$. The matrix A being nonsingular $\det(A) \neq 0$. Hence
$\det(AM) = \det(A)\det(M) \neq 0$ so that $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ is a linearly independent set of vectors, by Corollary 6.4 in Section 6.6.

Finally, if $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ is linearly independent and has n vectors (notice that $\dim(\mathbf{R}^n) = n$), it only remains to use Theorem 6.9 (a), p.312.

T13. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbf{R}^n$ and let A be a singular matrix. Prove or disprove that $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ is linearly independent.

Solution. **Disprove** (see also the previous Exercise).

Indeed, $\{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n\}$ can be linearly dependent: take for instance $A = \mathbf{0}$: a set of vectors which contains a zero vector is not linearly independent.

Page 337 - 338 (6 Bonus Exercises).

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_5\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{v}_5 = \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$. Find a basis for the subspace $V = \mathrm{span}S$ of $\mathbf{R}^4$.

Solution. We use the procedure given after the proof of Theorem 6.6, p. 308:

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 & 5 \\ 2 & 1 & 2 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & -3 & -4 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -4 & -3 & -7 \end{bmatrix} \sim$$

$$\begin{bmatrix} \mathbf{1} & 2 & 3 & 3 & 5 \\ 0 & \mathbf{1} & \frac{4}{3} & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ and so $\mathbf{v}_1$ and $\mathbf{v}_2$ is a basis.

11. Compute the row and column ranks of A verifying Theorem 6.11,
$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}.$$

Solution. We transform A into row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & -5 & -14 & -8 & -2 \\ 0 & -6 & -22 & -12 & -2 \end{bmatrix} \overset{-R_3+R_2}{\sim}$$
$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 8 & 4 & 0 \\ 0 & -6 & -22 & -12 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 8 & 4 & 0 \\ 0 & 0 & 26 & 12 & -2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 8 & 4 & 0 \\ 0 & 0 & 1 & \frac{6}{13} & -\frac{1}{13} \end{bmatrix}$$ and so the column rank is $= 3$.

Further, for the row rank, we transform in row echelon form the transpose A^T,
$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -5 & -6 \\ 0 & -14 & -22 \\ 0 & -8 & -12 \\ 0 & -2 & -2 \end{bmatrix} \overset{R_2 \leftrightarrow R_5}{\sim} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & -14 & -6 \\ 0 & -8 & -12 \\ 0 & -5 & -6 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ and so the row rank is (also) $= 3$.

18. If A is a 3×4 matrix, what is the largest possible value for rank(A) ?

Solution. The matrix A has three rows so the row rank is ≤ 3 and four columns, and so the column rank is ≤ 4. By Theorem 6.11, the rank of A is at most 3.

20. If A is a 5×3 matrix, show that the rows are linearly dependent.

Solution. The matrix having only 3 columns, the column rank is ≤ 3. Hence also the row rank must be ≤ 3 and 5 rows are necessarily dependent (otherwise the row rank would be ≥ 5).

25. Determine whether the matrix $A = \begin{bmatrix} 1 & 1 & 4 & -1 \\ 1 & 2 & 3 & 2 \\ -1 & 3 & 2 & 1 \\ -2 & 6 & 12 & -4 \end{bmatrix}$ is singular or nonsingular using Theorem 6.13.

Solution. We transform the matrix into row echelon form in order to compute its rank:

$$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 1 & 2 & 3 & 2 \\ -1 & 3 & 2 & 1 \\ -2 & 6 & 12 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & 6 & 0 \\ 0 & 8 & 20 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & -12 \\ 0 & 0 & 28 & -38 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 28 & -30 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Hence the rank (the number of nonzero rows) is 4 and the matrix is nonsingular.

29. Is $S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ a linearly independent set of vectors ?

Solution. The matrix whose columns are the vectors in S

$$\begin{bmatrix} 4 & 2 & 2 \\ 1 & 5 & -1 \\ 2 & -5 & 3 \end{bmatrix},$$

has a zero determinant (verify!). Hence, by Corollary 6.4, the set S of vectors is linearly dependent.

T7. Let A be an $m \times n$ matrix. Show that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every $m \times 1$ matrix $\mathbf{b}$ if and only if $\text{rank}(A) = m$.

Solution. Case 1: $m \leq n$. If the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every $m \times 1$ matrix $\mathbf{b}$, then every $m \times 1$ matrix $\mathbf{b}$ belongs to the column space of A. Hence this column space must be all $\mathbf{R}^m$ and has dimension m. Therefore $\text{rank}(A) = \text{column rank}(A) = \dim(\text{column space}(A)) = m$.

Conversely, if $\text{rank}(A) = m$, the equality $\text{rank}(A) = \text{rank}[A|\mathbf{b}]$ follows at once because, generally, $\text{rank}(A) \leq \text{rank}[A|\mathbf{b}]$ and $\text{rank}[A|\mathbf{b}] \leq m$ (because $[A|\mathbf{b}]$ has only m rows). Finally we use Theorem 6.14.

Case 2: $m > n$. We observe that in this case $\text{rank}(A) = m$ is impossible, because A has only $n < m$ rows and so $\text{rank}(A) \leq n$. But if $m > n$, the linear system $A\mathbf{x} = \mathbf{b}$ has no solution for every $m \times 1$ matrix $\mathbf{b}$ (indeed, if we have more equations than unknowns, and for a given $\mathbf{b}$, the system has a solution, it suffices to modify a coefficient in $\mathbf{b}$, and the corresponding system is no more verified by the same previous solution).

Chapter Test 6.

1. Consider the set W of all vectors in $\mathbf{R}^3$ of the form (a, b, c), where $a + b + c = 0$. Is W a subspace of $\mathbf{R}^3$?

Solution. Yes: $(0, 0, 0) \in W$ so that $W \neq \emptyset$.

For $(a, b, c), (a', b', c') \in W$ also $(a + a', b + b', c + c') \in W$ because $(a + a') + (b + b') + (c + c') = (a + b + c) + (a' + b' + c') = 0 + 0 = 0$.

Finally, for every $k \in \mathbf{R}$ and $(a, b, c) \in W$ also $k(a, b, c) \in W$ because $ka + kb + kc = k\,(a + b + c) = 0$.

2. Find a basis for the solution space of the homogeneous system

$$\begin{array}{rcl} x_1 + 3x_2 + 3x_3 - x_4 + 2x_5 & = & 0 \\ x_1 + 2x_2 + 2x_3 - 2x_4 + 2x_5 & = & 0 \\ x_1 + x_2 + x_3 - 3x_4 + 2x_5 & = & 0 \end{array}.$$

Solution. We transform the augmented matrix A in row echelon form:

$$\begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 0 \\ 1 & 2 & 2 & -2 & 2 & 0 \\ 1 & 1 & 1 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -2 & -2 & 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \mathbf{1} & 0 & 0 & -4 & 2 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$. The corresponding system is now

$$\begin{array}{rcl} x_1 & = & 4x_4 - 2x_5 \\ x_2 & = & -x_3 - x_4 \end{array}.$$

Hence $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ taking $x_3 = s$, $x_4 = t$, $x_5 = u$, so that $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the required basis.

3. Does the set of vectors $\{(1, -1, 1), (1, -3, 1), (1, 2, 2)\}$ form a basis for $\mathbf{R}^3$?

Solution. If $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ is the matrix consisting, as columns, of the given vectors, $\det(A) = -6 - 1 + 2 + 3 - 2 + 2 = -2 \neq 0$ so that the vectors are linearly independent. Using Theorem 6.9, (a), this is a basis in $\mathbf{R}^3$.

4. For what value(s) of λ is the set of vectors $\{(\lambda^2 - 5, 1, 0), (2, -2, 3), (2, 3, -3)\}$ linearly dependent ?

Solution. If $A = \begin{bmatrix} \lambda^2 - 5 & 2 & 2 \\ 1 & -2 & 3 \\ 0 & 3 & -3 \end{bmatrix}$ is the matrix consisting, as columns, of the given vectors, then $\det A = 6(\lambda^2 - 5) + 6 - 9(\lambda^2 - 5) + 6 = -3\lambda^2 + 27$. Thus $\det A = 0$ if and only if $\lambda \in \{\pm 3\}$.

5. Not required.

6. Answer each of the following as true or false.

(a) All vectors of the form $(a, 0, -a)$ form a subspace of $\mathbf{R}^3$.

(b) In $\mathbf{R}^n$, $\|c\mathbf{x}\| = c\,\|\mathbf{x}\|$.

(c) Every set of vectors in $\mathbf{R}^3$ containing two vectors is linearly independent.

(d) The solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is spanned by the columns.

(e) If the columns of an $n \times n$ matrix form a basis for $\mathbf{R}^n$, so do the rows.

(f) If A is an 8×8 matrix such that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then $\mathrm{rank}(A) < 8$.

(g) Not required.

(h) Every linearly independent set of vectors in $\mathbf{R}^3$ contains three vectors.

(i) If A is an $n \times n$ symmetric matrix, then $\mathrm{rank}(A) = n$.

(j) Every set of vectors spanning $\mathbf{R}^3$ contains at least three vectors.

Solution. (a) True. Indeed, these vectors form exactly span$\{(1,0,-1)\}$ which is a subspace (by Theorem 6.3, p. 285).

(b) False. Just take $c = -1$ and $\mathbf{x} \neq \mathbf{0}$. You contradict the fact that the length (norm) of any vector is ≥ 0.

(c) False. $\mathbf{x} = (1,1,1)$ and $\mathbf{y} = (2,2,2)$ are linearly dependent in $\mathbf{R}^3$ because $2\mathbf{x} - \mathbf{y} = \mathbf{0}$.

(d) False. The solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is spanned by the columns which correspond to the columns of the reduced row echelon form which do not contain the leading ones.

(e) True. In this case the column rank of A is n. But then also the row rank is n and so the rows form a basis.

(f) False. Just look to Corollary 6.5, p. 335, for $n = 8$.

(h) False. For instance, each nonzero vector alone in $\mathbf{R}^3$ forms a linearly independent set of vectors.

(i) False. For example, the zero $n \times n$ matrix is symmetric, but has not the rank $= n$ (it has zero determinant).

(j) True. The dimension of the subspace of $\mathbf{R}^3$ spanned by one or two vectors is ≤ 2, but $\dim \mathbf{R}^3 = 3$.

Chapter 8

Diagonalization

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T1. Let λ_j be a particular eigenvalue of A. Show that the set W of all the eigenvectors of A associated with λ_j, as well as the zero vector, is a subspace of $\mathbf{R}^n$ (called the *eigenspace associated with* λ_j).

Solution. First, $\mathbf{0} \in W$ and so $W \neq \emptyset$.

Secondly, if $\mathbf{x}, \mathbf{y} \in W$ then $A\mathbf{x} = \lambda_j \mathbf{x}$, $A\mathbf{y} = \lambda_j \mathbf{y}$ and consequently $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda_j \mathbf{x} + \lambda_j \mathbf{y} = \lambda_j(\mathbf{x}+\mathbf{y})$. Hence $\mathbf{x}+\mathbf{y} \in W$.

Finally, if $\mathbf{x} \in W$ then $A\mathbf{x} = \lambda_j \mathbf{x}$ and so $A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda_j \mathbf{x}) = \lambda_j(c\mathbf{x})$. Hence $c\mathbf{x} \in W$.

T3. Show that if A is an upper (lower) triangular matrix, then the eigenvalues of A are the elements on the main diagonal of A.

Solution. The corresponding matrix $\lambda I_n - A$ is also upper (lower) triangular and by Theorem 3.7 (Section 3.1, p. 188), the characteristic polynomial $f(\lambda)$ is given by:

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} & ... & -a_{1n} \\ 0 & \lambda - a_{22} & ... & -a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda - a_{nn} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22})...(\lambda - a_{nn})$$

(expanding successively along the first column). The corresponding characteristic equation has the solutions $a_{11}, a_{22}, ..., a_{nn}$.

T4. Show that A and A^T have the same eigenvalues.

Solution. Indeed, these two matrices have the same characteristic polynomials:
$\det(\lambda I_n - A^T) = \det((\lambda I_n)^T - A^T) = \det((\lambda I_n - A)^T) \overset{\text{Th.3.1}}{=} \det(\lambda I_n - A)$.

T7. Let A be an $n \times n$ matrix.

(a) Show that $\det A$ is the product of all the roots of the characteristic polynomial of A.

(b) Show that A is singular if and only if 0 is an eigenvalue of A.

Solution. The characteristic polynomial

$$f(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{vmatrix}$$

is a degree n polynomial in λ which has the form $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_{n-1}\lambda + c_n$ (one uses the definition of the determinant in order to check that the leading coefficient actually is 1).

If in the equality $f(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_{n-1}\lambda + c_n$ we let $\lambda = 0$ we obtain $c_n = \det(-A) = (-1)^n \det A$.

By the other way, if $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$ is the decomposition using the (real) eigenvalues of A, then (in the same way, $\lambda = 0$) we obtain $f(0) = c_n = (-1)^n\lambda_1\lambda_2...\lambda_n$. Hence $\lambda_1\lambda_2...\lambda_n = \det A$.

(b) By Theorem 3.12 (p. 203), we know that a matrix A is singular if and only if $\det A = 0$. Hence, using (a), A is singular if and only if $\lambda_1\lambda_2...\lambda_n = 0$, or, if and only if A has 0 as an eigenvalue.

Chapter Test 8.

1. If possible, find a nonsingular matrix P and a diagonal matrix D so that A is similar to D where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}.$$

Solution. The matrix A is just tested for diagonalizability. Since the matrix $\lambda I_3 - A$ is lower triangular (using Theorem 3.7, p. 188), the characteristic polynomial is $\det(\lambda I_3 - A) = (\lambda - 1)(\lambda - 2)^2$, so we have a simple eigenvalue $\lambda_1 = 1$ and a double eigenvalue $\lambda_2 = 2$.

For $\lambda_1 = 1$ the eigenvector is given by the system $(I_3 - A)\mathbf{x} = 0$ (solve it!): $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -5 \\ 11 \end{bmatrix}$

For the double eigenvalue $\lambda_2 = 2$ **only one** eigenvector is given by the system $(2I_3 - A)\mathbf{x} = 0$ (solve it!): $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore A is not diagonalizable.

2 and **3**: not required.

4. Answer each of the following as true or false.

(a) Not required.

(b) If A is diagonalizable, then each of its eigenvalues has multiplicity one.

(c) If none of the eigenvalues of A are zero, then $\det(A) \neq 0$.

(d) If A and B are similar, then $\det(A) = \det(B)$.

(e) If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A associated with the distinct eigenvalues λ_1 and λ_2, respectively, then $\mathbf{x} + \mathbf{y}$ is an eigenvector of A associated with the eigenvalue $\lambda_1 + \lambda_2$.

Solution. (b) False: see the Example 6, p. 427.

(c) True: according to T. Exercise 7, p. 421 (section 8.1), $\det(A)$ is the product of all roots of the characteristic polynomial. But these are the eigenvalues of A (see Theorem 8.2, p. 413). If none of these are zero, neither is their product.

(d) True: indeed, if $B = P^{-1}AP$ then

$\det(B) = \det(P^{-1}AP) = \det P^{-1} \det A \det P = \det P^{-1} \det P \det A = \det A$, since all these are real numbers.

(e) False: hypothesis imply $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$. By addition (of columns) $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}$ is not generally equal to $(\lambda_1 + \lambda_2)(\mathbf{x} + \mathbf{y})$. An easy **example**: any 2×2 matrix having two different nonzero eigenvalues. The sum is a real number different from both: a 2×2 matrix cannot have 3 eigenvalues (specific example, see Example 3, p. 424).

Bibliography

[1] Kolman B, Hill D.R., *Introductory Linear Algebra with Applications*, 8-th Edition, 2005, Prentice Hall Inc., New Jersey.