# UNIPOTENT-REGULAR MATRICES AND RINGS

GRIGORE CĂLUGĂREANU, HORIA F. POP

Abstract. An element $a$ in a ring $R$ is called unipotent-regular if there is a unipotent $u$ such that $a = aua$. A ring is unipotent-regular if so are all its elements. We show that a ring is unipotent-regular iff it is Boolean. Additionally, we characterize the unipotent-regular $2 \times 2$ matrices over Prüfer domains. Not all unipotent-regular matrices are idempotent.

## 1. Introduction

To the best of our knowledge, so far there is no characterization of unit-regular matrices (over commutative rings), not even for $2 \times 2$ matrices.

Therefore we introduce and characterize a naturally defined subset of unit-regular matrices.

**Definition.** An element $a \in R$ is called *unipotent-regular* if there exists a unipotent unit $u$ such that $a = aua$. Equivalently, $a$ is unipotent-regular if there exists a nilpotent $t$ such that $a = a(1 + t)a$. A ring is called *unipotent-regular* if so are all its elements.

Obviously, unipotent-regular elements are unit-regular and unit-regular elements are regular.

As in case of unit-regular elements, it is easy to see that *an element is unipotent-regular iff it is a product of an idempotent and a unipotent element (in either order).*

In this short note we first show that unipotent-regular *rings* are scarce: these are precisely the Boolean rings.

Secondly, we characterize the unipotent-regular $2 \times 2$ *matrices* over Prüfer domains.

The rings we consider are associative with identity. By $U(R)$ we denote the set of all units of $R$ and by $N(R)$ we denote the set of all nilpotents of $R$. The "regular" word for elements or rings means Von Neumann regular.

## 2. Unipotent-regular rings and $2 \times 2$ matrices

The so-called *UU-rings* (rings with only unipotent units) were defined and studied in [1]. Their study was further developed in [3].

Thus, a ring $R$ is UU iff $U(R) = 1 + N(R)$. For the sake of completeness we recall (see [5]) the following

**Definitions**. An element $r$ of a ring $R$ is called *clean* if $r = e + u$ with $e^2 = e$ and $u \in U(R)$. An element $x$ of a ring $R$ is called *left exchange* (initially left "suitable") if there exists $e^2 = e \in Rx$ such that $1 - e \in R(1 - x)$. Right exchange elements are defined similarly and it is proved that the exchange property is left-right symmetric (see [5] **Theorem 2.1**). A ring is called *Abelian* if all its idempotents are central.

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As already mentioned in the Introduction we have the following result. For reader's convenience we supply (almost) all the details.

**Theorem 1.** *A ring is unipotent-regular iff it is Boolean.*

*Proof.* First observe that a unit is unipotent-regular iff it is unipotent. Indeed, as inverses of unipotents are unipotent, one way is obvious. Conversely, suppose $u \in U(R)$ and $u = e(1+t)$ for an idempotent $e$ and a nilpotent $t$. Then $eu = u$ and so $e = 1$. Hence $u = 1+t$, as claimed.

As a consequence, every unipotent-regular ring is UU. Secondly, we show that $R$ is a regular UU-ring iff $R$ is a Boolean ring. Again, one way is obvious, so assume $R$ is a regular UU-ring. We first show that $R$ is reduced. The only existing proof goes like this. Assume $R$ is not reduced, i.e., there exists a nonzero nilpotent in $R$. Then there exists $e^2 = e$ in $R$ such that the corner $eRe$ is isomorphic to $\mathbb{M}_n(S)$ for some $S$ (by Levitzki, [4], Th. 2.1), but $eRe$ is UU while $\mathbb{M}_n(S)$ is not UU (see [1]), a contradiction.

Since $R$ is reduced, it is easy to see $R$ is Abelian (we just show $(er - ere)^2 = 0 = (re - ere)^2$).

Next, recall that every regular element is exchange. Indeed, if $a = axa$, write $f = xa = f^2$. Then take $e = f + (1-f)xf = e^2 \in Rx$ and so $1 - e = (1-f)(1-x)$ (see [5] **Proposition 1.6**). Hence regular rings are exchange.

Further, we show that any Abelian exchange ring is clean. As $R$ is exchange for any $x \in R$ we choose $e^2 = e \in Rx$ with $1 - e \in R(1-x)$. If $e = ax$ we may assume $ea = a$ so that $axa = a$. Since idempotents are central $xa = ax$. Similarly we write $1-e = b(1-x)$ where $(1-e)b = b$ and $b(1-x) = (1-x)b$. Then an easy calculation shows that $a - b$ is the inverse of $x - (1-e)$ (see [5] **Proposition 1.8**).

Finally, as $R$ is clean and $U(R) = \{1\}$, start with any element $r \in R$. Since $r+1$ is clean, $r + 1 = e + u$ with $e^2 = e$ and $u \in U(R)$ and so $r = e$, all elements of $R$ are idempotents. $\square$

The second part of the proof appeared in [3] as **Theorem 4.1**, (5) ⇒ (6) ⇒ (3). It would be nice to have a direct (somewhat elementary) proof for the statement: "*any regular UU ring is reduced*". For the time being, the one mentioned above (via Levitzky's result) is the only one known.

As for matrix rings, since *for any ring* $R \neq 0$ *and any integer* $n \geq 2$, $\mathbb{M}_n(R)$ *is not a UU-ring* (see [1]), we have the following result

**Proposition 2.** *For any ring* $R \neq 0$ *and any integer* $n \geq 2$, $\mathbb{M}_n(R)$ *is not unipotent-regular.*

In what follows we determine the $2 \times 2$ unipotent-regular matrices over a Prüfer domain.

First notice that *only zero determinant* $2 \times 2$ *matrices can be unipotent-regular.* Indeed, this follows at once since $\det(E(I_2 + T)) = \det(E)\det(I_2 + T) = 0 \cdot 1 = 0$, for any idempotent $E$ and nilpotent $T$.

Next, since in our characterization we use Prüfer domains, recall that a *Prüfer domain* is a semihereditary integral domain. Equivalently, an integral domain $R$ is a Prüfer domain if every nonzero finitely generated ideal of $R$ is invertible. Fields, PIDs and Bézout domains are Prüfer domains but UFDs may not be Prüfer.

In the next theorem we intend to use the Kronecker (Rouché) - Capelli theorem for compatible linear systems. As early as 1971 we recall from [2] the following characterization. In this characterization, the ideal $D_t(A)$ generated by the $t \times t$ minors of the matrix $A$ is called the *t-th determinantal ideal* of $A$ and we put $D_0 = 1$. As customarily, $[A, \mathbf{b}]$ denotes the augmented matrix.

**Theorem 3.** *Let $R$ be an integral domain, $A$ a matrix of rank $r$ over $R$ and $\mathbf{x}$ and $\mathbf{b}$ column vectors over $R$. The condition $D_r(A) = D_r[A, \mathbf{b}]$ is necessary and sufficient for the system $A\mathbf{x} = \mathbf{b}$ to be solvable iff $R$ is a Prüfer domain.*

Our characterization follows.

**Theorem 4.** *A (zero determinant) matrix $A = [a_{ij}]_{1 \leq i,j \leq 2}$ over a Prüfer domain is unipotent-regular iff there exist $a, c$ with $c \mid a(1-a)$ such that $\mathrm{crow}_1(A) = a\mathrm{row}_2(A)$ and if $bc = a(1-a)$ then $(a_{11} - a)^2, (a_{12} - b)^2$ and $(a_{11} - a)(a_{12} - b)$ are divisible by $ba_{11} - aa_{12}$. The divisibilities are equivalent with $(a_{21} - c)^2, (a_{22} + a - 1)^2$ and $(a_{21} - c)(a_{22} + a - 1)$ being divisible by $(1 - a)a_{21} - ca_{22}$.*

We discuss separately the cases $a \in \{0, 1\}$, so below we assume $a, b, c \neq 0$ and $a \neq 1$.

*Proof.* As noticed in the Introduction, an element is unipotent-regular iff it is a product of an idempotent and a unipotent element (in either order). Therefore, over any integral domain a unipotent-regular $2 \times 2$ matrix is of form $E(I_2 + T) = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix}$ with $a(1-a) = bc$ and $x^2 + yz = 0$. Denoting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the equality $A = E(I_2 + T)$ amounts to the system

$$\begin{array}{rcl} a(1+x) + bz & = & a_{11} \\ b(1-x) + ay & = & a_{12} \\ c(1+x) + (1-a)z & = & a_{21} \\ -(1-a)(1-x) + cy & = & a_{22} \end{array}$$

We write the linear system as follows

$$\begin{array}{rcl} ax + bz & = & a_{11} - a \\ -bx + ay & = & a_{12} - b \\ cx + (1-a)z & = & a_{21} - c \\ --(1-a)x + cy & = & a_{22} + a - 1 \end{array}.$$

The four equations form a linear system with 3 unknowns and 4 equations whose augmented matrix is

$$\begin{bmatrix} a & 0 & b & a_{11} - a \\ -b & a & 0 & a_{12} - b \\ c & 0 & 1-a & a_{21} - c \\ a-1 & c & 0 & a_{22} + a - 1 \end{bmatrix}.$$

An easy computation shows that the system matrix $\begin{bmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & 1-a \\ a-1 & c & 0 \end{bmatrix}$ has rank 2, as $a(1-a) = bc$.

Since the $3 \times 3$ minors of the system matrix are zero, so is the determinant of the augmented matrix.

Another easy computation shows that the remaining twelve $3 \times 3$ minors of the augmented matrix are zero iff $c\text{row}_1(A) = a\text{row}_2(A)$.

Thus, in order to find a solution we select (say) the first two equations i.e., $ax + bz = a_{11} - a$, $-bx + ay = a_{12} - b$. Then $x = \frac{a_{11} - a - bz}{a} - 1$ and $y = \frac{b(a_{11} - a - bz) + a(a_{12} - b)}{a^2}$ and by replacing in $x^2 + yz = 0$ we obtain $x = -\frac{(a_{11} - a)(a_{12} - b)}{ba_{11} - aa_{12}}$, $y = -\frac{(a_{12} - b)^2}{ba_{11} - aa_{12}}$ and $z = \frac{(a_{11} - a)^2}{ba_{11} - aa_{12}}$. Hence, the existence of this solution requires the divisibilities in the statement. □

**The case** $a = 1$. As $a(1 - a) = bc$, at least one of $b, c$ must be zero and (say) $E = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$. Then $a_{21} = a_{22} = 0$ are necessary conditions for a matrix $A = [a_{ij}]_{1 \le i,j \le 2}$ to be unipotent-regular. As in the previous proof, $x = a_{11} - bz - 1$, $y = a_{12} - b + bx = a_{12} - b + b(a_{11} - bz - 1)$ and $x^2 + yz = 0$ gives $x = -\frac{(a_{11} - 1)(a_{12} - b)}{ba_{11} - a_{12}}$, $y = -\frac{(a_{12} - b)^2}{ba_{11} - a_{12}}$ and $z = \frac{(a_{11} - 1)^2}{ba_{11} - a_{12}}$ with $(a_{11} - 1)^2, (a_{12} - b)^2$ and $(a_{11} - 1)(a_{12} - b)$ divisible by $ba_{11} - a_{12}$.

The case $b = 0$ follows by transpose.

**The case** $a = 0$. Again at least one of $b, c$ must be zero and (say) $E = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$. The first two equations of the linear system are $bz = a_{11}$, $b(1 - x) = a_{12}$. Therefore if both $a_{11}, a_{12}$ are divisible by $b$,we get $x = 1 - \frac{a_{12}}{b}$, $z = \frac{a_{11}}{b}$ and arbitrary $y$.

**Remark**. If $R$ is not an integral domain, we don't have a known form for $2 \times 2$ idempotent or nilpotent matrices and so the above proof is not suitable.

Same for $n \times n$ matrices with $n \ge 3$.

In view of Theorem 1, we could wonder whether there exist unipotent-regular matrices which are not idempotent. Such matrices do exist.

**Example**. The zero determinant integral matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is *not idempotent but is unipotent-regular*.

The rows are dependent so we can take $a = k$, $c = 2k$ for any $k$. To choose $b$, from $k(1 - k) = 2kb$ we need $2b = 1 - k$.

For $k = 1$, that is $a = 1$, $c = 2$, $c$ divides $a(1 - a) = 0$. Then $b = 0$ and $0^2, 2^2$ and $0 \cdot 2$ are divisible by $a_{12} = 2$. Indeed, $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is an idempotent-unipotent product.

The decomposition is unique. Since $2b = 1 - k$, $k$ must be odd, say $k = 2l - 1$. Then $b - 2a = 3 - 5l$ should divide $2(1 - l)^2$, $(1 - l)^2$ and $4(1 - l)^2$. Over $\mathbb{Z}$, this amounts to a quadratic Diophantine equation $l^2 + 5lm - 2l - 3m + 1 = 0$ which has only one solution: $(l, m) = (1, 0)$. Hence $k = 1$.

[The details. Solving the first two equations of the corresponding linear system and replacing in $x^2 + yz = 0$, we get $x = -\frac{(1 - a)(2 - b)}{b - 2a}$, $y = -\frac{(2 - b)^2}{b - 2a}$ and

$z = \frac{(1-a)^2}{b-2a}$. For $a = 1$, we obtain $x = z = 0$, $y = 2-b$. Finally, as $a(1-a) = 0 = bc$ we have $b = 0$].

## References


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Department of Mathematics, Babeş-Bolyai University, 1 Kogălniceanu Street, Cluj-Napoca, Romania;, Department of Computer Science, Babeş-Bolyai University, 1 Kogălniceanu Street, Cluj-Napoca, Romania

*Email address*: calu@math.ubbcluj.ro; horia.pop@ubbcluj.ro