A zero determinant 2×2 matrix which has not stable range one

G. Călugăreanu, H. F. Pop

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Abstract

In an attempt to answer a question stated by T. Y. Lam, we tried, without success to find such an example over $\mathbb{Z}[i\sqrt{5}]$.

1 Introduction

In [1] it is shown that over $\mathbb{Z}$, the only sr1 2×2 matrices have determinant in $\{-1, 0, 1\}$. In a forthcoming paper ([3]), T. Y. Lam generalized this result, proving

Theorem 1 *Let $A \in R = M_n(S)$, where $n \geq 1$ and S is a commutative elementary divisor domain (EDD). If $\det(A) \in \{0\} \cup U(S)$, then $sr_R(A) = 1$. In the case where $S = \mathbb{Z}$, the converse of this statement also holds.*

Accordingly, Lam also asks the following

Question. Let $R = M_n(S)$ where S is any commutative ring, and $n \geq 1$. If $A \in R$ is such that $\det(A) = 0$, does it follow that $sr_R(A) = 1$?

2 The example

In an attempt to provide an example justifying a negative answer, we thought of the commutative ring $R = \mathbb{Z}[i\sqrt{5}]$, well-known of not being UFD and at the (also well-known) zero determinant 2×2 matrix $M = \begin{bmatrix} 3 & 1 - i\sqrt{5} \\ 1 + i\sqrt{5} & 2 \end{bmatrix}$. Notice that R is not an EDD: if R would be EDD, it should be a PID and so UFD, a contradiction.

We first recall the following result (see [2] Cor. 5 (a), p. 13).

Proposition 2 *$sr(R) = 1$ iff for each $b \in R$, the homomorphism $U(R) \to U(R/Rb)$ is surjective .*

Then we have

Lemma 3 *$sr(R) = sr(\mathbb{M}_n(R)) = 2$ for any $n \geq 1$.*

Proof. (A. Vasiu) Since R is the normalization of $\mathbb{Z}$ in a totally imaginary quadratic extension K of $\mathbb{Q}$, it follows that $U(R)$ is finite.

Using the previous proposition, $sr(R) \neq 1$; as $sr(R) \leq 2$, we conclude that $sr(R) = 2$. Finally, a matrix ring $\mathbb{M}_n(R)$ (for a fixed positive integer n) has stable range one iff the base ring R has stable range one (see [4]). ■

Summarizing, $\mathbb{Z}[i\sqrt{5}]$ and $\mathbb{M}_n(\mathbb{Z}[i\sqrt{5}])$ both have stable range two, but some elements in each of these may have stable range one.

The question stated above asks to what extent zero determinant 2×2 matrices have sr1. As will follow below, the matrix M above, has sr1, so is not a suitable counterexample. A simple sufficient [so far...] condition for zero determinant 2×2 matrices will be found.

We next recall the characterization theorem proved in [1].

Theorem 4 *Let R be a commutative ring and $A \in \mathbb{M}_2(R)$. Then A has left stable range 1 iff for any $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that*

$$\det(Y)(\det(X)\det(A) - \mathrm{Tr}(XA) + 1) + \det(A(\mathrm{Tr}(XY) + 1)) - \mathrm{Tr}(A\mathrm{adj}(Y))$$

is a unit of R.

Here $\mathrm{adj}(Y)$ is the classical adjoint (the adjugate).

The following consequences will be useful.

Corollary 5 *Let R be a commutative ring and $A \in \mathbb{M}_2(R)$. If $\det(A) = 0$, then $sr_R(A) = 1$ iff for any $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that $\det(Y)(1 - Tr(XA) - Tr(Aadj(Y)) = 1$.*

Corollary 6 *Let R be a commutative ring, $0_2 \neq A \in \mathbb{M}_2(R)$ and $\det(A) = 0$. If there exist $Y \in \mathbb{M}_2(R)$ such that $\det(Y) = 0$ and $Tr(Aadj(Y)) = -1$ then $sr_R(A) = 1$.*

Corollary 7 *The matrix $M = \begin{bmatrix} 3 & 1 - i\sqrt{5} \\ 1 + i\sqrt{5} & 2 \end{bmatrix}$ has sr1. Moreover, a unitizer exists which is independent of X.*

Proof. Since $\det(M) = 0$, using the last corollary, it suffices to take $Y = \begin{bmatrix} 2 + 2i\sqrt{5} & 3 \\ -2 + 2i\sqrt{5} & 2 + i\sqrt{5} \end{bmatrix}$. One verifies that $\det(Y) = 0$ and $Tr(Madj(Y)) = -1$. ■

We can rephrase Corollary 6 as follows.

Theorem 8 *Let R be a commutative ring, $0_2 \neq A \in \mathbb{M}_2(R)$ and $\det(A) = 0$. If there exists a matrix Y such that $\det(Y) = 0$ and AY is a nontrivial idempotent, then $sr_R(A) = 1$.*

Proof. We just replace in Corollary 6, $adj(Y)$ with Y and then Y by $-Y$ (this can be done as $\det(Y) = \det(adj(Y))$ for 2×2 matrices, and $Tr(-B) = -Tr(B)$) and then use Cayley-Hamilton theorem (as $\det(AY) = 0$ and $Tr(AY) = 1$). ■

Example. For the matrix $M = \begin{bmatrix} 3 & 1 - i\sqrt{5} \\ 1 + i\sqrt{5} & 2 \end{bmatrix}$ we started with and $Y = \begin{bmatrix} 2 + 2i\sqrt{5} & 3 \\ -2 + 2i\sqrt{5} & 2 + i\sqrt{5} \end{bmatrix}$ we have $-adj(Y) = \begin{bmatrix} -2 - i\sqrt{5} & 3 \\ -2 + 2i\sqrt{5} & -2 - 2i\sqrt{5} \end{bmatrix}$. Indeed, $M(-adj(Y)) = \begin{bmatrix} 2 + i\sqrt{5} & -3 \\ -1 + i\sqrt{5} & -1 - i\sqrt{5} \end{bmatrix}$ is idempotent (zero determinant and trace = 1) and $\det \begin{bmatrix} -2 - i\sqrt{5} & 3 \\ -2 + 2i\sqrt{5} & -2 - 2i\sqrt{5} \end{bmatrix} = 0$.

3 Computer aid

First, everything is about zero determinant 2×2 matrices over $\mathbb{Z}[i\sqrt{5}]$; the computer will browse the zero determinant nonzero 2×2 matrices A.

If there exists Y such that $\det(Y) = 0$ and AY is a nontrivial idempotent, computer displays nothing and continues browsing the A's.

Computer displays only the matrices A for which **NO** zero determinant matrix Y with idempotent AY exists.

Computer displays $A = -(1 + i\sqrt{5}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, since for $z = 1$ it is the first matrix considered. It is easy to generalize this example.

Lemma 9 *In the conditions of Theorem 8, if* $\gcd(A) \notin \{\pm 1\} = U(\mathbb{Z}[i\sqrt{5}])$ *then a (zero determinant) matrix* Y *does not exist.*

Proof. Let $\alpha \in \mathbb{Z}[i\sqrt{5}]$ with $\alpha \notin \{\pm 1\}$ and $A = \alpha A'$. Then $Tr(AY) = \alpha Tr(A'Y) = 1$ implies α is a unit, a contradiction. ■

Indeed, displays also $(1 + i\sqrt{5}) \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, $\pm(1 + i\sqrt{5}) \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\pm(1 + i\sqrt{5}) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and others similar.

In the general case, if $A = \alpha A'$, then $\det(Y)(1 - Tr(XA)) - Tr(Aadj(Y) = \det(Y) - \alpha\beta = 1$ implies $\gcd(\det(Y), \alpha) = 1$.

...

Computer displays also $A = \begin{bmatrix} \alpha & \alpha \\ \overline{\alpha} & \overline{\alpha} \end{bmatrix}$ with $\alpha = 1 + i\sqrt{5}$. Both α, $\overline{\alpha}$ are irreducible in $\mathbb{Z}[i\sqrt{5}]$ so $\gcd(\alpha, \overline{\alpha}) = 1$.

Also transpose and other two similar like $\begin{bmatrix} \alpha & \overline{\alpha} \\ -\alpha & -\overline{\alpha} \end{bmatrix}$.

Explanation: $Tr(AY) = \alpha(y_{11} + y_{21}) + \overline{\alpha}(y_{12} + y_{22}) = 1$. See proof below: there is no linear combination of α, $\overline{\alpha}$ equal to 1.

Also $\pm\begin{bmatrix}\alpha & 0\\ \overline{\alpha} & 0\end{bmatrix}$ or $\pm\begin{bmatrix}\alpha & \overline{\alpha}\\ 0 & 0\end{bmatrix}$ or with opposite signs. Again, of course zero determinant.

$Tr(AY) = \alpha y_{11} + \overline{\alpha} y_{12} = 1$.

The explanation: $\mathbb{Z}[i\sqrt{5}]$ *is not Bézout !* $\gcd(\alpha, \overline{\alpha}) = 1$ but there is no linear combination of α and $\overline{\alpha}$ equal to 1.

Proof. $(1+i\sqrt{5})(a+bi\sqrt{5}) + (1-i\sqrt{5})(c+di\sqrt{5}) = 1$ amounts to $a+c-5(b-d) = 1$ and $a+b-c+d = 0$. Replacing $c = a+b+d$ gives $2a-4b+6c = 1$ with no integer solutions.

Like in the above proof, $(1+i\sqrt{5})(a+bi\sqrt{5}) + (1-i\sqrt{5})(c+di\sqrt{5}) = x$ is equivalent to $2 \mid x$.

We abandoned the program: after 3 days and a half only matrices with $z = 1$ and upper left corner $-(1+i\sqrt{5})$ were partly covered.

4 Attempts

First attempt. Coming back to the general case, $\det(Y)[1 - Tr(XA)] - Tr(Aadj(Y) = 1$, we consider $A = \begin{bmatrix}1+i\sqrt{5} & 0\\ 1-i\sqrt{5} & 0\end{bmatrix}$.

By computation (the usual notations), for every X we should (not) find an Y [we no more suppose $\det(Y) = 0$] such that

$$\det(Y)[1 - x_{11}(1+i\sqrt{5}) - x_{12}(1-i\sqrt{5})] - y_{22}(1+i\sqrt{5}) + y_{12}(1-i\sqrt{5}) = 1$$

of course with $\det(Y) = y_{11}y_{22} - y_{12}y_{21}$.

The benefit is now that computer has to browse only two entries of X: $x_{11}, x_{12} \in \mathbb{Z}[i\sqrt{5}]$. It displays only when a matrix $Y = \begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}$ is **not** found.

Summary. Computer browses $x_{11}, x_{12} \in \mathbb{Z}[i\sqrt{5}]$. For each such pair, it searches for 4 entries $y_{11}, y_{12}, y_{21}, y_{22} \in \mathbb{Z}[i\sqrt{5}]$ such that

$$(y_{11}y_{22} - y_{12}y_{21})[1 - x_{11}(1+i\sqrt{5}) - x_{12}(1-i\sqrt{5})] - y_{22}(1+i\sqrt{5}) + y_{12}(1-i\sqrt{5}) = 1.$$

That is, an equation in $\mathbb{Z}[i\sqrt{5}]$, without matrices, determinants or traces.

Second attempt. In the first attempt, replace $1+i\sqrt{5}$ by 2. That is

$$(y_{11}y_{22} - y_{12}y_{21})[1 - 2x_{11} - x_{12}(1-i\sqrt{5})] - 2y_{22} + y_{12}(1-i\sqrt{5}) = 1.$$

Third attempt. In the first attempt replace $1+i\sqrt{5}$ by $2+i\sqrt{5}$. That is

$$(y_{11}y_{22} - y_{12}y_{21})[1 - x_{11}(2+i\sqrt{5}) - x_{12}(1-i\sqrt{5})] - y_{22}(2+i\sqrt{5}) + y_{12}(1-i\sqrt{5}) = 1.$$

Forth attempt. Consider $A = \begin{bmatrix} 2 & 2 \\ 1+i\sqrt{5} & 1+i\sqrt{5} \end{bmatrix}$. Then $Aadj(Y) = \begin{bmatrix} 2(y_{11}+y_{21}) & * \\ * & (1+i\sqrt{5})(y_{12}+y_{22}) \end{bmatrix}$ and

$XA = \begin{bmatrix} 2x_{11}+(1+i\sqrt{5})x_{12} & * \\ * & 2x_{21}+(1+i\sqrt{5})x_{22} \end{bmatrix}$. Now

$\det(Y)[1-2x_{11}-(1+i\sqrt{5})x_{12}-2x_{21}-(1+i\sqrt{5})x_{22})]-2(y_{11}+y_{21})-(1+i\sqrt{5})(y_{12}+y_{22})=1$.

After days, no matrix was displayed.

5 Another idea

Would be to vanish (instead of $\det(Y)$) $1-Tr(XA)$. That is, to find an X such that $Tr(XA)=1$ but there is no Y with $Tr(Aadj(Y))=-1$.

Unfortunately, this is not possible as $Tr(CD)=Tr(DC)$ for any matrices C, D.

Hence, if a matrix X exists with $Tr(XA)=1$ then for $Y=-adj(X)$ we get $Tr(Aadj(Y)=Tr(Aadj(-adj(X)))=-Tr(AX)=-Tr(XA)=-1$.

In general, if we take $Y=-adj(X)$, the characterization gives $\det(X)-)(1-Tr(XA))=0$, so this unitizer choice works only when $\det(X)=1$ or else $Tr(XA)=1$.

6 Final comment

As considerable efforts were made, without success, to find a zero determinant 2×2 matrix over $\mathbb{Z}[i\sqrt{5}]$ which has not sr1, it remains *plausible* that the initial question has an affirmative answer, i.e., *zero determinant* 2×2 *matrices over* $\mathbb{Z}[i\sqrt{5}]$ *have stable range one.*

References

[1] G. Călugăreanu, H. F. Pop *On stable range one matrices*. Bull. Math. Soc. Sci. Math. Roumanie Tome **65** (**113**) (3) (2022), 317-327.

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[4] L. N. Vaserstein *The stable range of rings and dimension of topological spaces*. Func. Anal. Appl. **5** (1971), 17-27.