EXAMPLES OF EQUIVALENT NILPOTENT MATRICES THAT ARE NOT SIMILAR

Abstract. An example proving the statement in the title is given in $\mathbb{M}_3(\mathbb{Z}(8))$. As for 2×2 matrices, it is proved that over commutative GCD (every pair of elements has a greatest common divisor) domains, two matrices are equivalent if and only if these are similar. This fails over commutative rings with zero divisors, as a final example shows.

1. Introduction

For any unital ring R, we denote by $U(R)$, the set of units of R. In any matrix ring, E_{ij} denotes the matrix all whose entries are zero excepting the (i,j) entry, which is 1. As usual, for a ring R, $R^* = R - \{0\}$. If R is an arbitrary unital ring and $n \geq 2$ is a positive integer, by $\mathbb{M}_n(R)$ we denote the ring of all $n \times n$ matrices with entries in R. In a ring R, two elements a, b are *equivalent* if there exist units $p, q \in U(R)$ such that $paq = b$. Two elements are *associated* if one element is a unit multiple of the other. Finally, two elements are *conjugate* if there exists a unit $u \in U(R)$ such that $uau^{-1} = b$.

If $t^k = 0$ and $t^{k-1} \neq 0$ for $t \in R$ and $k \geq 2$, we say that k is the *nilpotence index* of t in R. In the sequel the word "index" will always mean "nilpotence index". It is easy to see that *conjugate nilpotents have the same index*. In our discussion, the zero (nilpotent) element of a ring can be excepted. Indeed, if a, b are equivalent and one of these elements is zero, so is the other and a, b are also conjugated.

In [1], a nice elementary proof shows that *equivalent idempotents must be conjugate* in any (unital) ring, i.e. the restrictions to idempotents of (all) these (binary) relations coincide. It is easy to see that *the nilpotent analogue*, that is, equivalent nilpotents must be conjugate, generally *fails*.

Suggested by Jordan block decompositions, one can give the following

Example. Over any ring consider the 4×4 matrices $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (i.e., $J_2(0) \oplus J_2(0)$ and $J_3(0) \oplus 0$ where $J_n(0)$ denotes the

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Jordan $n \times n$ block associated to the eigenvalue 0). Since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = B,$$

it follows that these matrices are equivalent. A straightforward computation shows that any matrix U with the property $AU = UB$ has the first column equal to zero. Hence no such invertible U exists and thus A and B are not similar. Moreover, this example can be extended to matrices of sizes $n \geq 4$ as well.

Over fields, it suffices to observe that A and B have different minimal polynomials and so are not similar.

In this short note we discuss the 3×3 and the 2×2 cases. More precisely, we give a 3×3 example of equivalent matrices which are not similar over $\mathbb{Z}(8)$, and for 2×2 matrices, we prove that over a (commutative) Bézout (every pair of elements has a greatest common divisor which can be expressed as a linear combination of the elements) domain, two matrices are equivalent if and only if these are similar. This fails over commutative rings with zero divisors, as a final example shows.

We also mention that an element in a ring which is associated to a nilpotent element, may not be nilpotent: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is an idempotent for 2×2 matrices over any unital ring.

2. The 3×3 example

In order to construct an example, we recall that conjugate nilpotents have the same index. Hence it suffices to construct two equivalent (actually, associated) nilpotents of different indexes.

The starting point was to check whether 3×3 matrices of form $M = \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{bmatrix}$ can be nilpotent. Since $M^3 = \begin{bmatrix} 0 & ac^2 & acd \\ 0 & c^3 & c^2d \\ 0 & 0 & 0 \end{bmatrix}$, we may want to vanish the $(2, 2)$ entry, without vanishing the other entries. Hence it is natural to consider $R = \mathbb{Z}(8)$ and $c = 2$ (no special notation for classes mod 8).

Take $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ for which $T^2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $T^3 = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $T^4 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T^5 = 0_3$. Hence T has index 5 in $\mathbb{M}_3(\mathbb{Z}(8))$.

Now consider $S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It is easily checked that S has index 3 (over any unital ring), so the matrices T and S are not similar. However since S can be obtained from T by an elementary row operation, these two matrices are associated (and so equivalent): $UT = S$ for $U = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. 2×2 MATRICES

A commutative domain R is called *GCD* if every pair of elements has a greatest common divisor, denoted by $\gcd(a; b)$. GCD domains include unique factorization domains, Bézout domains and valuation domains. An element a is called *primal* if whenever a divides bc (for $a, b, c \in R$), there exist $a_1, a_2 \in R$ such that $a = a_1a_2$, a_1 divides b and a_2 divides c. A basic well-known property of a GCD domain is needed for the next (lemma and) proposition: *in any GCD domain every nonzero element is primal.*

Lemma 1. *In any GCD domain,*
(1) if a divides bc and $\gcd(a; b) = 1$, then a divides c.
(2) if $\gcd(x; y) = 1$ then $\gcd(x^2; y) = 1$.

Proof. (1) In fact, $\gcd(a; b) = 1$ implies $\gcd(ac; bc) = c$. As a is a common divisor of ac and bc, a divides $\gcd(ac; bc)$. That is, a divides c.

(2) Suppose $1 \neq d = \gcd(x^2; y)$. By the primal property, $d|x^2$ implies $d = d_1^2$ with $d_1|x$ and clearly $1 \neq d_1$. Since also $d_1|y$ we get $\gcd(x; y) \neq 1$. □

Proposition 2. *Every nonzero nilpotent 2×2 matrix over a Bézout domain R is similar to rE_{12}, for some $r \in R$.*

Proof. We are looking for an invertible matrix U such that, for a given nilpotent matrix $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$ (trace and determinant equal zero), $TU = U(rE_{12})$.

Let $d = \gcd(x; y)$. Then $x = dx_1$, $y = dy_1$ with $\gcd(x_1; y_1) = 1$. From $x^2 = -yz$ we get $dx_1^2 = -y_1z$ and using the previous lemma, $y_1|d$. Set $d = y_1y_2$ and so $-z = x_1^2y_2$. Now $T = y_2T'$ with $T' = \begin{bmatrix} x_1y_1 & y_1^2 \\ -x_1^2 & -x_1y_1 \end{bmatrix}$.

Since $\gcd(x_1; y_1) = 1$, there are elements $u, v \in R$ such that $ux_1 + vy_1 = 1$.

For $U = \begin{bmatrix} y_1 & u \\ -x_1 & v \end{bmatrix}$ it is readily checked that $T'U = UE_{12}$, i.e. $TU = U(rE_{12})$ with $r = y_2$, as desired. □

Remark. 1) Conjugation with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ shows that over any ring R, rE_{12} is similar to $-rE_{12}$.

2) Conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ shows that over any ring R, rE_{12} is also similar to $rE_{21} = \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}$.

Hence, for instance if $R = \mathbb{Z}$, in order to have non-similar representatives, it suffices to take rE_{12} with $r \in \mathbb{N}$.

Proposition 3. *Let R be a GCD domain. For $r, s \in R^*$, the following conditions are (logically) equivalent:*

(i) rE_{12} *and* sE_{12} *are equivalent*
(ii) r, s *are associate*
(iii) rE_{12} *and* sE_{12} *are similar.*

Proof. (i) ⇒ (ii) Start with units $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Q = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, i.e., $ad - bc, xt - yz \in U(R)$ and require $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. So $\begin{bmatrix} 0 & ar \\ 0 & cr \end{bmatrix} = \begin{bmatrix} sz & st \\ 0 & 0 \end{bmatrix}$, or $z = c = 0$, $ar = st$. From above $ad, xt \in U(R)$, that is, $a, d, x, t \in U(R)$. Since $a, t \in U(R)$ we get associate r, s.

(ii) ⇒ (iii) If r, s are associate, say $r = us$ with $u \in U(R)$, then for $U = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ we obtain $U(rE_{12}) = (sE_{12})U$, as desired. □

Moreover

Proposition 4. *Over any Bézout domain, two nilpotent* 2×2 *matrices are equivalent if and only if these are similar.*

Proof. One way is obvious. Conversely, take two not similar nilpotent 2×2 matrices A, B (i.e. $A^2 = B^2 = 0_2$). By Proposition 2, there exist units U, V such that $UAU^{-1} = rE_{12}$, $VBV^{-1} = sE_{12}$, with not associate $r, s \in R$.

By contradiction, assume A and B are equivalent, that is, there exist units P, Q such that $PAQ = B$. By replacement, $rPU^{-1}E_{12}UQ = sV^{-1}E_{12}V$ (because both $= B$). If $d = \gcd(r; s)$ and $r = dr'$, $s = ds'$, we have $r'PU^{-1}E_{12}UQ = s'V^{-1}E_{12}V$ with $\gcd(r'; s') = 1$. Notice that both r', s' cannot be units (otherwise r, s are associate).

This is possible only if $PU^{-1}E_{12}UQ = s'C$ and $V^{-1}E_{12}V = r'D$ for some matrices C, D. Hence $E_{12} = r'VDV^{-1} = s'UP^{-1}CQ^{-1}U^{-1}$. If for instance r' is not a unit, $E_{12} = r'VDV^{-1}$ fails, a contradiction. □

4. 2×2 matrices over rings with zero divisors

We use the following example: $S = \mathbb{Z}[x]$ with $x^2 = 0$ and $R = \mathbb{M}_2(S)$. Take $T = \begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}$. Then $T^2 = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ and $T^3 = 0_2$. So T has index 3.

Next consider the unit $U = \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}$. Then $UT = \begin{bmatrix} x & 1 \\ 0 & -x \end{bmatrix}$ and $(UT)^2 = 0_2$ (indeed, $\det(UT) = \mathrm{Tr}(UT) = 0$), so UT has index 2.

Hence T and UT are associate (and so equivalent) nilpotents, of different indexes, which are not similar.

References

[1] G. Song, X. Guo *Diagonability of idempotent matrices over noncommutative rings*. Linear Algebra and its Applications **297** (1999), 1-7.