

# MODULES WITH ABELIAN ENDOMORPHISM RINGS.

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ABSTRACT. The results of Szele and Szendrei characterizing Abelian groups with commutative endomorphism rings are generalized to modules whose endomorphism rings have various restrictions on their idempotents. Such properties include central or commuting idempotents, and one-sided ideals being two-sided. Related properties include direct summands having unique complements, or being fully invariant.

## 1. INTRODUCTION

Throughout this paper, unless specifically noted, the word *ring* denotes a unital associative ring, *group* denotes an abelian group, and *module* denotes a unital right  $R$ -module. It is a long standing problem in abelian group theory to describe those groups  $G$  whose endomorphism ring  $\text{End}(G)$  is commutative and those commutative rings which are isomorphic to the endomorphism ring of an abelian group. Krylov, Mikhalev and Tuganbaev [KMT03] present an authoritative survey of the history of the problems. The known results are of two sorts: if  $G$  is torsion, then a complete classification is known for both parts of the problem [KMT03, Corollary 19.3]; if  $G$  is torsion-free, then no classification is feasible; instead, there are realization theorems stating that large classes of commutative rings can be realized as  $\text{End}(G)$  for some group  $G$  [KMT03, Section 29]. Naturally, mixed groups lie between these two extremes.

The purpose of this paper is to generalize the problem by considering modules whose endomorphism ring is ‘almost commutative’. For example, we describe classes of modules whose endomorphism rings have commuting idempotents, or in which one-sided ideals are two-sided. We also consider modules for which direct summands have unique complements or are fully invariant.

Let us first establish some notation. Elements of a ring or module are denoted by lower case Latin letters, and homomorphisms by lower case Greek letters. Functions are written on the left of their arguments, and matrices act by left multiplication on column vectors representing module elements. We denote a submodule  $N$  of  $M$  by  $N \leq M$  and a direct summand  $N$  of  $M$  by  $N \leq_{\oplus} M$ . In particular, if  $R$  is a ring, then  $I \leq R$  means that  $I$  is a right ideal of  $R$ . We denote the (not necessarily abelian) group of units of  $R$  by  $U = U(R)$  and the centre of a ring  $R$  by  $Z = Z(R)$ .

Let  $R$  be a ring and let  $E = E(R)$  denote the set of idempotents in  $R$ . If  $e \in E$ , then  $e$  is called *proper* if  $e \neq 0$  or  $1$ . There is a natural partial order, denoted  $\leq$ , on  $E$  defined by  $e \leq f$  if  $ef = fe = e$ , and non-zero minimal elements of  $E$  with

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respect to this order are called *primitive* idempotents. A ring  $R$  is called *Abelian* if  $E(R) \subseteq Z(R)$ , that is, if all idempotents are central.

If  $e \in E$ , then  $\bar{e} = 1 - e \in E$  is called the *complementary* idempotent and  $e\bar{e} = \bar{e}e = 0$ . In general,  $e$  and  $f \in E$  are called *orthogonal* if  $ef = fe = 0$ . If  $e \in E$  is proper, then  $R = eR \oplus \bar{e}R$  is a proper direct decomposition of  $R$  considered as right  $R$ -module. In particular,  $e$  is primitive if and only if  $eR$  is a minimal right ideal summand. Similar remarks are true of course for left modules  $Re$  and  $R\bar{e}$ . If  $e \in E$  is central, then so is  $\bar{e}$  and the decomposition is into a direct sum of ideals.

For right  $R$ -modules  $M$  and  $N$ ,  $\text{Hom}(M, N) = \text{Hom}_R(M, N)$  denotes the additive group of  $R$ -homomorphisms of  $M$  into  $N$  and  $\text{End}(M) = \text{End}_R(M)$  the ring of  $R$ -endomorphisms of  $M$ . The following facts concerning idempotents  $\varepsilon \in \text{End}(M)$  are well known [F73, Section 106]. If  $\varepsilon$  is a proper idempotent then  $\varepsilon$  determines a non-trivial direct decomposition  $M = \text{im}(\varepsilon) \oplus \text{ker}(\varepsilon)$ . Conversely, if  $M = N \oplus K$  is a decomposition of  $M$  into non zero summands, then there is a unique idempotent  $\varepsilon \in \text{End}(M)$  such that  $\varepsilon|_N$  is the identity map and  $\varepsilon|_K$  is the zero map. We say that  $N \oplus K$  is the decomposition corresponding to  $\varepsilon$  and that  $\varepsilon$  is the idempotent corresponding to the decomposition  $M = N \oplus K$ .

The paper is organised as follows. In Section 2, we describe the rings in which direct summand right ideals are uniquely complemented and relate this property to subcommutativity and certain one-sided ideals being two sided. In Section 3, we show that these properties are equivalent to  $R$  being Abelian.

In the remainder of the paper, the results of Sections 2 and 3 are applied to endomorphism rings of modules. For example, in Section 4 we show how unique complementation of summands of  $\text{End}(M)$  implies unique complementation of summands of  $M$ . We also find a module theoretic generalisation of the main result of Szele and Szendrei [SzSz51] on abelian groups with commutative endomorphism rings. Section 5 is devoted to characterizations of the fully invariant submodules and direct summands of a module.

## 2. UNIQUELY COMPLEMENTED SUMMANDS OF RINGS

Suppose that  $R = I \oplus J$  is a non-trivial decomposition of the ring  $R$  into right ideals. Then the idempotent endomorphisms considered above can be realised by multiplications by elements of  $E$ , the set of idempotents of  $R$ . In this Section, we examine the connection between the centrality of  $e \in E$  and uniqueness of complements of the direct summand  $eR$ . We begin with some known facts and some elementary results.

**Proposition 2.1.** (Folklore) *Let  $R_R = I \oplus J$  be a right ideal direct decomposition of  $R$ . There exists  $e \in E$  such that  $I = eR$  and  $J = \bar{e}R$*   $\square$

Much is known about idempotents  $e$  and  $f$  for which  $eR = fR$ . For example, here is a selection from Lam [Lam95, Section 21]:

**Proposition 2.2.** *Let  $e$  and  $f \in E$ . The following are equivalent:*

- (1)  $eR = fR$
- (2)  $ef = f$  and  $fe = e$
- (3) *there exists  $r \in R$  such that  $f = e + er\bar{e}$*
- (4) *there exists  $u \in U$  such that  $f = eu$*
- (5)  $R\bar{e} = R\bar{f}$ .

$\square$

The direct complements of a right ideal direct summand may be characterized as follows

**Lemma 2.3.** *Let  $R = eR \oplus gR$ . Then  $gR = (\bar{e} - er\bar{e})R$  for a suitable  $r \in R$ . Conversely, for every  $r \in R$ ,  $(\bar{e} - er\bar{e})R$  is a direct complement for  $eR$ .*

*Proof.* According to Proposition 2.1, there is an idempotent  $f$  such that  $eR = fR$  and  $gR = \bar{f}R$ . By Proposition 2.2,  $f = e + er\bar{e}$  and  $g = \bar{f} + \bar{f}sf$  for suitable  $r, s \in R$ . Thus  $g = \bar{f}(1 + \bar{f}sf)$  and  $u = 1 + \bar{f}sf$  is invertible in  $R$  with inverse  $1 - \bar{f}sf$ . Hence  $g = (\bar{e} - er\bar{e})u$  so that  $gR = (\bar{e} - er\bar{e})R$ .

As for the converse, again by Proposition 2.1,  $eR \oplus (\bar{e} - er\bar{e})R = R$  if there is an idempotent  $f \in R$  such that  $eR = fR$  and  $(\bar{e} - er\bar{e})R = \bar{f}R$ . Clearly  $f = e + er\bar{e}$  suits well for this.  $\square$

Since  $eR$  always has the direct complement  $\bar{e}R$ , we say that  $eR$  has a unique complement if  $R = eR \oplus gR$  implies  $\bar{e}R = gR$ .

**Proposition 2.4.**  *$eR$  has a unique complement if and only if  $eR\bar{e} = 0$ .*

*Proof.* Since by Lemma 2.3 all direct complements of  $eR$  are of the form  $(\bar{e} - er\bar{e})R$ ,  $eR\bar{e} = 0$  implies that  $eR$  has a unique complement.

Conversely, if for all  $r \in R$ ,  $(\bar{e} - er\bar{e})R = \bar{e}R$ , then by Proposition 2.2, (2),  $(\bar{e} - er\bar{e})\bar{e} = \bar{e}$  so  $er\bar{e} = 0$  for all  $r \in R$ .  $\square$

Similarly,  $eR$  is the unique direct complement of  $\bar{e}R$  if and only if  $\bar{e}Re = 0$ .

We now consider the question of when not just the complement of  $eR$  but even its idempotent generator  $\bar{e}$  is unique.

**Proposition 2.5.** *Let  $R = eR \oplus fR$  with  $e, f \in E$ . Then  $f = \bar{e}$  if and only if  $ef = fe$ .*

*Proof.* If  $f = \bar{e}$  then  $ef = e(1 - e) = (1 - e)e = fe$ .

Conversely, since  $ef = fe$ ,  $\bar{e}f = f\bar{e}$  so that  $efR$  and  $\bar{e}fR$  are both contained in  $fR$ . Decompose  $fR$  as  $fR = efR \oplus \bar{e}fR$ . Since  $efR \subseteq eR \cap fR = 0$ ,  $ef = 0$  so that  $fR = \bar{e}fR \subseteq \bar{e}R$ . Similarly,  $\bar{e}R \subseteq fR$  so that  $\bar{e}R = fR$ .

It follows that  $\bar{e} = fr$  for some  $r \in R$  and  $f = \bar{e}s$  for some  $s \in R$ . Hence  $f\bar{e} = f^2r = fr = \bar{e}$  and similarly  $\bar{e}f = f$ . By commutativity,  $\bar{e} = f$ .  $\square$

A similar property concerns commuting idempotent generated right ideals of  $R$ .

**Proposition 2.6.** *Let  $e, f \in E$  satisfy  $R = eR \oplus fR$ . If  $\bar{e}RfR = fR\bar{e}R$ , then  $fR = \bar{e}R$ .*

*Proof.* Since  $\bar{e}R \leq_{\oplus} eR \oplus fR$ ,  $\bar{e}R = \bar{e}^2R \leq \bar{e}fR \leq \bar{e}RfR = fR\bar{e}R \leq fR$ , so that  $\bar{e}R \leq fR$ . Similarly,  $fR \leq \bar{e}R$  so we have equality.  $\square$

A property of idempotents related to unique complementation is subcommutativity.

**Definition 2.7.** [R65] An element  $a \in R$  is *right (left) subcommutative* if  $Ra \subseteq aR$  ( $aR \subseteq Ra$ ).  $a$  is *subcommutative* if it is both left and right subcommutative. A subset of  $R$  is (left) (right) subcommutative if each of its elements is.

The concept of one-sided subcommutativity frequently occurs in the ring theory literature under different names. For example, Birkenmeier [B83, p. 569] defines

an idempotent  $e \in E$  to be left (right) semicentral if  $Re = eRe$  ( $eR = eRe$ ). It is readily seen that left semicentral is equivalent to our right subcommutative.

The importance of subcommutativity in the present context is that if  $\alpha \in \text{End}(M)$  is right subcommutative, then for all  $\beta \in \text{End}(M)$ ,  $\beta\alpha(M) \subseteq \alpha(M)$ , i.e.,  $\alpha(M)$  is fully invariant. Reid [R65, Section 3] gives an example of a non-commutative left subcommutative endomorphism ring. However, we shall show in Section 3 that subcommutative idempotents are actually central.

We show now that an idempotent  $e$  is left subcommutative if and only if  $eR$  has a unique complement.

**Proposition 2.8.** *Let  $e \in E$ . The following are equivalent:*

- (1)  $e$  is left subcommutative.
- (2)  $eR\bar{e} = 0$
- (3)  $\bar{e}$  is right subcommutative.
- (4)  $eR$  has a unique complement.

*Proof.* (1)  $\Leftrightarrow$  (2) For all  $r \in R$ , (1) implies that  $er \in Re$ , so  $eR\bar{e} \subseteq Re\bar{e} = 0$ . Conversely, for all  $r \in R$ ,  $r = re + r\bar{e}$ , so  $er = ere + er\bar{e} = ere \in Re$ .

(2)  $\Leftrightarrow$  (3) For all  $r \in R$ ,  $r = er + \bar{e}r$ . Hence (2) implies that  $r\bar{e} = \bar{e}r\bar{e} \in \bar{e}R$ .

Conversely, for all  $r \in R$ ,  $r\bar{e} \in \bar{e}R$ , so  $er\bar{e} \in e\bar{e}R = 0$ .

(2)  $\Leftrightarrow$  (4) This is Proposition 2.4. □

A related commutativity condition on rings is that one-sided ideals are two-sided.

**Definition 2.9.** A ring is called *right (left) duo* if every right (left) ideal is two-sided.

**Proposition 2.10.** *A ring is right (left) duo if and only if it is right (left) subcommutative.*

*Proof.* ( $\Rightarrow$ ) Let  $a \in R$ . Then  $aR$  is a right ideal, which by hypothesis must be also left ideal. Since rings have identity,  $a \in aR$  and so  $Ra \subseteq aR$ .

( $\Leftarrow$ ) Let  $I$  be a right ideal of  $R$ . In order to check that  $I$  is also left ideal, we have to show that for all  $a \in R$ ,  $Ra \subseteq I$ . But this is immediate, since  $Ra \subseteq aR \subseteq I$  by right subcommutativity of  $a$ . □

### 3. ABELIAN RINGS

Recall that  $R$  is called Abelian if  $E \subseteq Z(R)$ . A non-empty subset of  $R$  is called commutative if each pair of its elements commute. A surprising result noticed by Lam [Lam95, Ex. 22.3A, with solution] is that  $R$  is Abelian if and only if  $E$  is commutative and more generally, if and only if every  $e \in E$  commutes with the idempotents  $f$  for which  $eR \cong fR$ .

Consequently, we have:

**Theorem 3.1.** *Let  $R$  be ring with idempotent set  $E$ . The following statements are equivalent.*

- (1)  $E$  is commutative.
- (2)  $R$  is Abelian.
- (3) Every idempotent generated right ideal is uniquely complemented.
- (4)  $E$  is subcommutative.

*Proof.* (1)  $\Rightarrow$  (2) By the remark in the previous paragraph, every idempotent is central.

(2)  $\Rightarrow$  (3) is an immediate consequence of Proposition 2.8.

(3)  $\Rightarrow$  (4) is also an immediate consequence of Proposition 2.8.

(4)  $\Rightarrow$  (1) By Proposition 2.8, for all  $e \in E$ ,  $eR\bar{e} = 0 = \bar{e}Re$ . Hence  $fe = efe + \bar{e}fe = efe = efe + ef\bar{e} = ef$ .  $\square$

**Example 3.2.** For an example of an Abelian but non-commutative ring, let  $\{D_i : i \in I\}$  be a set of non-isomorphic division rings indexed by an arbitrary set  $I$ , not all of which are fields. Let  $R = \prod_{i \in I} D_i$  and consider  $R$  as right  $R$ -module. It is clear that the only idempotents in  $R$  are the sequences  $(a_i : i \in I)$  where each  $a_i$  is either the identity or the zero of  $D_i$ . Thus all right ideals of  $R$  are uniquely complemented, so  $R$  is an Abelian ring.

Abelian rings have many properties in common with commutative rings, for example, we have seen that complements and idempotent generators of idempotent generated ideals are unique. Another such property is:

**Proposition 3.3.** *If  $R$  is Abelian then  $E$  is a Boolean algebra with respect to the natural order  $ef \leq e$ .*

*Proof.* For all  $e$  and  $f \in E$ ,  $ef$  is an idempotent satisfying  $ef \leq e$  and  $ef \leq f$ . Furthermore, if  $g \in E$  satisfies  $g \leq e$  and  $g \leq f$ , then  $g \leq ef$ . Hence  $E$  contains  $ef = \inf\{e, f\}$ .

Next,  $e + f - ef$  is an idempotent satisfying  $e \leq e + f - ef$  and  $f \leq e + f - ef$ . If  $g \in E$  satisfies  $e \leq g$  and  $f \leq g$ , then  $e + f - ef \leq g$ . Hence  $E$  contains  $e + f - ef = \sup\{e, f\}$ .

Each  $e \in E$  has complement  $\bar{e} \in E$ , and for  $e, f$  and  $g \in E$ ,  $(e + f - ef)g = eg + fg - egfg$  while  $ef + g - efg = (e + g - eg)(f + g - fg)$  so  $E$  is a distributive lattice.  $\square$

The following corollary is immediate.

**Corollary 3.4.** *Let  $R$  be an Abelian ring and let  $\mathcal{L}$  be the set of idempotent generated right ideals of  $R$ , ordered by inclusion. Then  $\mathcal{L}$  is a Boolean algebra and the map  $e \mapsto eR$  from  $E$  to  $\mathcal{L}$  is a Boolean algebra isomorphism.*  $\square$

Other related properties are that Abelian rings are Dedekind finite and products and intersections of idempotent generated right ideals coincide.

**Proposition 3.5.** *If  $R$  is Abelian, then  $R$  is Dedekind finite. The converse is false.*

*Proof.* Recall that  $R$  is Dedekind finite if for all  $a, b \in R$ ,  $ab = 1$  implies  $ba = 1$ . Let  $ab = 1$ . Then  $ba = b(ab)a = (ba)(ba)$  is an idempotent. Since  $R$  is Abelian,  $baR(1 - ba) = 0$  so in particular,  $bab(1 - ba) = 0$ . Hence  $1 - ba = (ab)^2(1 - ba) = 0$  so  $ba = 1$ .

To see that the converse is false, let  $R$  be the  $2 \times 2$  matrix ring over a field.  $\square$

**Proposition 3.6.** *Let  $R$  be an Abelian ring and let  $e, f \in E$ . Then  $efR = eRfR = eR \cap fR$ .*

*Proof.* Clearly  $efR \leq eRfR \leq eR \cap fR$ . Conversely, let  $x = er = fs \in eR \cap fR$ . Then  $x = e^2r = efs \in efR$ .  $\square$

Finally, we consider the semi-commutativity of  $E$ . A subset  $X$  of a ring  $R$  is called *semi-commutative* [CN08, Definition 1.2] if for all  $a, b \in X$ ,  $ab = 0$  implies that  $aRb = 0$ .

**Proposition 3.7.**  *$E$  is semi-commutative if and only if  $R$  is Abelian.*

*Proof.* For all  $e \in E$ ,  $e\bar{e} = 0$  so that  $eR\bar{e} = 0$ . Hence by Proposition 2.4  $eR$  is uniquely complemented so  $e$  is central.

Conversely, if  $e$  is central and  $ef = 0$ , then obviously  $eRf = 0$ .  $\square$

#### 4. MODULES WITH ABELIAN ENDOMORPHISM RINGS

We now show how the results of the previous section apply to endomorphism rings of modules. Throughout,  $R$  is a unital ring and all modules are unital right  $R$ -modules. Homomorphisms are written on the left of their arguments. To avoid confusion, we write  $\mathcal{E}$  for  $E(\text{End}(M))$ , the set of idempotents in  $\text{End}(M)$ . A module  $M$  is an *endabelian module* if its endomorphism ring  $\text{End}(M)$  is Abelian.

**Remark 4.1.** Since left multiplications from  $R$  are in  $\text{End}(M)$ , if  $M$  is an endabelian module then  $R$  is an Abelian ring.

For a direct decomposition  $J \oplus K$  of  $M$ , the *associated projections*  $\varepsilon_J$  and  $\varepsilon_K$  are the uniquely determined idempotents in  $\mathcal{E}$  for which  $J = \text{im } \varepsilon_J = \ker \varepsilon_K$  and  $K = \ker \varepsilon_J = \text{im } \varepsilon_K$ . Thus  $\varepsilon_K = \bar{\varepsilon}_J$ . We show that summands of  $M$  are uniquely determined by right ideals generated by idempotents and *vice versa*.

**Lemma 4.2.** *Let  $M = J \oplus K$ . Then  $\varepsilon_J \text{End}(M) = \{f \in \text{End}(M) : f(M) \subseteq J\}$ .*

*Conversely, let  $\varepsilon \in \mathcal{E}$  and let  $J = \text{trace}(\varepsilon \text{End}(M))$ , the submodule of  $M$  generated by  $\{\varepsilon f(M) : f \in \text{End}(M)\}$ . Then  $M = J \oplus K$ ,  $\varepsilon_J = \varepsilon$  and  $\varepsilon_K = \bar{\varepsilon}$ .*

*Proof.* 1. Clearly for all  $f \in \text{End}(M)$ ,  $\varepsilon_J f$  maps  $M$  into  $J$ . Also, if  $f \in \text{End}(M)$  maps  $M$  into  $J$ , then  $f = \varepsilon_J f \in \varepsilon_J \text{End}(M)$ .

2. Let  $m \in M$ . Then  $m = \varepsilon m + (m - \varepsilon m)$  where  $\varepsilon m \in \text{trace}(\varepsilon \text{End}(M))$  and  $m - \varepsilon m \in \ker \varepsilon = \bar{\varepsilon}(M)$ .  $\square$

**Lemma 4.3.** *Let  $M = J \oplus K$ . The following are equivalent:*

- (1)  $J$  is uniquely complemented in  $M$
- (2)  $\varepsilon_J \text{End}(M)$  is uniquely complemented in  $\text{End}(M)$ , considered as a right  $\text{End}(M)$  module.
- (3)  $\varepsilon_J \text{End}(M) \varepsilon_K = 0$
- (4)  $K$  is fully invariant in  $M$

*Proof.* (1)  $\Leftrightarrow$  (2) By Lemma 4.2, for any submodule  $L$  of  $M$ ,  $L$  is a complementary summand for  $J$  if and only if  $\varepsilon_L \text{End}(M)$  is a complementary summand for  $\varepsilon_J \text{End}(M)$ . Hence the uniqueness of  $K$  implies the uniqueness of  $\varepsilon_K \text{End}(M)$  and *vice versa*.

(2)  $\Leftrightarrow$  (3) By Proposition 2.4,  $\varepsilon_J \text{End}(M)$  is uniquely complemented in  $\text{End}(M)$  if and only if  $\varepsilon_J \text{End}(M) \varepsilon_K = 0$

(3)  $\Leftrightarrow$  (4) Since  $\text{End}(M) = \text{Hom}(J, J) \oplus \text{Hom}(J, K) \oplus \text{Hom}(K, J) \oplus \text{Hom}(K, K)$ ,  $K$  is fully invariant if and only if  $\text{Hom}(K, J) = 0$ . But since every  $f \in \text{Hom}(K, J)$  satisfies  $f = \varepsilon_J g \varepsilon_K$  for some  $g \in \text{End}(M)$ ,  $K$  is fully invariant if and only if  $\varepsilon_J \text{End}(M) \varepsilon_K = 0$ .  $\square$

**Theorem 4.4.** *Let  $M$  be an  $R$ -module. The following are equivalent:*

- (1) All direct summands of  $M$  are uniquely complemented.
- (2)  $M$  is an endabelian module.
- (3) All direct summands of  $M$  are fully invariant.

*Proof.* In view of Lemma 4.3, it suffices to show that  $\text{End}(M)$  is Abelian if and only if for all  $\varepsilon \in \mathcal{E}$ ,  $\varepsilon \text{End}(M)$  is uniquely complemented. But this precisely the content of Theorem 3.1.  $\square$

In the theory of commutative endomorphism rings of mixed abelian groups, a major rôle is played by groups  $G$  of the form

$$\bigoplus_{p \in S} G_p \leq G \leq \prod_{p \in S} G_p$$

where  $S$  is an infinite set of primes, for all  $p \in S$ ,  $G_p \cong \mathbb{Z}(p^{k_p})$  with  $k_p$  a positive integer or  $\infty$ , and  $G$  is  $S$ -pure in  $\prod_{p \in S} G_p$ .

Such groups have endomorphism rings of the form

$$\bigoplus_{p \in S} \text{End}(G_p) \subseteq \text{End}(G) \subseteq \prod_{p \in S} \text{End}(G_p)$$

where  $\text{End}(G_p) \cong \mathbb{Z}(p^{k_p})$  if  $k_p$  is finite and  $\text{End}(G_p) \cong \mathbb{Z}_p$ , the ring of  $p$ -adic integers, if  $k_p = \infty$ .

In particular,  $\text{End}(G)$  is commutative. This result appears for example in [SzSz51], [S73] and [KMT03].

In the torsion-free context, a similar result is obtained by replacing the set  $S$  of primes by a set  $T$  of incomparable idempotent types  $\tau$  (i.e., subrings of  $\mathbb{Q}$ ), the groups  $G_p$  by the types  $\tau \in T$  and the  $S$ -purity condition by  $\text{Hom}(G / \bigoplus_{\tau \in T} \tau, G) = 0$ .

In both cases, the commutativity of  $\text{End}(G)$  follows from the commutativity of each  $\text{End}(G_p)$  and  $\text{End}(\tau)$ , together with full invariance of the indecomposable summands  $G_p$  and  $\tau$ . However the structure theorem can be generalised as follows without any commutativity assumption on the endomorphism rings of the summands.

**Definition 4.5.** A ring  $R$  with idempotent set  $E$  is called *nearly commutative* if

- (1)  $R$  is Abelian.
- (2) For every  $0 \neq a \in R$ , there is a primitive idempotent  $e \in E$  such that  $ea \neq 0$ .

**Remark 4.6.** (a) We say a module  $M$  is an NC module if  $\text{End}(M)$  is nearly commutative.

- (b) Condition (2) implies that every non-zero idempotent in  $E$  dominates a primitive idempotent, i.e., for all  $0 \neq e \in E$  there is a primitive idempotent  $f$  such that  $fe = f = ef$ .
- (c) The abelian group example above shows that (2) may hold even when the dcc on idempotents does not.
- (d) To see that the conditions are independent, note that the ring  $\mathbb{Z}(p)^2$  satisfies (2) but not (1), and the semigroup ring  $\mathbb{Z}[S]$ , where  $S$  is the semigroup  $\{s_i : i \in \mathbb{Z}\}$  with  $s_i s_j = s_{\min(i,j)}$  satisfies (1) but not (2).

The following theorem describes the structure of nearly commutative rings

**Theorem 4.7.** *Let  $R$  be a nearly commutative ring, and let  $F$  be a maximal orthogonal set of primitive idempotents in  $R$ . Then  $R$  is a subdirect product of  $\{eR : e \in F\}$  containing  $\bigoplus_{e \in F} eR$ . That is to say,*

$$\bigoplus_{e \in F} eR \leq R \leq \prod_{e \in F} eR$$

*Proof.* First note that condition (2) of near commutativity implies that  $R$  has nonzero primitive idempotents and Zorn's Lemma implies that maximal orthogonal sets of primitive idempotents exist in  $R$ .

Let  $F$  be a maximal orthogonal set of primitive idempotents in  $R$ , and define  $\theta : R \rightarrow \prod_{e \in F} eR$  by  $x \mapsto (ex)$ . It is clear that

- (1)  $\theta$  is a module homomorphism;
- (2)  $\theta$  preserves multiplication;
- (3) the image of  $\theta$  contains all  $eR$ ,  $e \in F$  and hence  $\bigoplus_{e \in F} eR$ .

Let  $K = \ker(\theta)$  and let  $x \in K$ . By condition (2), if  $x \neq 0$ , then there is a primitive idempotent  $0 \neq f \in R$  such that  $fx \neq 0$  and hence  $f \notin F$ . But for all  $e \in F$ , either  $fe = e = f$  or  $fe = 0$ . Thus  $fe = 0$  for all  $e \in F$ . Therefore  $F \cup \{f\}$  is an orthogonal set of primitive idempotents strictly containing  $F$ , a contradiction.

Hence  $K = 0$  so that  $\theta$  is an embedding.  $\square$

**Corollary 4.8.** *If  $R$  is a nearly commutative ring, then  $R$  is a unique subdirect product of indecomposable ideals.*  $\square$

**Remark 4.9.** (1) It is easy to find countable reduced torsion-free rings satisfying the hypotheses of Theorem 4.7. For example, take any countable poset  $P$  satisfying the descending chain condition, let  $S$  be the semigroup on the set  $P$  with operation  $xy = \min\{x, y\}$  and let  $R$  be the semigroup ring  $\mathbb{Z}[S]$ . Then  $R$  satisfies Theorem 4.7 and by Corner's Theorem, there is a torsion-free abelian group  $G$  with  $\text{End}(G) \cong R$ .

- (2) The condition Abelian may be weakened to primitive idempotents are central.

Our main structure theorem is:

**Theorem 4.10.** *Let  $M$  be an  $R$ -module.  $M$  is NC if and only if there is a rigid family  $\mathcal{M}$  of indecomposable  $R$ -modules such that  $\bigoplus_{N \in \mathcal{M}} N \leq M \leq \prod_{N \in \mathcal{M}} N$  and  $\text{Hom}(M/(\bigoplus_{N \in \mathcal{M}} N), M) = 0$ .*

*Proof.* Recall that a family  $\mathcal{M}$  of modules is rigid if for all  $L, N \in \mathcal{M}$ ,  $\text{Hom}(L, N) = 0$ .

( $\Rightarrow$ ) Since  $\text{End}(M)$  is a subdirect product of indecomposable ideals generated by primitive idempotents,  $M$  contains a unique rigid maximal family  $\mathcal{M}$  of fully invariant indecomposable summands and hence their direct sum  $\bigoplus_{N \in \mathcal{M}} N$ .

The projections of  $M$  onto its summands  $N$  induce a homomorphism of  $M$  into  $\prod_{N \in \mathcal{M}} N$ . Since this homomorphism has kernel zero,  $\text{Hom}(M/(\bigoplus N), M) = 0$ .

( $\Leftarrow$ ) Since each  $N$  is fully invariant in  $M$ , each  $\text{End}(N)$  is an ideal summand in  $\text{End}(M)$  and their sum is direct. The homomorphism  $\Theta : \text{End}(M) \rightarrow \prod$  defined by  $f \mapsto (\varepsilon_N f)$  has kernel  $\{f \in \text{End}(M) : f(\bigoplus N) = 0\}$ . But since  $\text{Hom}_R(M/(\bigoplus N), M) = 0$  this kernel is 0, so  $\Theta$  is injective.  $\square$

5. FULLY INVARIANT SUBMODULES OF  $M$

We are concerned with properties of  $\text{End}(M)$  which imply that certain classes of submodules of  $M$  are fully invariant. For example, we have seen that direct summands are fully invariant if and only if  $M$  is an endabelian module, which implies that  $R$  is an Abelian ring. With this in mind, we define  $M$  to be *stable* if all endomorphic images are fully invariant, and *substable* if all kernels of endomorphisms are fully invariant. We noted in the paragraph following Definition 2.7 that if  $\alpha \in \text{End}(M)$  is right subcommutative, then  $\alpha(M)$  is fully invariant, so if  $\text{End}(M)$  is right subcommutative, then  $M$  is stable.

It is clear that if  $\text{End}(M)$  is commutative then  $M$  is stable and substable. It has long been known that the converse is false, even for abelian groups, but to find a counterexample, one must construct a torsion-free countable reduced ring which is subcommutative but not commutative and then invoke Corner's Theorem to show it is an endomorphism ring. This was done independently by Orsatti and Reid in 1965 [KMT03, p 123]. Other examples have been constructed since, but they all rely on the same principle.

Our aim here is to elucidate the relations between (sub)stability of  $M$  and subcommutativity of  $\text{End}(M)$

**Proposition 5.1.** *Let  $\alpha \in \text{End}(M)$ .*

- (1) *If  $\alpha$  is left subcommutative then  $\ker \alpha$  is fully invariant.*
- (2) *If  $\text{im } \alpha$  is a summand of  $M$  and  $\ker \alpha$  is fully invariant, then  $\alpha$  is left subcommutative.*

*Proof.* 1. Let  $\beta \in \text{End}(M)$ . There exists  $\gamma \in \text{End}(M)$  such that  $\alpha\beta = \gamma\alpha$ . Let  $a \in \ker \alpha$ , so  $\alpha\beta(a) = \gamma\alpha(a) = 0$ . Hence  $\beta a \in \ker \alpha$ .

2. Let  $\alpha, \beta \in \text{End}(M)$ . Suppose  $\text{im } \alpha$  is a summand of  $M$ . Define  $\gamma \in \text{End}(G)$  by  $\gamma(\alpha a) = \alpha\beta a$  on  $\text{im } \alpha$  and  $\gamma = 0$  on a complement.  $\gamma$  is well defined since if  $\alpha a = \alpha b$  then  $a - b \in \ker \alpha$  so  $\beta a - \beta b \in \ker \alpha$  and hence  $\alpha\beta a = \alpha\beta b$ .

It follows that  $\alpha\beta = \gamma\beta$  as required. □

**Corollary 5.2.** *If  $\alpha \in \text{End}(M)$  is idempotent or an epimorphism, then  $\alpha$  is left subcommutative if and only if  $\ker \alpha$  is fully invariant.* □

On the other hand, we have:

**Proposition 5.3.** *Let  $\alpha \in \text{End}(M)$ .*

- (1) *If  $\alpha$  is right subcommutative then  $\text{im } \alpha$  is fully invariant.*
- (2) *If  $\alpha$  is a monomorphism and  $\text{im } \alpha$  is fully invariant, then  $\alpha$  is right subcommutative.*

*Proof.* 1. Let  $\beta \in \text{End}(M)$ . There exists  $\gamma \in \text{End}(M)$  such that  $\beta\alpha = \alpha\gamma$ . Let  $a \in \text{im } \alpha$ , say  $a = \alpha b$ . Then  $\beta a = \beta\alpha b = \alpha\gamma(a) \in \text{im } \alpha$ .

2. Suppose  $\alpha$  is monic. Since  $\text{im } \alpha$  is fully invariant, for every  $a \in M$ , there is a unique  $b \in \text{im } \alpha$  such that  $\alpha b = \beta\alpha(a)$ . Let  $\gamma$  be defined by  $\gamma(a) = b$ . It follows that  $\gamma \in \text{End}(M)$  and  $\beta\alpha = \alpha\gamma$  as required. □

**Corollary 5.4.** *If  $\alpha \in \text{End}(M)$  is idempotent or a monomorphism, then  $\alpha$  is right subcommutative if and only if  $\text{im } \alpha$  is fully invariant.* □

**Corollary 5.5.** *Let  $M$  be an  $R$ -module.*

- (1) If  $\text{End}(M)$  is right subcommutative then  $M$  is stable. The converse holds if every endomorphism is monic.
- (2) If  $\text{End}(M)$  is left subcommutative then  $M$  is substable. The converse holds if every endomorphism is epic.  $\square$

We have seen that if  $\varepsilon \in \text{End}(M)$  is a central idempotent, then  $M = \text{im } \varepsilon \oplus \ker \varepsilon$  where each summand is fully invariant in  $M$  and hence uniquely complemented. We now address the converse of this result, i.e., if  $M$  is the direct sum of two fully invariant submodules, are they the image and kernel of a central idempotent in  $\text{End}(M)$ ? More generally, if  $H$  is a fully invariant submodule of  $M$ , when is  $H$  a summand with a fully invariant complement?

We first consider the relationship between fully invariant submodules of  $M$  and (two-sided) ideals of  $\text{End}(M)$ . Let  $\mathcal{H}$  be the set of fully invariant submodules of  $M$  and  $\mathcal{I}$  the set of ideals of  $\text{End}(M)$ , It is routine to check that  $\mathcal{H}$  and  $\mathcal{I}$  are complete lattices under inclusion. Denote by  $\mathcal{H}^{op}$  the opposite lattice of  $\mathcal{H}$ .

For all  $H \in \mathcal{H}$ , let  $H' = \text{Ann}(H) = \{\alpha \in \text{End}(M) : \alpha(H) = 0\}$  and let  $H^* = \text{Ann}(M/H) = \{\alpha \in \text{End}(M) : \alpha(M) \subseteq H\}$ . It is routine to check that  $H'$  and  $H^* \in \mathcal{I}$ . Similarly, for all  $I \in \mathcal{I}$ , let  $I' = \{a \in M : Ia = 0\}$  and let  $I^* = IM$ , the submodule of  $M$  generated by  $\{\alpha(a) : \alpha \in I \text{ and } a \in M\}$ , so that  $I'$  and  $I^* \in \mathcal{H}$ .

It was shown in [AS04] that the mappings  $\Theta' : \mathcal{H} \rightarrow \mathcal{I}$  defined by  $H \mapsto H'$  and  $\Theta^* : \mathcal{H}^{op} \rightarrow \mathcal{I}$  defined by  $H \mapsto H^*$  are Galois Correspondences, with inverses defined by  $I \mapsto I'$  and  $I \mapsto I^*$  respectively. In particular, for all  $H \in \mathcal{H}$ ,  $H \leq H''$  and  $H^{**} \leq H$  and for all  $I \in \mathcal{I}$ ,  $I \leq I''$  and  $I \leq I^{**}$ .

An  $H \in \mathcal{H}$  is called  $'$ -closed if  $H = H''$  and  $*$ -closed if  $H = H^{**}$ . Similarly,  $I \in \mathcal{I}$  is  $'$ -closed if  $I = I''$  and  $*$ -closed if  $I = I^{**}$ . It follows that  $H$  is  $'$ -closed if and only if  $H = I'$  for some ideal  $I \in \mathcal{I}$ ,  $I \in \mathcal{I}$  is  $'$ -closed if and only if  $I = H'$  for some  $H \in \mathcal{H}$ ,  $H$  is  $*$ -closed if and only if  $H = I^*$  for some  $I \in \mathcal{I}$  and  $I$  is  $*$ -closed if and only if  $I = H^*$  for some  $H \in \mathcal{H}$ .

We have seen that if  $I \in \mathcal{I}$  is a summand of  $\text{End}(M)$  then it is generated by a central idempotent and is uniquely complemented, while it is clear that if  $H \in \mathcal{H}$  is a summand of  $M$ , then it is the image of a central idempotent and is uniquely complemented.

We now describe conditions on  $I \in \mathcal{I}$  and on  $H \in \mathcal{H}$  which ensure that they are summands.

**Proposition 5.6.** *Let  $I \in \mathcal{I}$ . The following conditions are equivalent:*

- (1)  $I$  is a summand of  $\text{End}(M)$ ;
- (2)  $I$  is  $'$ -closed,  $I \cap (I')^* = 0$  and every endomorphism of  $I'$  extends to an endomorphism of  $M$ ;
- (3)  $I$  is  $*$ -closed,  $I \cap (I^*)' = 0$  and every endomorphism of  $M/I^*$  lifts to an endomorphism of  $M$ .

*If these conditions are satisfied, then  $M = I' \oplus I^*$ .*

*Proof.* (1)  $\Leftrightarrow$  (2) Suppose  $I$  is a summand of  $\text{End}(M)$ , say  $\text{End}(M) = \varepsilon \text{End}(M) \oplus \bar{\varepsilon} \text{End}(M)$ , where  $I = \varepsilon \text{End}(M)$  for some central idempotent  $\varepsilon$ . Then  $I' = \bar{\varepsilon}(M) = H$ , say and  $H' = I$ , so  $I = I''$ . Hence  $H = \ker \varepsilon = I'$  is a summand of  $M$  so that  $I \cap (I')^* = 0$  and every endomorphism of  $H$  extends to  $\text{End}(M)$ .

Conversely, Proposition 2.4 of [AS04] implies that if  $I \in \mathcal{I}$  with  $I \cap (I')^* = 0$ , then there is a short exact sequence  $I'' \rightarrow \text{End}(M) \xrightarrow{\eta} \text{End}(M)^H$  where the latter

ring is the set of endomorphisms of  $H$  which extend to endomorphisms of  $M$ ,  $\eta$  being the restriction map. If every such endomorphism extends, then the sequence splits so  $\text{End}(M) \cong I \oplus \text{End}(H)$  as a direct sum of ideals.

(1)  $\Leftrightarrow$  (3) The proof is similar, so only sketched. If  $I$  is a summand of  $\text{End}(M)$ , with central idempotent generator  $\varepsilon$ , let  $H = \varepsilon(M) = I^*$ . Then  $I = H^*$  and  $M = H \oplus I'$ , so that  $I = I^{**}$  and  $I \cap (I^*)' = 0$ . Since  $H$  is a summand of  $M$ , every endomorphism of  $M/H = I'$  lifts to  $\text{End}(M)$ .

Conversely, Proposition 2.4 of [AS04] implies that the hypotheses force  $\text{End}(M) = H' \oplus H^* = (I^*)' \oplus I$ .  $\square$

**Proposition 5.7.** *Let  $H \in \mathcal{H}$ . The following conditions are equivalent:*

- (1)  $H$  is a summand of  $M$ ;
- (2)  $H$  is  $'$ -closed,  $H' \cap H^* = 0$  and every endomorphism of  $H$  extends to an endomorphism of  $M$ ;
- (3)  $H$  is  $*$ -closed,  $H^* \cap H' = 0$  and every endomorphism of  $M/H$  lifts to an endomorphism of  $M$ .

If these conditions are satisfied, then  $\text{End}(M) = H' \oplus H^*$ .

*Proof.* (1)  $\Leftrightarrow$  (2) Suppose  $M = H \oplus K$ , and let  $\varepsilon$  be the corresponding projection onto  $K$ . Since  $H$  is fully invariant,  $\bar{\varepsilon}$  and its complement  $\varepsilon$  are central idempotents in  $\text{End}(M)$ . Let  $I = \varepsilon \text{End}(M)$  and  $L = \bar{\varepsilon} \text{End}(M)$ . Then  $H = I'$  is  $'$ -closed and  $L = H^*$  so that  $H' \cap H^* = 0$ . Clearly every endomorphism of  $H$  extends to an endomorphism of  $M$ .

Conversely, the hypotheses and Proposition 2.4 of [AS04] imply that  $\text{End}(M) = H' \oplus H^*$  with central idempotent generators  $\varepsilon$  and  $\bar{\varepsilon}$  so that  $H = \ker \varepsilon$  is a summand with complement  $\text{im } \varepsilon$ .

(1)  $\Leftrightarrow$  (3) The proof in both directions is similar.  $\square$

The criteria of Propositions 5.6 and 5.7 simplify if  $\text{End}(M)$  is Abelian since in that case, by Theorem 4.4, all direct summands are fully invariant. The following theorem is immediate.

**Theorem 5.8.** *Let  $R$  be a ring and  $M$  an endabelian  $R$ -module. Let  $H$  be a submodule of  $M$  and  $I$  an ideal of  $\text{End}(M)$ .*

- (1) *The following are equivalent:*
  - (a)  $H$  is a direct summand of  $M$ ;
  - (b)  $H$  is fully invariant and  $'$ -closed,  $H' \cap H^* = 0$  and every endomorphism of  $H$  extends to an endomorphism of  $M$ ;
  - (c)  $H$  is fully invariant and  $*$ -closed,  $H^* \cap H' = 0$  and every endomorphism of  $M/H$  lifts to an endomorphism of  $M$

*In that case,  $\text{End}(M) = H' \oplus H^*$*

- (2) *The following are equivalent:*
  - (a)  $I$  is a direct summand of  $\text{End}(M)$ ;
  - (b)  $I$  is  $'$ -closed,  $I \cap (I')^* = 0$  and every endomorphism of  $I'$  extends to an endomorphism of  $M$ ;
  - (c)  $I$  is  $*$ -closed,  $I \cap (I^*)' = 0$  and every endomorphism of  $M/I^*$  lifts to an endomorphism of  $M$ .

*In that case,  $M = I' \oplus I^*$*   $\square$

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