The centroid of some generalized pedal configurations

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... all meaning comes from analogies.
Douglas Hofstadter

A mathematician is a person who can find analogies between theorems, a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories, the ultimate mathematician is one who can see analogies between analogies.
Stefan Banach

Abstract. The main goal of this paper is to give possible generalizations, analogues of the following property:

If $M_1, M_2$ and $M_3$ are the orthogonal projections of a point $M$ to the sides $A_2A_3$, $A_3A_1$ and $A_1A_2$ of an equilateral triangle $A_1A_2A_3$, then the centroid of the triangle $M_1M_2M_3$ is the midpoint of the segment $OM$, where $O$ is the center of the triangle $A_1A_2A_3$.

In the first step we extend this property to regular $n$-gons, regular tetrahedrons and regular $n$ simplices. In the second part we give a general affine version for triangles and simplices.

It is also our objective to analyze the possibility of using such properties in teaching problem solving strategies for students and mathematics teachers. Theorem 2, 3, 4, 6 and Conjecture 1 and 2 were discovered/rediscovered during a training course for mathematics teachers.

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Introduction

In this note we analyze some possible generalizations of the following property:

Theorem 1. If we denote by $M_1, M_2$ and $M_3$ the orthogonal projections of a point $M$ to the sides $A_1A_2$, $A_2A_3$ and $A_3A_1$ of an equilateral triangle $A_1A_2A_3$, then the centroid $G$ of the triangle $M_1M_2M_3$ is the midpoint of the line segment $OM$, where $O$ is the center of triangle $A_1A_2A_3$.

Figure 1: The centroid of a podar triangle relative to an equilateral triangle

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This problem was studied on a teacher training course at the Babeș-Bolyai University from the perspective of the inquiry based learning. The starting point was not only the problem, but also a solution of this problem using complex numbers. The aim of the instructors was to support the participants (secondary school mathematics teachers) in experimenting, developing and proving generalizations of this property. This approach was used in order to prove that inquiry based learning can also be used in problem solving activities and in the framework of the existing curricula. The main tools for experimenting were dynamic geometry softwares (Geogebra, Geonext and Cabri) and the teachers were working in groups. In the first round each group had to present some ideas about the possible generalizations, in the second round each group had to formulate conjectures and had to experiment his own conjectures and in the final round they had to prove the conjectures which seemed to be valid.

**Conjectures and proofs**

As a first step we can replace the equilateral triangle with something more general: an arbitrary triangle, a regular polygon, a regular tetrahedron or a regular simplex. Or we can use some more general projections in constructing the pedal triangle. In order to formulate some more general properties we recall a proof for theorem 1.

**Proof.** Without loosing generality we can assume that $M$ is in the plane of the triangle $A_1A_2A_3$, and the circumscribed circle is of radius 1. Let’s consider $O$ as the origin of the coordinate system and $OA_1$ as the $OX$ axis. The complex numbers corresponding to the vertices $A_1$, $A_2$ and $A_3$ are

$$a_1 = \varepsilon, \quad a_2 = \varepsilon^2 \text{ and } a_3 = \varepsilon^3,$$

where $\varepsilon^3 = 1$ and $\varepsilon \neq 1$. Due to our assumptions, $m_1$ is easy to find, because it’s real part is $-\frac{1}{2}$ and it’s imaginary part is the same as $m$’s. So

$$m_1 = \frac{1}{2} + \frac{m - \overline{m}}{2}.$$

To calculate $m_2$ we’ll use a counterclockwise rotation of angle $\alpha = \frac{4\pi}{3}$. This rotation transforms the point $M_2$ to the orthogonal projection of the point $Q(\varepsilon^2 \cdot m)$ to the side $A_1A_2$. Hence we have

$$m_2 \cdot \varepsilon^2 = \frac{1}{2} \cdot \varepsilon^2 \cdot m - \frac{(\varepsilon^2 \cdot m)}{2},$$

and so

$$m_2 = \frac{1}{2} \cdot \varepsilon + \frac{m - \varepsilon^2 \cdot \overline{m}}{2}.$$

By the same argument we deduce

$$m_3 = \frac{1}{2} \cdot \varepsilon^2 + \frac{m - \varepsilon \cdot \overline{m}}{2}.$$

From these relations we get

$$\frac{m_1 + m_2 + m_3}{3} = \frac{m}{2},$$

which expresses the desired property. \qed

The above presented proof suggests that something similar holds also for a regular $n$-gon. A little experience with a dynamic geometry software helps us to formulate the following property:

**Theorem 2.** If we denote by $M_i$ ($1 \leq i \leq n$) the orthogonal projections of a point $M$ to the sides $A_iA_{i+1}$ ($1 \leq i \leq n$ and $A_{n+1} = A_1$) of a regular $n$-gon $A_1A_2A_3 \ldots A_n$ and by $O$ the center of the polygon, then the centroid of the $n$-gon $M_1M_1M_2 \ldots M_n$ is the midpoint of the line segment $OM$. 

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Figure 2: The centroid of a podar polygon relative to a regular n-gon

Proof. First we consider the case when $n$ is odd. The vertices of the regular $n$-gon are represented by

$$a_j = \varepsilon^j = \cos \frac{2j\pi}{n} + i \sin \frac{2j\pi}{n}, \quad 1 \leq j \leq n.$$  

If $n = 2k + 1$, we have $A_kA_{k+1} \parallel OY$ and hence

$$m_k = \cos \frac{2k\pi}{2k+1} + \frac{m - \overline{m}}{2}.$$  

Using rotations again we can deduce

$$m_j \cdot \varepsilon^{k-j} = \cos \frac{2k\pi}{2k+1} + \frac{m \cdot \varepsilon^{k-j} - (m \cdot \varepsilon^{k-j})}{2}$$

for $1 \leq j \leq 2k + 1$. But $\varepsilon = \varepsilon^{2k} = \varepsilon^{-1}$ and hence we have the following relations:

$$m_j = \varepsilon^{k+j+1} \cos \frac{2k\pi}{2k+1} + \frac{m - \varepsilon^{2j+1} \overline{m}}{2} \quad \text{for} \quad 1 \leq j \leq 2k + 1.$$  

These relations and the $\sum_{v=0}^{n-1} \varepsilon^v = 0$ equality imply

$$\frac{1}{n} \cdot \sum_{j=1}^{n} m_j = \frac{m}{2}.$$  

If $n$ is even ($n = 2k$) we take

$$a_j = z_0 \varepsilon^j = z_0 \left( \cos \frac{2j\pi}{n} + i \cdot \sin \frac{2j\pi}{n} \right), \quad 1 \leq j \leq n,$$

where

$$z_0 = \cos \frac{\pi}{2k} + i \cdot \sin \frac{\pi}{2k}.$$
and we obtain
\[
m_{k-1} = \cos \frac{(2k-1)\pi}{2k} + \frac{m-m_2}{2},
\]
\[
m_{j} \cdot \varepsilon^{k-j-1} = \cos \frac{(2k-1)\pi}{2k} + \frac{m \cdot \varepsilon^{k-j-1} - (m \cdot \varepsilon^{k-j-1})}{2}
\]
for \(1 \leq j \leq 2k\). These relations imply \(\frac{1}{n} \sum_{j=1}^{n} m_j = \frac{m_2}{2}\) and hence the proof is completed.

**Remark 1.** If \(n = 2k\), the points corresponding to the complex numbers \(\frac{m_j + m_{j+k}}{2}\) are on the axis of symmetry parallel to \(A_jA_{j+1}\). So the relation
\[
\frac{1}{n} \cdot \sum_{j=1}^{n} m_j = \frac{1}{k} \cdot \sum_{j=1}^{k} \frac{m_j + m_{j+k}}{2} = \frac{m}{2}
\]
means that the centroid of the \(k\)-gon determined by the projections of \(M\) to the these axis of symmetry is the midpoint of \(OM\). In this case the property can be viewed as theorem 2 with a degenerated regular \(k\)-gon (which degenerates into the center of symmetry).

The previous remark suggests that a similar property holds if the points \(M_1, M_2, \ldots, M_n\) are the projections of \(M\) to the axis of symmetry. To explore this possibility and to formulate a proper conjecture we can construct figures using a geometry software. Finally we obtain the following property:

**Theorem 3.** The centroid of the \(n\)-gon determined by the orthogonal projections of a point \(M\) to the axis of symmetry of a regular \(n\)-gon is the midpoint of \(OM\) (\(O\) is the center of symmetry).

![Figure 3: The centroid of a podar polygon relative to the symmetry axes of a regular n-gon](image)

Figure 3: The centroid of a podar polygon relative to the symmetry axes of a regular n-gon
Proof. The method and the calculations are almost the same. We can assume that one of the symmetry axes is the $OY$ axes, so the real part of the projections affix is 0 while the imaginary part is the same as the imaginary part of the projected point’s affix. This implies that if we repeat all the calculations instead of $\cos \left( 2k - \frac{1}{2} \right) \pi$ or $\cos \frac{2k \pi}{2k+1}$ we have 0.

\textbf{Remark 2.} All groups discovered/rediscovered independently theorem 2 while theorem 3 was observed only by one group when they analyzed their proof.

In order to generalize the previous results to higher dimensions as a first step we analyze the case of a regular tetrahedron.

\textbf{Theorem 4.} Let $A_1A_2A_3A_4$ be a regular tetrahedron with center $O,$ $M_i$ (1 ≤ $i$ ≤ 4) the orthogonal projections of a point $M$ to its faces and $Q_i$ (1 ≤ $i$ ≤ 6) the projections to its sides. The following two statements are true:

a) The centroid $G_2$ of the tetrahedron $M_1M_2M_3M_4$ is on $OM$ and satisfies $\frac{OG_2}{OM} = \frac{2}{3}$.

b) The centroid $G_1$ of the system $Q_1Q_2Q_3Q_4Q_5Q_6$ is on $OM$ and satisfies $\frac{OG_1}{OM} = \frac{1}{3}$.

![Figure 4: The centroid of a podar tetrahedron relative to the faces and the sides of a regular tetrahedron](image)

Proof. First we prove that the two statements are equivalent. Due to theorem 1, we have (see figure 4):

$$\frac{\overrightarrow{OQ_1} + \overrightarrow{Q_2}}{2} = \frac{\overrightarrow{OQ_4} + \overrightarrow{OQ_5} + \overrightarrow{OQ_6}}{3},$$

$$\frac{\overrightarrow{OQ_2} + \overrightarrow{Q_3}}{2} = \frac{\overrightarrow{OQ_4} + \overrightarrow{OQ_1} + \overrightarrow{OQ_2}}{3},$$

$$\frac{\overrightarrow{OQ_3} + \overrightarrow{Q_4}}{2} = \frac{\overrightarrow{OQ_3} + \overrightarrow{OQ_5} + \overrightarrow{OQ_2}}{3},$$

and

$$\frac{\overrightarrow{OQ_4} + \overrightarrow{Q_5}}{2} = \frac{\overrightarrow{OQ_1} + \overrightarrow{OQ_3} + \overrightarrow{OQ_6}}{3},$$

(where $Q_1 \in A_1A_3, Q_2 \in A_1A_4, Q_3 \in A_1A_2, Q_4 \in A_1A_3, Q_5 \in A_2A_4, Q_6 \in A_2A_3, \overrightarrow{AB}$ means the vector from $A$ to $B,$ and $O_i$ are the centroids of the faces). From these equalities we deduce

$$\frac{1}{2} \sum_{i=1}^{4} OM_i = \frac{2}{3} \sum_{i=1}^{6} OQ_i.$$
(because \( \frac{1}{2} \cdot \sum_{i=1}^{4} \overrightarrow{O O_i} = 0 \) ) and so
\[
\frac{1}{2} \cdot \overrightarrow{O G_2} = \overrightarrow{O G_1}.
\]
This relation implies the equivalence of the two statements.

In order to prove the first statement we consider a regular tetrahedron with vertices \( A_1 \left( 0, 0, \frac{\sqrt{2}}{3} \right), \)
\( A_2 \left( -\frac{1}{6}, -\frac{\sqrt{3}}{6}, 0 \right), A_3 \left( \frac{1}{6}, -\frac{\sqrt{3}}{6}, 0 \right), A_4 \left( 0, \frac{\sqrt{3}}{3}, 0 \right) \) and we calculate the coordinates of the points \( M_i \). The equations of the faces are:

\[
\begin{align*}
A_1 A_2 A_3 & : \quad -2\sqrt{3} y + \sqrt{3} z - \sqrt{2} = 0, \\
A_1 A_2 A_4 & : \quad 3\sqrt{2} x - \sqrt{6} y - \sqrt{3} z + \sqrt{2} = 0, \\
A_1 A_3 A_4 & : \quad 3\sqrt{2} x + \sqrt{6} y + \sqrt{3} z - \sqrt{2} = 0 \quad \text{and} \\
A_3 A_2 A_4 & : \quad z = 0
\end{align*}
\]

Using the formula
\[
x = x_0 - A \cdot \frac{A \cdot x_0 + B \cdot y_0 + C \cdot z_0 + D}{A^2 + B^2 + C^2},
\]
which gives the \( x \) coordinate of the orthogonal projection of the point \((x_0, y_0, z_0)\) to the plane with equation \( A \cdot x + B \cdot y + C \cdot z + D = 0 \) we obtain:

\[
\begin{align*}
x_1 &= x_4 = x_0, \\
x_2 &= x_0 - 3\sqrt{2} \cdot \frac{3\sqrt{2} \cdot x_0 + \sqrt{6} \cdot y_0 + \sqrt{3} \cdot z_0 - \sqrt{2}}{27} \quad \text{and} \\
x_3 &= x_0 - 3\sqrt{2} \cdot \frac{3\sqrt{2} \cdot x_0 - \sqrt{6} \cdot y_0 - \sqrt{3} \cdot z_0 + \sqrt{2}}{27}
\end{align*}
\]

From these equalities we can easily deduce the relation
\[
\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{2x_0}{3}.
\]

A similar calculation shows the same relation for the \( y \) and the \( z \) coordinates, hence \( G_2 \) lies on \( OP \) and satisfies the desired equality. This completes the proof. \( \square \)

Based on this property we can formulate the following conjecture for the \( n \)-dimensional euclidian space:

**Conjecture 1.** If we denote by \( G_k \) the centroid of the system determined by the orthogonal projections of a point \( M \) to the \( k \)-dimensional faces of a regular \( n \)-simplex \( A_1 A_2 A_3 \ldots A_n A_{n+1} \), then \( G_k \in OM \) and satisfies \( \frac{OG_k}{OM} = \frac{k}{n} \), \( 1 \leq k \leq n-1 \), where \( O \) is the center of the simplex.

**Remark 3.** Theorem 4 was discovered/rediscovered by all the groups while conjecture 1 was formulated (without proof) only by one group. After an analysis of the proof of theorem 4 all groups formulated the necessity of a different approach in attacking the higher dimensional problem.

This conjecture can be proved using the same ideas as in the proof of theorem 4, but the calculations are more complicated. In order to simplify the proof of this conjecture first we try to give an affine version of the 2 dimensional property and then by extending this affine version to higher dimensions we find a property which is more general than conjecture 1 and admits a simpler proof. For this we need to observe that in a regular polygon (polyhedra or simplex) the segment joining the centroid \( O \) of the polygon with the midpoint \( O_i \) of a side (the centroid of a face) is perpendicular to this side (face). Hence the construction of the orthogonal projection of \( M \) to a side \( d \) can be viewed as the construction of a projection in the direction \( OO_i \). This can be done for an arbitrary triangle or even for an arbitrary polygon, so we can formulate the following conjectures:
Figure 5: The centroid of a generalized podar triangle relative to an arbitrary triangle

**Conjecture 2.** In the triangle $A_1A_2A_3$ $O_1 \in A_2A_3$, $O_2 \in A_3A_1$ and $O_3 \in A_1A_2$ are the midpoints of the sides and $A_1O_1 \cap A_2O_2 \cap A_3O_3 = \{O\}$. If for an arbitrary point $M$ we consider the points $M_1 \in A_2A_3$, $M_2 \in A_3A_1$ and $M_3 \in A_1A_2$ such that $MM_1||OO_1$, $MM_2||OO_2$ and $MM_3||OO_3$, then the centroid of the triangle $M_1M_2M_3$ is the midpoint of the segment $OM$.

**Conjecture 3.** Denote by $O_1, O_2, \ldots, O_n$ the midpoints of the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ in the polygon $A_1A_2\ldots A_n$ and by $O$ the centroid of the polygon. If for an arbitrary point $M$ we consider the points $M_i \in A_iA_{i+1}$, $1 \leq i \leq n$ such that $MM_i||OO_i$, $1 \leq i \leq n$, then the centroid of the polygon $M_1M_2\ldots M_n$ is the midpoint of the segment $OM$.

Using a dynamic geometry software we can explore the validity of these conjectures. Conjecture 2 seems to be true but unfortunately Conjecture 3 is not true for all polygons.

**Remark 4.** It is an open question to assure conditions for which conjecture 3 becomes a theorem. One of the groups observed that conjecture 3 becomes a theorem for $n = 4$ if $A_1A_2A_3A_4$ is a trapezoid.

**Proof of conjecture 2.** If we denote by the corresponding small letter the position vector of each point, we have the following relations: $a = \frac{1}{3}(a_1 + a_2 + a_3)$, $o_1 = \frac{1}{2}(a_2 + a_3)$, $o_2 = \frac{1}{2}(a_3 + a_1)$ and $o_3 = \frac{1}{2}(a_1 + a_2)$. $M$ is an arbitrary point in the plane so there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$m = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Since $M_1 \in A_2A_3$ there exists $\lambda_1 \in \mathbb{R}$ with the property $m_1 = \lambda_1 a_2 + (1 - \lambda_1) a_3$, hence the condition $MM_1||OO_1$ can be expressed as

$$m - m_1 = c(o - o_1)$$

with $c \in \mathbb{R}$. This implies

$$\alpha_1 a_1 + (\alpha_2 - \lambda_1) a_2 + (\alpha_3 - 1 + \lambda_1) a_3 = c \left( \frac{1}{3} a_1 - \frac{1}{6} a_2 - \frac{1}{6} a_3 \right).$$

If the origin of the position vectors is not in the same plane as the vertices of the initial triangle, than $a_1, a_2$ and $a_3$ are linearly independent vectors, so we obtain the following system

$$\begin{align*}
\alpha_1 &= \frac{1}{3} c \\
\alpha_2 - \lambda_1 &= \frac{-c}{6} \\
\alpha_3 - 1 + \lambda_1 &= \frac{-c}{6}
\end{align*}$$

and so

$$m_1 = m - 3\alpha_1(o - o_1) = \left( \alpha_2 + \frac{1}{2} \alpha_1 \right) a_2 + \left( \alpha_3 + \frac{1}{2} \alpha_1 \right) a_3.$$
Using a similar argument we obtain

\[ m_2 = \left( \alpha_3 + \frac{1}{2} \alpha_2 \right) a_3 + \left( \alpha_1 + \frac{1}{2} \alpha_2 \right) a_1 \] and \[ m_3 = \left( \alpha_1 + \frac{1}{2} \alpha_3 \right) a_1 + \left( \alpha_2 + \frac{1}{2} \alpha_3 \right) a_2 \]

so

\[ g = \frac{1}{3} (m_1 + m_2 + m_3) = \frac{1}{2} (m + o). \]

Analyzing theorem 4 and conjecture 2 we can formulate the following property:

**Theorem 5.** (Zsolt Szilágyi, Szilárd András) Consider the \( n \)-simplex \( A_0 \ldots A_n \), an arbitrary point \( M \) and \( O \) the centroid of the simplex. Denote by \( O_{i_0 \ldots i_k} \) the centroid of the face \( A_{i_0} \ldots A_{i_k} \) and by \( M_{i_0 \ldots i_k} \) the intersection of the face \( A_{i_0} \ldots A_{i_k} \) with the line drawn through \( M \) and parallel to \( O_{i_0 \ldots i_k} A_{i_k+1} \ldots A_{i_n} \). If \( G'_k \) denotes the centroid of the system \( M_{i_0 \ldots i_k} \), where \( k \) is fixed and \( i_0 \ldots i_k \) takes all possible values, then for each \( k \in \{1, 2, 3 \ldots, n-1\} \) the points \( M, O, G'_k \) are on the same line and

\[ \frac{OG'_k}{OM} = \frac{k}{n}. \]

**Proof.** Denote by the corresponding small letters the position vectors (in \( \mathbb{R}^n \)) of the points. \( M \) is an arbitrary point in \( \mathbb{R}^n \), so there exist \( \alpha_0, \ldots, \alpha_n \in \mathbb{R} \) such that \( \sum_{i=0}^{n} \alpha_i = 1 \) and

\[ m = \sum_{i=0}^{n} \alpha_i a_i. \]

If \( \overline{m} := m_{i_0 \ldots i_k} = \sum_{j=0}^{k} c_j a_{i_j} \) (with \( \sum_{j=0}^{k} c_j = 1 \)) is the intersection of the face \( A_{i_0} \ldots A_{i_k} \) with the parallel line to \( O_{i_0 \ldots i_k} A_{i_k+1} \ldots A_{i_n} \), and \( a_{i_0 \ldots i_k} = \frac{1}{k+1} \sum_{j=0}^{k} a_{i_j} \) is the centroid of the face \( A_{i_0} \ldots A_{i_k} \), then the condition

\[ g = \frac{1}{3} (m_1 + m_2 + m_3) = \frac{1}{2} (m + o). \]
The number of combinations $i$ for which $l \in \{i_0, \ldots, i_k\}$, where $l$ is a fixed index, is $C_n^k$, while the number of combinations $i$ for which $l \in \{i_0, \ldots, i_k\}$ and $j \in \{i_{k+1}, \ldots, i_n\}$ is $C_n^{k-1}$. This implies

$$g_k = \sum_{l=0}^{n} \left[ C_n^k \alpha_l + \sum_{j \neq l} \frac{C_n^{k-1}}{k+1} \alpha_j \right] a_l$$

The centroid is $o = \frac{1}{n+1} \sum_{i=0}^{n} a_i$, so

$$\overrightarrow{OG}_k = \sum_{l=0}^{n} \left( -\frac{1}{n+1} + \frac{k}{n} \alpha_l + \frac{n-k}{n(n+1)} \right) a_l$$

$$= \sum_{l=0}^{n} \left( -\frac{k}{n(n+1)} + \frac{k}{n} \right) a_l. \quad (1)$$

But $\overrightarrow{OM} = \sum_{l=0}^{n} \left( \alpha_l - \frac{1}{n+1} \right) a_l$, hence $\overrightarrow{OG}_k = \frac{1}{n} \overrightarrow{OM}$, which completes the proof. \(\square\)
Remark 5. Theorem 5 proves Conjecture 1, because in a regular simplex $O_{i_0...i_k}A_{i_0+1}...A_{i_k}$ is orthogonal to the face $A_{i_0}...A_{i_k}$. At the training course this theorem and its proof was presented by the instructors.

Examining theorem 2 the following natural question arise: can we replace the midpoints of the faces by other points? The following theorem answers this question.

**Theorem 6.** Consider a triangle $A_1A_2A_3$ and the real numbers $w_1, w_2, w_3$ with $w_1 + w_2 + w_3 = 1$. Let $O$ be the point with barycentric coordinates $(w_1, w_2, w_3)$ and $M$ an arbitrary point in the plane of the triangle. If we construct the points $M_1 \in A_2A_3$, $M_2 \in A_3A_1$ and $M_3 \in A_1A_2$ such that $MM_1 || OA_1$, $MM_2 || OA_3$ and $MM_3 || OA_3$, then the centroid of the triangle $M_1M_2M_3$ coincides with the centroid of the triangle $MOP$, where the barycentric coordinates of $P$ relative to $M_1M_2M_3$ are $(w_1, w_2, w_3)$.

**Remark 6.** If $w_1 = w_2 = w_3$, then $P$ is the centroid of $M_1M_2M_3$, so the above theorem reduces to conjecture 2.

![Figure 7: The centroid of a generalized podar triangle relative to an arbitrary triangle](image)

**Proof.** If $\gamma_1, \gamma_2, \gamma_3$ denote the barycentric coordinates of $M$, then

$$m = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

and

$$o = w_1 a_1 + w_2 a_2 + w_3 a_3,$$

where $a_1, a_2, a_3$ are the position vectors of the vertices and $m, o$ the corresponding position vectors of the points $M$ and $O$. $M_1 \in A_2A_3$, so there exists $\lambda_1 \in \mathbb{R}$ such that $m_1 = \lambda_1 a_2 + (1 - \lambda_1) a_3$. The condition $MM_1 || OA_1$ can be expressed as $m_1 - m = c(o - a_1)$, with some $c \in \mathbb{R}$. This implies

$$\lambda_1 a_2 + (1 - \lambda_1) a_3 - \gamma_1 a_1 - \gamma_2 a_2 - \gamma_3 a_3 = c(w_1 a_1 + w_2 a_2 + w_3 a_3 - a_1)$$

(2)

If we suppose that the starting point of the position vectors is outside the plane $A_1A_2A_3$, then $a_1, a_2, a_3$ are linearly independent, hence (2) implies $-\gamma_1 = c(w_1 - 1)$, $\lambda_1 - \gamma_2 = cw_2$ and $1 - \lambda_1 - \gamma_3 = cw_3$. From these relations we obtain

$$m_1 = \left(\gamma_2 + \gamma_1 \frac{w_2}{w_2 + w_3}\right) a_2 + \left(\gamma_3 + \gamma_1 \frac{w_3}{w_2 + w_3}\right) a_3.$$  \hspace{1cm} (3)

By a similar reasoning we deduce

$$m_2 = \left(\gamma_3 + \gamma_2 \frac{w_3}{w_1 + w_3}\right) a_3 + \left(\gamma_1 + \gamma_2 \frac{w_1}{w_1 + w_3}\right) a_1.$$  \hspace{1cm} (4)

$$m_3 = \left(\gamma_1 + \gamma_3 \frac{w_1}{w_2 + w_1}\right) a_1 + \left(\gamma_2 + \gamma_3 \frac{w_2}{w_2 + w_1}\right) a_2.$$  \hspace{1cm} (5)
From (3), (4) and (5) we deduce

\[ m_1(1 - w_1) + m_2(1 - w_2) + m_3(1 - w_3) = m + a, \]

thus

\[ \frac{1}{3}(m_1 + m_2 + m_3) = \frac{1}{3}(m + o + p), \]

which is the desired property.

Remark 7. Theorem 6 can be extended also to higher dimensional simplices.

Remark 8. It would be interesting to find an affine version which generalizes all the previous properties (including theorem 2 too).

Concluding remarks

- From scientific point of view the training course was fruitful because the participants discovered new properties of the studied configurations and they developed new generalizations, which clarify some aspects of the problem.
- Remark 4 and remark 8 shows that interesting questions arose, they can constitute a starting point for further study.
- From a teaching point of view the main aim of the training course was achieved, the participants got familiar with the inquiry based methods, they realized that a deeper inquiry of the problems can constitute a solid motivational base for further individual study. They also perceived that in many cases the understanding of the background implicitly requires a deeper inquiry and this can be done even in the traditional framework and context.
- Some participants pointed out that the major inference of this training course was that they realized the importance of simultaneous alternatives in solving a problem and the importance of selecting the tools they use in designing the proofs of well known properties.
- The participants emphasized that a major task in preparing and designing an inquiry based lesson is the selection of a sufficiently rich problem.

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