RADII OF REGULAR POLYTOPES

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Abstract. For the first time complete lists of two pairs of inner and outer radii classes of the three types of regular polytopes which exist in all dimensions are presented. A new access using isotropic polytopes provides easier understanding of the underlying geometry and helps unifying the results.

1. Introduction

There are three types of regular polytopes which exist in every dimension d: regular simplices, (hyper-) cubes, and regular cross-polytopes. In this paper we investigate two pairs of inner and outer j-radii (r_j, R_j) and $(\bar{r}_j, \bar{R}_j)$ of these polytopes (inner and outer radii classes are almost always considered in pairs, such that for a 0-symmetric body K and its dual K° the inner (outer) radii of K are the reciprocal values of the outer (inner) radii of K° [9]).

The inner j-radii r_j and $\bar{r}_j$ of a body K are defined as the radii of largest j-balls contained in j-dimensional slices $K \cap F$ of K, whereby the value of r_j is obtained from maximizing and the value of $\bar{r}_j$ is obtained from minimizing over the possible directions of F. The outer j-radii R_j and $\bar{R}_j$ of a body K are defined as the radii of smallest j-balls containing the projection of K onto j-dimensional subspaces F, whereby the value of R_j is obtained from minimizing and the value of $\bar{R}_j$ is obtained from maximizing over the possible directions of F. One should note that $r_d = \bar{r}_d$ is the usual inradius and $R_d = \bar{R}_d$ is the usual outer radius. Moreover, it is well known [3] that $R_1 = \bar{r}_1$ is the half width and $r_1 = \bar{R}_1$ is the half diameter. In some Russian papers the inner radii r_j are also called Bernstein diameters and the outer radii R_j Kolmogorov diameters (or sometimes Kolmogorov width).

The inner radii r_j of regular simplices were studied in [1]. Ball uses a well known result of John [10] in his proof, which also plays an important role in our computation of the outer radii R_j of regular simplices.

Until recently we thought that besides the classical results of Steinhagen [15] and Jung [11] about the outer 1- and the outer d-radii, respectively, the R_j's of regular simplices were computed only in the case that $j = d - 1$ by Weißbach [16, 17]. The pretended open cases originally stimulated our work. However, on

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the one hand it turned out that already Pukhov [13] computed the R_j's of regular simplices in the remaining cases. On the other hand, in [4] it was shown that the proof of Weißbach for the $(d-1)$-case with even d contained a crucial error.

The $\bar{R}_j$'s and the $\bar{r}_j$'s of regular simplices were considered in [9] and [2], respectively. While the $\bar{R}_j$'s were completely listed, the $\bar{r}_j$'s could only be computed in several special cases. However, a lower bound was given and a criterion when this bound is attained. We show that this criterion is fulfilled in all remaining cases, which means that we can now complete this list.

As the last piece to complete the radii of regular simplices, the result about $R_{d-1}(T^d)$ for even d could recently be reestablished in [5].

If we turn to the other two types of regular polytopes, it follows immediately from their (central) symmetry that $r_d = R_1$ and $r_1 = R_d$ and therefore that the $\bar{r}_j$'s and the $\bar{R}_j$'s do not depend on j. Hence we concentrate our attention on (r_j, R_j) in case of symmetric bodies.

Pukhov gives references for papers in which the R_j's of regular cross-polytopes are computed, from which it is possible to deduce the r_j's of cubes via polarization [8]. Everett, Stojmenovíc, Valtr, and Whitesides [6] give a recursive formula for the inner radii of general d-dimensional boxes, which generalizes the cited result about cubes. However, Everett et al. obviously did not know the papers cited by Pukhov, since they thought that even the inner radii of cubes were not known previously, except from trivial cases and the outer 3-radius of a cube, computed by Shklarsky, Chentzov, and Yaglom [14].

We do not know a reference on the outer radii of boxes (and/or the inner radii of general cross-polytopes). This gap is closed by showing that these radii are the circumradii of smallest j-faces of boxes. Table 1 summarizes the results about regular polytopes.

However, instead of just putting together the radii from all the authors cited above, this paper unifies the different papers. We show that all the (r_j, R_j) radii of regular polytopes, apart from the r_j's of regular simplices, can be obtained from the R_j's of regular simplices. Moreover, the $\bar{r}_j$'s of regular simplices can be obtained from the R_j's of regular simplices, in almost all cases.

Pukhov used the result about the R_j's of regular cross-polytopes in his computation of the R_j's of regular simplices, which would lead to an improper circular closing of our chain of proof. This is one reason, to resettle a complete new proof. Another is the strong connection to isotropic polytopes (Kawashima called them π-polytopes [12], but we prefer to call them isotropic as they are in an isotropic position in the sense of [7]). Specifically, we show that the existence of a $(j, d+1)$-isotropic polytope is equivalent to the existence of a j-dimensional projection of the regular d-simplex such that a previously computed general lower bound for the outer radii is attained. Afterwards we state a way of constructing $(j, d+1)$-isotropic polytopes for arbitrary pairs (j, d), except for the cases where d is even

	regular simplex	cube	regular cross-polytope
R_j	$\sqrt{\frac{j}{d}}$, $j \notin \{1, d-1\}$ or d odd $\frac{d+1}{d}\sqrt{\frac{1}{d+2}}$, $j=1$ and d even $\frac{2d-1}{2d}$, $j=d-1$ and d even	$\sqrt{\frac{j}{d}}$	$\sqrt{\frac{j}{d}}$
r_j	$\sqrt{\frac{d+1}{j(j+1)d}}$	$\sqrt{\frac{1}{j(d+1)}}$	$\sqrt{\frac{1}{j(d+1)}}$
$\bar{R}_j$	$\sqrt{\frac{j(d+1)}{(j+1)d}}$	1	1
$\bar{r}_j$	$\sqrt{\frac{1}{j(d+1)}}$, $j \notin \{1, d-1\}$ or d odd $\frac{d+1}{d}\sqrt{\frac{1}{d+2}}$, $j=1$ and d even $\frac{2}{\sqrt{d(d+2)}+\sqrt{d(d-2)}}$, $j=d-1$ and d even	$\sqrt{\frac{1}{d(d+1)}}$	$\sqrt{\frac{1}{d(d+1)}}$

TABLE 1. For the first time, a complete table of the radii of the three types of regular polytopes can be given. The polytopes are scaled such that their circumradius is 1.

and $j \in \{1, d-1\}$, showing that the lower bound is tight in all but the two exceptional cases.

We will then show that the lower bound criterion of [2] is fulfilled in almost all cases (all open cases), such that the $\bar{r}_j$'s of regular simplices can be completed.

Finally, it is shown how to deduce the radii of cubes and regular cross-polytopes from the results about the R_j's of regular simplices and formulas for the radii of general boxes and cross-polytopes, as mentioned above, are stated.

2. Preliminaries

Let $\mathbb{E}^d = (\mathbb{R}^d, \|\cdot\|)$ denote the d-dimensional *Euclidean space*, $d \geq 2$, $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$ the *unit ball* and the *unit sphere* in $\mathbb{E}^d$, and $\langle \cdot\,,\cdot \rangle$ the usual *scalar product* $\langle x, y \rangle = x^T y$. Furthermore, we use $\{e_1, \ldots, e_d\}$ for the *standard basis* of $\mathbb{E}^d$. A set $K \subset \mathbb{E}^d$ is called a *body* if it is bounded, closed, convex and contains an inner point. For every body $K \subset \mathbb{E}^d$ let $K^\circ = \{y \in \mathbb{E}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$ denote the *polar* of K.

By $\mathcal{L}_{j,d}$ and $\mathcal{A}_{j,d}$ we denote the set of all j-dimensional *linear subspaces* and all j-dimensional *affine subspaces* of $\mathbb{E}^d$, respectively. For any $E \in \mathcal{L}_{j,d}$ let $E^\perp \in \mathcal{L}_{d-j,d}$ be the *orthogonal complement* of E. Let $\mathrm{lin}\{s_1, \ldots, s_j\}$ denote the *linear span* $\{x \in \mathbb{E}^d : x = \sum_{k=1}^{j} \lambda_k s_k,\ \lambda_k \in \mathbb{R}\}$ of $s_1, \ldots, s_j \in \mathbb{S}^{d-1}$. For any set

$A \in \mathbb{E}^d$, $A|E$ denotes the *(orthogonal) projection* of A onto $E \in \mathcal{L}_{j,d}$. For any $x \in \mathbb{E}^{d_1}$ and $y \in \mathbb{E}^{d_2}$ let $x \otimes y$ denote the rank 1 matrix with elements $x_i y_j$, $i = 1, \dots, d_1$ and note that for any set of orthonormal vectors $\{s_1, \dots, s_j\}$ the projection P of $\mathbb{E}^d$ onto $\mathrm{lin}\{s_1, \dots, s_j\}$ can be represented by the matrix $\sum_{l=1}^{j} s_l \otimes s_l$. For any two sets $A, B \subset \mathbb{E}^d$ the *Minkowski sum* $A + B$ is defined as $A + B = \{a + b \in \mathbb{E}^d : a \in A, b \in B\}$.

For any convex set K let $r(K)$ and $R(K)$ denote the inner and outer radius of K, respectively. Now, for any $j \in \{1, \dots, d\}$ the *inner j-radii* of K are defined by

$$r_j(K) = \max_{E \in \mathcal{L}_{j,d}} \max_{q \in \mathbb{E}^d} r(K \cap (E + q)),$$

$$\bar{r}_j(K) = \min_{E \in \mathcal{L}_{j,d}} \max_{q \in \mathbb{E}^d} r(K \cap (E + q)),$$

and the *outer j-radii* by

$$R_j(K) = \min_{E \in \mathcal{L}_{j,d}} R(K|E),$$

$$\bar{R}_j(K) = \max_{E \in \mathcal{L}_{j,d}} R(K|E).$$

The outer radii often are also introduced in terms of enclosing cylinders. That means, defining a *j-cylinder* as the set $F + q + \rho(\mathbb{B} \cap F^\perp)$, for $F \in \mathcal{A}_{d-j,d}$, $q \in \mathbb{E}^d$ and radius $\rho > 0$, then, e.g., $R_j(K)$ is the minimal radius of a K enclosing j-cylinder.

Surely, for all $1 \leq j \leq d$ the inner and outer j-radii are invariant under translation and rotation. Furthermore, if the convex body is scaled by a factor ρ, so are its radii. For this reason, we use the term 'ball' to signify any body similar (in the above sense) to $\mathbb{B}^d$, and the same we do for simplices, cross-polytopes and boxes.

Let T^d denote the *regular d-simplex* of circumradius $R(T^d) = 1$, which we assume to be embedded in $\mathbb{E}^{d+1}$ as $T^d = \sqrt{\frac{d+1}{d}}\,\mathrm{conv}\{e_1, \dots, e_{d+1}\}$. By $B_{a_1,\dots,a_d}$ we denote a d-dimensional *box* of the form $\left\{x \in \mathbb{E}^d : -a_i \leq x_i \leq a_i, i \in \{1, \dots, d\}\right\}$ and the *cube* $\sqrt{\frac{1}{d}} B_{1,\dots,1}$ is denoted by C^d. Finally, a general *cross-polytope* $X_{a_1,\dots,a_d} = \mathrm{conv}\{\pm a_1 e_1, \dots, \pm a_d e_d\}$ is just the polar of $B_{\frac{1}{a_1},\dots,\frac{1}{a_d}}$ and especially the *regular cross-polytope* $X^d = \mathrm{conv}\{\pm e_1, \dots, \pm e_d\}$ is the polar of $\sqrt{d}\,C^d$.

3. Regular simplices

The following results about the r_j's and the $\bar{R}_j$'s of regular simplices are taken from [1] and [9].

Proposition 3.1. *For all* $1 \leq j \leq d$

(i) $r_j(T^d) = \sqrt{\frac{d+1}{j(j+1)d}}$,

(ii) $\bar{R}_j(T^d) = \sqrt{\frac{j(d+1)}{(j+1)d}}$, *and*

in both cases the extreme j-spaces are parallel to the j-faces of T^d.

Definition 3.2. *We call any set of orthonormal vectors $\{s_1, \ldots, s_j\} \subset \mathbb{E}^{d+1}$, $1 \leq j \leq d$*

(i) a valid subset basis *(*vsb *for short) if $\sum_{k=1}^{d+1} s_{lk} = 0$ for all $l \in \{1, \ldots, j\}$, and*

(ii) a good subset basis *(*gsb *for short) if it is a vsb and $\sum_{l=1}^{j} s_{lk}^2 = \frac{j}{d+1}$ for all $k \in \{1, \ldots, d+1\}$.*

Note that any set of orthonormal vectors $\{s_1, \ldots, s_j\}$ is called a vsb if it spans a j-dimensional subspace of $\mathbb{E}_0^d = \left\{x \in \mathbb{E}^{d+1} : \sum_{k=1}^{d+1} x_k = 0\right\}$, the d-dimensional linear subspace of $\mathbb{E}^{d+1}$ parallel to the hyperplane in which we have embedded T^d.

The projection of T^d onto $\mathbb{E}_0^d$ can be written as $I^{d+1} - \frac{1}{d+1}\mathbf{1}^{d+1}$, where I^{d+1} denotes the identity matrix in $\mathbb{E}^{(d+1)\times(d+1)}$ and $\mathbf{1}^{d+1}$ the matrix in $\mathbb{E}^{(d+1)\times(d+1)}$ consisting only of 1's. Hence $\sum_{l=1}^{d} s_l \otimes s_l = I^{d+1} - \frac{1}{d+1}\mathbf{1}^{d+1}$, for every vsb of d elements. This implies the important fact that each vsb is a gsb if $j = d$, which we use in Corollary 3.4.

Now we start computing the outer radii of regular simplices by giving a general lower bound, which we prove to be tight in almost all cases further on. This theorem will also show the reason why we call a vsb good if it fulfills the condition (ii) in Definition 3.2.

Lemma 3.3. *$R_j(T^d) \geq \sqrt{\frac{j}{d}}$ for all $j \in \{1, \ldots, d\}$ and equality holds if and only if there exists a gsb $\{s_1, \ldots, s_j\}$ in $\mathbb{E}^{d+1}$.*

Proof. Let P denote the projection onto a subspace spanned by a vsb $\{s_1, \ldots, s_j\}$. It follows

$$\|Pe_k\|^2 = \langle Pe_k, e_k \rangle = \left\langle \sum_{l=1}^{j} s_{lk} s_l, e_k \right\rangle = \sum_{l=1}^{j} s_{lk}^2.$$

Now assume there exists any $x \in \mathbb{E}^{d+1}$ such that $\|x - Pe_k\|^2 < \frac{j}{d+1}$ for all $k = 1, \ldots, d+1$. Summing over the k's it follows

$$\begin{aligned}
j &> \sum_{k=1}^{d+1} \|x - Pe_k\|^2 \\
&= \sum_{k=1}^{d+1} (\|x\|^2 - 2\langle x, Pe_k \rangle + \|Pe_k\|^2) \\
&= (d+1)\|x\|^2 - 2\left\langle x, \sum_{k=1}^{d+1}\sum_{l=1}^{j} s_{lk} s_l \right\rangle + \sum_{k=1}^{d+1}\sum_{l=1}^{j} s_{lk}^2
\end{aligned}$$

and since $\sum_{k=1}^{d+1} s_{lk} = 0$ and $\sum_{k=1}^{d+1} s_{lk}^2 = 1$

$$= (d+1)\|x\|^2 + j$$
$$\geq j$$

which is a contradiction. This proves the first part of the lemma. In order to prove the other part we note that equality in $\|x - Pe_k\|^2 \leq \frac{j}{d+1}$ for all k can only be obtained if $x = 0$ and $\sum_{l=1}^{j} s_{lk}^2 = \frac{j}{d+1}$ for each k. □

As every vsb of d vectors is already a gsb we obtain the following corollary from Lemma 3.3 by the basis extension property (used on $\mathbb{E}_0^d$):

Corollary 3.4. *For any dimension d and any $j \in \{1, \dots, d-1\}$ it holds $R_j(T^d) = \sqrt{\frac{j}{d}}$ if and only if $R_{d-j}(T^d) = \sqrt{\frac{d-j}{d}}$ holds. Moreover, there always exists a pair of corresponding optimal projections which take place in orthogonal subspaces.*

Since Steinhagen [15] showed that

$$R_1(T^d) = \begin{cases} \sqrt{\frac{1}{d}}, & \text{if } d \text{ odd} \\ \frac{d+1}{d}\sqrt{\frac{1}{d+2}}, & \text{if } d \text{ even.} \end{cases} \tag{1}$$

Corollary 3.4 implies for the outer $(d-1)$-radius that the lower bound of Lemma 3.3 is attained for odd dimensions, but that the bound is not attained for even dimensions. The following formula was claimed in [13] and first shown in [17]. The proof in that paper contained a crucial error, but the correctness of the formula could be reestablished lately [5].

Proposition 3.5. *For even d*

$$R_{d-1}(T^d) = \frac{2d-1}{2d}.$$

We will soon see that $j \in \{1, d-1\}$ for even d are the only cases where the lower bound is not attained.

The following proposition is a polar version of John's Theorem [10] (see also [1]).

Proposition 3.6. *$\mathbb{B}^j$ is the ellipsoid of minimal volume containing some body $K \subset \mathbb{E}^j$ if and only if $K \subset \mathbb{B}^j$ and for some $m \geq j$ there are unit vectors $u_1, \dots, u_m$ on the boundary of K, and positive numbers $c_1, \dots, c_m$ summing to j such that*

(i) $\sum_{i=1}^{m} c_i u_i = 0$, *and*

(ii) $\sum_{i=1}^{m} c_i u_i \otimes u_i = I^j$.

It is obvious that if K is a regular polytope all c_i can be chosen as $\frac{j}{m}$ were m is the number of vertices of K. But it is not obvious which other polytopes fulfill this property. Nevertheless, according to [7] these polytopes are in an isotropic position, corresponding to the discrete measure μ^* on $\mathbb{S}^{d-1}$ that gives mass $\frac{j}{m}$ to all vertices u_i (see [7, Section 5] for more details). This is the source for the following definition:

Definition 3.7. *Let $u_1, \dots, u_{d+1} \in \mathbb{S}^{j-1}$ (not necessarily different) and $K = \mathrm{conv}\{u_1, \dots, u_{d+1}\}$. K is called* $(j, d+1)$-isotropic, *if all the c_i's in Proposition 3.6 can be taken as $\frac{j}{d+1}$.*

Lemma 3.8. *There exists a gsb $s_1, \dots, s_j$ of $\mathbb{E}^{d+1}$ if and only if there exists a $(j, d+1)$-isotropic polytope $K = \mathrm{conv}\{u_1, \dots, u_{d+1}\} \subset \mathbb{E}^j$, $j \leq d$. Moreover, if we project T^d onto $\mathrm{lin}\{s_1, \dots, s_j\}$ the projection equals the corresponding K up to rotation and dilatation.*

Proof. If $K = \mathrm{conv}\{u_1, \dots, u_{d+1}\}$ is a $(j, d+1)$-isotropic polytope then

(i) $\|u_k\| = 1$,
(ii) $\sum_{k=1}^{d+1} u_k = 0$, and
(iii) $\sum_{k=1}^{d+1} u_k \otimes u_k = \frac{d+1}{j} I^j$.

Now let $s_l = \sqrt{j/(d+1)}(u_{1,l}, \dots, u_{d+1,l})^T$, $l = 1, \dots, j$. This defines a gsb. For showing this it is necessary that the s_l form an orthonormal set, but this is the case because of (iii). $\sum_{k=1}^{d+1} s_{lk}$ has to be 0, but this follows from (ii), and finally we need $\sum_{l=1}^{j} s_{lk}^2 = \frac{j}{d+1}$ for all k, but this is true because of (i). The other direction can be shown using a similar reasoning.

Now, if we project the vertices of T^d onto $\mathrm{lin}\{s_1, \dots, s_j\}$ we get

$$P\left(\sqrt{\frac{d+1}{d}} e_k\right) = \sum_{l=1}^{j} \sqrt{\frac{d+1}{d}} s_{lk} s_l = \sum_{l=1}^{j} \sqrt{\frac{j}{d}} u_{kl} s_l.$$

Hence the values $\sqrt{j/d}\, u_{kl}$ are just the coordinates of the vertices of the projection in terms of the basis $s_1, \dots, s_j$. □

Lemma 3.8 can be used in two ways:

(i) We know that $R_j(T^d) = \sqrt{j/d}$ whenever we find a $(j, d+1)$-isotropic polytope and vice versa. Hence there cannot exist $(1, d+1)$-isotropic polytopes nor $(d-1, d+1)$-isotropic polytopes if d is even).
(ii) We know that $R_k(K) \geq \sqrt{k/j}$ for any $(j, d+1)$-isotropic polytope K and any $k \leq j$ and equality holds if and only if the corresponding gsb $\{s_1, \dots, s_j\}$ can be split into two gsb's $\{s_1, \dots, s_k\}$ and $\{s_{k+1}, \dots, s_j\}$.

We will first concentrate our attention to (i) but come back to (ii) later.

The following lemma states a rule, how to construct higher dimensional isotropic polytopes from lower dimensional ones. We call it the *additive rule*.

Lemma 3.9. *Let $0 \le j_i < m_i$, $i = 1, 2$ such that $m_2 j_1 > m_1 j_2$. Let $j = j_1 + j_2$, $m = m_1 + m_2$, $\alpha = \sqrt{(m_2 j_1 - m_1 j_2)/m_2 j}$, and $\beta = \sqrt{m j_2 / m_2 j}$, and suppose there exists a (j_1, m_1)-isotropic polytope $K_1 = \operatorname{conv}\{u_1, \dots, u_{m_1}\}$, a (j_1, m_2)-isotropic polytope $K_2 = \operatorname{conv}\{v_1, \dots, v_{m_2}\}$, and a (j_2, m_2)-isotropic polytope $K_3 = \operatorname{conv}\{w_1, \dots, w_{m_2}\}$, such that $K' = \operatorname{conv}\left\{\sqrt{\tfrac{1}{2}}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \dots, \sqrt{\tfrac{1}{2}}\begin{pmatrix} v_{m_2} \\ w_{m_2} \end{pmatrix}\right\}$ is a (j, m_2)-isotropic polytope. Then there exists a (j, m)-isotropic polytope*

$$K = \operatorname{conv}\left\{\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{m_1} \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta w_1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha v_{m_2} \\ \beta w_{m_2} \end{pmatrix}\right\}$$

Proof. Since $\alpha^2 + \beta^2 = 1$ all vertices of K are situated on $\mathbb{S}^{j-1}$ and obviously 0 is the centroid of K. Hence we only have to show that condition (ii) from Proposition 3.6 holds with $c_i = j/m$, $i = 1, \dots, m$.

$$\begin{aligned}
&\sum_{i=1}^{m_1} \begin{pmatrix} u_i \\ 0 \end{pmatrix}\begin{pmatrix} u_i \\ 0 \end{pmatrix}^T + \sum_{i=1}^{m_2} \begin{pmatrix} \alpha v_i \\ \beta w_i \end{pmatrix}\begin{pmatrix} \alpha v_i \\ \beta w_i \end{pmatrix}^T \\
&= \begin{pmatrix} \frac{m_1}{j_1} I_{j_1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{m_2}{j_1}\alpha^2 I_{j_1} & 0 \\ 0 & \frac{m_2}{j_2}\beta^2 I_{j_2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{m_1 j + m_2 j_1 - m_1 j_2}{j_1 j_2} I_{j_1} & 0 \\ 0 & \frac{m}{j} I_{j_2} \end{pmatrix} \\
&= \frac{m}{j} I_j \,.
\end{aligned}$$

□

The reader may convince himself that neither it is possible to construct a $(1, d+1)$-isotropic polytope by the additive rule if d is even, nor it is possible to construct a $(d-1, d+1)$-isotropic polytope by the additive rule at all.

If m_2 is even, a good choice for K' is often a prism

$$\operatorname{conv}\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} v_{\frac{m_2}{2}} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ -1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} v_{\frac{m_2}{2}} \\ -1 \end{pmatrix}\right\}$$

or anti prism

$$\operatorname{conv}\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} v_{\frac{m_2}{2}} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} -v_1 \\ -1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} -v_{\frac{m_2}{2}} \\ -1 \end{pmatrix}\right\}$$

built from an $(j-1, m_2/2)$-isotropic base $K_2 = \operatorname{conv}\{v_1, \dots, v_{m_2/2}\}$.

Lemma 3.10. *For every pair (j, d), with $1 \le j \le d$ and such that if d is even $j \notin \{1, d-1\}$ there exists a $(j, d+1)$-isotropic polytope.*

Proof. We do an inductive proof over j and d. From Equation 1 and since every regular $(d+1)$-gon with vertices on $\mathbb{S}^1$ is $(2, d+1)$-isotropic we see that the claim is true for pairs (j,d) with $j \leq 2$. Moreover, the claim is true for $j \geq d-2$ because of Corollary 3.4.

Now, assume that the claim is true for every pair (j',d') with $j' < j$, $d' \leq d$ or $j' \leq j$, $d' < d$. Regarding to the initial statements we can assume $j \geq 3$ and because of Corollary 3.4 that $j < (d+1)/2$. We start with the case $(j,d+1) = (3,9)$. In this case we choose $j_1 = 2, j_2 = 1, m_1 = 3, m_2 = 6$. For sure $K_1 = K_2 = T^2$ are $(2,3)$-isotropic and also $(2,6)$-isotropic by duplicating every vertex. Now, $K_3 = T^1 = [-1,1]$ is $(1,6)$-isotropic (triplicating the two vertices) and

$$K' = \operatorname{conv}\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} v_2 \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} v_3 \\ 1 \end{pmatrix},\right.$$
$$\left.\sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} v_2 \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} v_3 \\ -1 \end{pmatrix}\right\}$$

is $(3,6)$-isotropic. Hence K_1, K_2 and K_3 fulfill the conditions of the additive rule and therefore there exist a $(3,9)$-isotropic polytope.

Next we assume that $j \geq 5$ is odd and that (as in the case before) $m = d+1 = 2j+3$. Then we choose $j_1 = j-2, j_2 = 2, m_1 = m-j-1, m_2 = j+1$. Since $j < m/2$ it holds $j_1 < m_1$ and since $j_1 = j-2 \neq j = m-j-3 = m_1-2$ there exists a (j_1,m_1)-isotropic polytope K_1. Completing the conditions of the additive rule we choose an m_2-gon for K_3 and the projection of T^j onto $(\operatorname{lin} K_3)^\perp$ as K_2 (thus $K' = T^j$). One should notice that $m_2 j_1 = j^2 + j > 2m = m_1 j_2$ since $j \geq 5$

Finally, let j be even or $m \neq 2j+3$. Then we set $j_1 = j, j_2 = 0, m_1 = j+1$ and $m_2 = m-j-1$. Since $j < m/2$ it holds $m_2 > j$ and if $j+2$ is odd $m_2 \neq j+2$ since $m \neq 2j+3$. Hence there exists a (j,m_2)-isotropic polytope K_2 by the induction hypothesis and $K_1 = T^{j+1}$ is a (j,m_1)-isotropic polytope, which obviously fulfills the conditions of the additive rule. □

The following Proposition is taken from [2]. It gives a lower bound for $\bar{r}_j(T^d)$ and a criterion when this lower bound is attained. For the purpose of the Proposition let $a_1,\ldots,a_{d+1} \in \mathbb{S}^{d-1}$ such that $T^d = \sqrt{\frac{1}{d(d+1)}}\{x \in \mathbb{E}^d : \langle x, a_i\rangle \leq 1, i = 1,\ldots,d+1\}$.

Proposition 3.11. *$\bar{r}_j(T^d) \geq \sqrt{\frac{1}{j(d+1)}}$ for all $1 \leq j \leq d$ and equality holds if and only if there exists an $E \in \mathcal{L}_{j,d}$ such that $\|a_i|E| = \sqrt{\frac{j}{d}}$ for all $i = 1,\ldots,d+1$.*

It follows from the self-duality of the regular-simplex

$$(T^d)^\circ = \sqrt{d(d+1)}T^d$$

that the criterion for equality in Proposition 3.11 is fulfilled if and only if $R_j(T^d) = \sqrt{\frac{j}{d}}$.

Theorem 3.12. *For every $1 \le j \le d$, such that d is odd or $j \notin \{1, d-1\}$*

(i) $R_j(T^d) = \sqrt{\frac{j}{d}}$, *and*

(ii) $\bar{r}_j(T^d) = \sqrt{\frac{1}{j(d+1)}}$.

Proof. Part (i) follows directly from Lemmas 3.3, 3.8, and 3.10. Part (ii) follows from part (i) and Proposition 3.11. ☐

One should mention that for every pair (j, d) where Theorem 3.12 holds, we have $R_j(T^d)\bar{r}_j((T^d)^\circ) = 1$. Hence, it follows from the self-duality of T^d (if 0 is the centroid and up to dilatation) that the result above combined with the results in [2] show that the minimal j-balls of T^d in the sense of $\bar{r}_j$ are at the same time the maximal j-balls with center 0 contained in T^d.

For completeness we should state the two remaining inner radii of regular simplices [2, Theorem 3].

Proposition 3.13. *For even d*

$$\bar{r}_1(T^d) = R_1(T^d) = \frac{d+1}{d}\sqrt{\frac{1}{d+2}}, \text{ and}$$

$$\bar{r}_{d-1}(T^d) = \frac{2}{\sqrt{d(d+2)} + \sqrt{d(d-2)}}.$$

4. Boxes and cross-polytopes

As already mentioned in the introduction, for all symmetric bodies K it holds $\bar{r}_1(K) = \cdots = \bar{r}_d(K)$ and $\bar{R}_1(K) = \bar{R}_d(K)$. Hence, we can draw our attention in this section to (r_j, R_j).

A proof of the following proposition can be found in [8]:

Proposition 4.1. *If K is a 0-symmetric body and $1 \le j \le d$ then $r_j(K)R_j(K^\circ) = 1$ and $R_j(K)r_j(K^\circ) = 1$.*

Now, we come back to the second statement after Lemma 3.8, saying that the k-radius of any (j, m)-isotropic polytope K is $\sqrt{k/j}$ if the gsb $\{s_1, \dots, s_j\}$ corresponding to K can be split into a gsb $\{s_1, \dots, s_k\}$ and a gsb $\{s_{k+1}, \dots, s_j\}$ – in other words, if K can be split into a (k, m)-isotropic polytope K_1 and a $(j-k, m)$-isotropic polytope K_2. From applying this on cubes and regular cross-polytopes we obtain the following corollary.

Corollary 4.2. *For all $1 \le j \le d$*

(i) $R_j(C^d) = R_j(X^d) = \sqrt{\frac{j}{d}}$, *and*

(ii) $r_j(C^d) = r_j(X^d) = \sqrt{\frac{1}{j(d+1)}}$, *and*

Proof. It suffices to show that for every $1 \leq j \leq d$ both C^d and X^d have (j,m)-isotropic projections (up to dilatation), since then (i) follows from the argument before the corollary and (ii) from Proposition 4.1.

For the cube every j-tuple of coordinate rows of its vertices describes an isotropic j-face, which is a cube and therefore isotropic.

Now we turn to X^d. First, project T^{2d-1} onto $\sqrt{\frac{d}{2d-1}}X^d$ by using the gsb

$$\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_{d-1} \\ -s_{d-1} \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix},$$

where $s_1, \dots, s_{d-1}$ is an arbitrary gsb for T^{d-1}. It follows from Lemma 3.10 that for every $j \notin \{1, d-2\}$ there exists a subset of size j of $s_1, \dots, s_{d-1}$ that is again a gsb, without loss of generality $s_1, \dots, s_j$, or if $j = d-2$, $s_1, \dots, s_{j-1}$. Hence the set

$$\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_j \\ -s_j \end{pmatrix}$$

or if $j \in \{1, d-2\}$ the set

$$\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_{j-1} \\ -s_{j-1} \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix}$$

is a gsb in $\mathbb{E}^{2d}$ projecting T^{2d-1} onto a $(j, 2d)$-isotropic polytope K. Since this gsb is a subset of the one projecting T^{2d-1} onto $\sqrt{\frac{d}{2d-1}}X^d$, the $(j, 2d)$-isotropic polytope K is a projection of X^d. $\square$

Corollary 4.2 can be generalized to obtain the inner and outer radii of general cross-polytopes and boxes.

The inner radii of boxes were computed in [6]. The part about outer radii of cross-polytopes follows from Proposition 4.1.

Proposition 4.3. *Let* $0 < a_1 \leq \dots \leq a_d$. *Then*

(i)

$$r_j(B_{a_1,\dots,a_d}) = \sqrt{\frac{a_1^2 + \dots + a_{d-k}^2}{j-k}},$$

where k is the smallest of the integers $0, \dots, j-1$ that satisfies

$$a_{d-k} \leq \sqrt{\frac{a_1^2 + \dots + a_{d-k-1}^2}{j-k-1}},$$

and

(ii)

$$R_j(X_{a_1,\dots,a_d}) = \sqrt{\frac{(j-k)\prod_{i=k}^{d} a_i^2}{\sum_{i=k}^{d}\prod_{l \neq i} a_l^2}},$$

where k is the smallest of the integers $0,\dots,j-1$ that satisfies

$$a_k \geq \sqrt{\frac{(j-k-1)\prod_{i=k+1}^{d} a_i^2}{\sum_{i=k+1}^{d}\prod_{l\neq i} a_l^2}}.$$

The corresponding result about the outer radii of boxes seems to be very intuitive. It says that one should just project the box through one of its smallest faces.

Theorem 4.4. *Let $0 < a_1 \leq \cdots \leq a_d$. Then*

(i) $R_j(B_{a_1,\dots,a_d}) = \sqrt{a_1^2+\cdots+a_j^2}$, *and*

(ii) $r_j(X_{a_1,\dots,a_d}) = \frac{\prod_{i=d-j+1}^{d} a_i}{\sqrt{\sum_{i=d-j+1}^{d}\prod_{l\neq i} a_l^2}}$.

Proof. It suffices to show part (i), since then part (ii) follows from Proposition 4.1. Moreover, as the result is obvious if $d=1$ we can assume that $d\geq 2$.

Any vertex v of $B_{a_1,\dots,a_d}$ can be written in the form $v=\sum_{k=1}^{d}\pm a_k e_k$ and all possible choices of the plus and minuses in that formula lead to a vertex of $B_{a_1,\dots,a_d}$. Hence, for every projection $P=\sum_{l=1}^{j} s_l\otimes s_l$ with pairwise orthogonal unit-vectors $s_l\in\mathbb{E}^d$, it holds that $\|Pv\|^2=\sum_{l=1}^{j}\langle v,s_l\rangle^2=\sum_{l=1}^{j}(\sum_{k=1}^{d}\pm a_k s_{lk})^2$. However, since the average value of $\|Pv\|^2$ over all vertices v is $\sum_{k=1}^{d} a_k^2\sum_{l=1}^{j} s_{lk}^2$, there exists a vertex of $B_{a_1,\dots,a_d}$ such that $\|Pv\|^2\geq\sum_{k=1}^{d} a_k^2\sum_{l=1}^{j} s_{lk}^2$.

Now extend the set $\{s_1,\dots,s_j\}$ to an orthonormal basis of $\mathbb{E}^d$. Since $\sum_{l=1}^{d} s_l\otimes s_l=I$ it follows that $\sum_{k=1}^{d} s_{lk}^2=\sum_{l=1}^{d} s_{lk}^2=1$, for all $k=1,\dots,d$ and all $l=1,\dots,d$, respectively. Hence $t_k:=\sum_{l=1}^{j} s_{lk}^2\in[0,1]$ and since $\sum_{k=1}^{d} t_k=\sum_{l=1}^{j}\sum_{k=1}^{d} s_{lk}^2$ has to equal j, the minimum value of $\sum_{k=1}^{d} t_k a_k^2$ is obtained for $t_1=\cdots=t_j=1$ and $t_{j+1}=\cdots=t_d=0$. Hence $R_j(B_{a_1,\dots,a_d})\geq\sqrt{a_1^2+\cdots+a_j^2}$. The projection of $B_{a_1,\dots,a_d}$ through its j-face $B_{a_1,\dots,a_j}$ achieves this value and so we get the desired result. □

Compared to the radii of boxes and general cross-polytopes very less can be stated about general simplices. Since Gritzmann and Klee [8] showed that the computation of $R_j(S)$ is $\mathbb{NP}$-hard for general simplices and many j a general formula is not expectable. However, in [5] it could be shown that in 'typical' configurations all vertices of the simplex are projected onto the minimal enclosing sphere in an optimal projection, and in [4] solution methods and a formula for a special case are given for $j=2$ and $d=3$.

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