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Abstract

This paper deals with the problem of packing convex polytopes into a parallelepiped of minimal height. It is assumed that the polytopes are oriented, i. e. rotations are not permitted. A mathematical model of the problem is developed and peculiarities of them are addressed. Based on these peculiarities a method to compute local optimal solutions is constructed. Both an approximate and an exact method to search for local minima of the problem are discussed. The exact method is a special modification of the duel simplex method. Some examples are also given.

Key words: mathematical modelling, packing, polytope, optimization

1 Introduction

Let convex 3D polytopes P_i, $i = 1, 2, ..., n$, and a 3D parallelepiped P be given. The parallelepiped P is assumed to be

$$P = \{(x, y, z,) \in I\!R^3 : \ 0 \le x \le l, \ 0 \le y \le w, \ 0 \le z \le h\}$$

where the length l and the width w of P are considered to be constants, and the height h is variable. Polytopes P_i, $i = 1, 2, ..., n$, are given by their vertices $v_{ik} = (x_{ik}, y_{ik}, z_{ik})$, $k = 1, \dots, n_i$, with respect to their eigen coordinate systems, $i = 1, 2, ..., n$. As usual, it is assumed that for any polytope the origin of the eigen coordinate system belong to its interior (cf. [7, 12, 13, 21]). The origin of the eigen coordinate system of polytope P_i is also called the *pole* of P_i, $i = 1, 2, ..., n$.

Within this paper, the polytopes are only allowed to be translated, i.e. rotation is not permitted. As usual, the translation of a polytope

$$P_i = \{v \in I\!R^3 : v = \sum_{k=1}^{n_i} v_{ik}\lambda_k, \ \sum_{k=1}^{n_i} \lambda_k = 1, \ \lambda_k \ge 0, \ k = 1, \dots, n_i\}$$

by a vector $u_i = (x_i, y_i, z_i)$ is denoted as

$$P_i(u_i) = \{v \in I\!R^3 : v = u_i + \overline{v}, \ \overline{v} \in P_i\},$$

$i = 1, ..., n$. The vector u_i is called *placement parameter vector* of polytope P_i.

The optimization problem under consideration is the following: Find placement parameter vectors u_1, u_2, ..., u_n such that all (translated) polytopes $P_i(u_i)$, $i = 1, 2, ..., n$, fit completely within parallelepiped P, that they do not overlap each other, and that height h of the occupied part of P attains minimal value.

Placement (optimization) problems concerning the packing of 3D geometric objects which can be formulated similar to above, arise in various branches of industry. As examples, the following applications are pointed out: powder metallurgy (laser 3D cutting), problems connected with arranging and loading containers for aviation, cosmic shipment, sea shipment, railroad transportation, cutting different natural and artificial crystals, layout of computers, buildings, ships, plants, etc.

Investigations concerning 3D problems are devoted basically to packing various parallelepipeds into containers (see, for example, [1, 2, 3, 6, 9, 10, 11, 5, 27, **?**]). Significantly fewer research is concerned with problems of placing polytopes into a given 3D domain.

We point out some of them.

In [14, 15, 16] a genetic algorithm based approach is presented which uses three kinds of information: the direction of object movement, the orientation of each object and tangent points of each object. Objects can be rotated by an angle divisible by 45° with respect to each coordinate axes. The objective in the investigations is either the number of objects placed into a container, or the sum of distances between centers of parallelepipeds which circumscribes round objects and a center axis of the container.

Various methods to solve 3D problems are considered in [4, 17, 25] which are based on a simulated annealing approach. The methods are applied to pack parallelepipeds and cylinders. In addition objects can be rotated by an angle divisible by 90°. Later on in [4] the same approach was applied for packing of arbitrary shaped objects.

In [4, 18] the objects are placed sequentially one by one without restrictions on rotation of objects. Utilizing a simulated annealing algorithm objects are assigned sequentially so that the objective is minimized in each moment.

In the papers [23, 24] the class of engineering systems of block design is defined, an informal statement of the arrangement problem is given and a mathematical model of the problem is offered taking into account the peculiarities arising at early stages of design. The analysis and choice of methods for solving layout design problems are carried out and a CAD-packing programming system is described which makes possible the automation of process the synthesis of layout solutions for engineering systems with modular structure at early stages of design.

At present all investigations devoted to 3D optimization problems do not construct mathematical models of the problems and offer only heuristic solution methods to solve them. The investigation presented here suggests a construction of a mathematical model for the problem stated and analyses peculiarities of the model. Based on these peculiarities both a starting point is constructed and a solution strategy is offered that realizes the search for a local minimum of the objective.

2 Modelling

In order to construct a mathematical model of the problem stated above it is necessary to formulate conditions in analytical form which ensure the non-intersection of the polytopes and their placement within the parallelepiped. Such analytical description can be realized using the concept of Φ-functions [22]. Applied to the current packing problem, any everywhere defined and continuous function $\Phi_{ij}(u_i, u_j) : I\!\!R^6 \to I\!\!R^1$ which satisfies the characteristic properties

$$\Phi_{ij}(u_i, u_j) = \begin{cases} > 0 & \text{if } \mathrm{cl}P_i(u_i) \cap \mathrm{cl}P_j(u_j) = \emptyset, \\ = 0 & \text{if } \mathrm{int}P_i(u_i) \cap \mathrm{int}P_j(u_j) = \emptyset, \text{ and } \mathrm{fr}P_i(u_i) \cap \mathrm{fr}P_j(u_j) \neq \emptyset, \\ < 0 & \text{if } \mathrm{int}P_i(u_i) \cap \mathrm{int}P_j(u_j) \neq \emptyset \end{cases}$$

(cl, int, fr denote closure, interior and frontier, respectively) is said to be a Φ-*function* of the pair of polytopes P_i and P_j. Hence, translated polytopes $P_i(u_i)$ and $P_j(u_j)$ do not overlap if and only if $\Phi_{ij}(u_i, u_j) \geq 0$.

Moreover, the containment conditions (P_i has to fit within P) can also be modelled by means of Φ-functions when the non-intersection of P_i and the domain

$$P_0 = \mathrm{cl}(I\!\!R^3 \setminus P)$$

is considered. Since the object P_0 is assumed to be immovable, the condition of non-intersection of P_i and P_0 can be described by an appropriate inequality

$$\Phi_i(u_i, h) \geq 0. \tag{1}$$

Let $u = (u_1, \ldots, u_n) \in I\!\!R^{3n}$. Then the objective function f ($f : I\!\!R^{3n} \to I\!\!R$) has the form

$$f(u) = h = \max\{z_1 + h_1^+, z_2 + h_2^+, ..., z_n + h_n^+\} \tag{2}$$

where $h_i^+ = \max\ \{z_{i1}, z_{i2}, ..., z_{in_i}\}$, $i = 1, \ldots, n$. Since height h of P (and of P_0) will be considered as variable another objective function F ($F : I\!\!R^{3n+1} \to I\!\!R$)

$$F(u, h) = h \tag{3}$$

is used together with corresponding restrictions of the form

$$h - z_i - h_i^+ \geq 0,\ \ i = 1, 2, ..., n. \tag{4}$$

Thus, a mathematical model of the placement problem stated can be formulated as follows:

$$\min F(\overline{u}) \qquad \text{subject to } \overline{u} \in \Omega \tag{5}$$

where $\overline{u} = (u, h)$,

$$\Omega = \Big\{(u, h) \in I\!\!R^{3n+1} : \Phi_{ij}(u_i, u_j) \geq 0,\ 1 \leq i < j \leq n,\ \Phi_i(u_i, h) \geq 0,\ i = 1, ..., n\Big\}. \tag{6}$$

and

$$\Phi_i(u_i,h) := \min\{f_{ik}(u_i,h) : k = 1,\dots,6\} \tag{7}$$

with

$$\begin{array}{lll} f_{i1}(u_i,h) = y_i - y_i^-, & f_{i3}(u_i,h) = x_i - x_i^-, & f_{i5}(u_i,h) = z_i - h_i^-, \\ f_{i2}(u_i,h) = w - y_i - y_i^+, & f_{i4}(u_i,h) = l - x_i - x_i^+, & f_{i6}(u_i,h) = h - z_i - h_i^+, \end{array} \tag{8}$$

$$x_i^+ = \max_{k=1,\dots,n_i} x_{ik},\ x_i^- = \min_{k=1,\dots,n_i} x_{ik},\ y_i^+ = \max_{k=1,\dots,n_i} y_{ik},\ y_i^- = \min_{k=1,\dots,n_i} y_{ik},\ h_i^- = \min_{k=1,\dots,n_i} z_{ik},$$
$$i = 1,2,...,n.$$

As it is known from [26], if P_i and P_j are convex polytopes then there exists a corresponding Φ-function of the form

$$\Phi_{ij}(u_i,u_j) = \max\{\varphi_{ij}^k(u_i-u_j) : k \in K_{ij} = \{1,\dots,k_{ij}\}\} \tag{9}$$

where k_{ij} denotes the number of linear functions $\varphi_{ij}^k : I\!R^3 \to I\!R$ needed in this representation (number of facets of the Minkovski sum $T_1 + (-T_2)$). Note, since any of the half spaces

$$\varphi_{ij}^k(u) = \varphi_{ij}^k(x,y,z) = a_{ij}^k x + b_{ij}^k y + c_{ij}^k z + d_{ij}^k \geq 0 \tag{10}$$

is determined by some combination of faces of P_i and P_j, such a representation of Φ_{ij} can be computed easily (cf. [26]). Furthermore, int$P_i \cap P_j(u) \neq \emptyset$ if and only if $\varphi_{ij}^k(u) < 0$ for all $k \in K_{ij}$.

Because of (9), for any pair $(u_i,u_j) \in I\!R^6$ with $\Phi_{ij}(u_i,u_j) \geq 0$ there exists at least one function φ_{ij}^κ with $\kappa \in K_{ij}$ and $\varphi_{ij}^\kappa(u_i-u_j) \geq 0$.

For the sake of simplicity in the description, we denote the Ω defining inequalities (determined by formulas (7) and (9)) consecutive by

$$\Psi_t(\overline{u}) \geq 0, \quad t \in T = \{1,...,\tau\}$$

where $\overline{u} = (u.h) \in I\!R^{3n+1}$ and τ is the total number of inequalities occurring.

Notice, whereas every of the $6n$ inequalities corresponding to (7) has to be fulfilled, for any pair $i \neq j$ at least one of the k_{ij} inequalities must be met.

3 Structure analysis

In the following some peculiarities of model (5) will be analyzed.

1. The feasible region Ω can, for some feasible $\overline{u}$, be described by a linear inequality system consisting of $C_n^2 = n(n-1)/2$ linear restrictions, one of kind $\varphi_{ij}^\kappa(u_i-u_j) \geq 0$ for every pair (i,j) of polytopes, and n linear inequality systems (determined by $\Phi_i(\overline{u}) \geq 0$) to ensure the containment within P. Thus, Ω is some polyhedral set in the space $I\!R^{3n+1}$.

2. The total number of inequalities $\Psi_t(\overline{u}) \geq 0$ needed for the description of feasible region Ω is greater than or equal to $4C_n^2$ $+6n$ $= 2n(n+2)$. This is easy to verify since each

collection of inequalities, needed for Φ_{ij} has a cardinality not smaller than 4, i. e. index set K_{ij} has at least four elements for any pair of convex polytopes.

3. Feasible region Ω is in general a disconnected polyhedral set, i. e. there is always a representation in form of

$$\Omega = \cup_{j=1}^{\nu} \Omega_j, \quad \Omega_j \cap \Omega_k = \emptyset \text{ for } j \neq k$$

where the sets Ω_j represent connected components. Disconnectedness occur if two polytopes are too large to fit side by side within the parallelepiped.

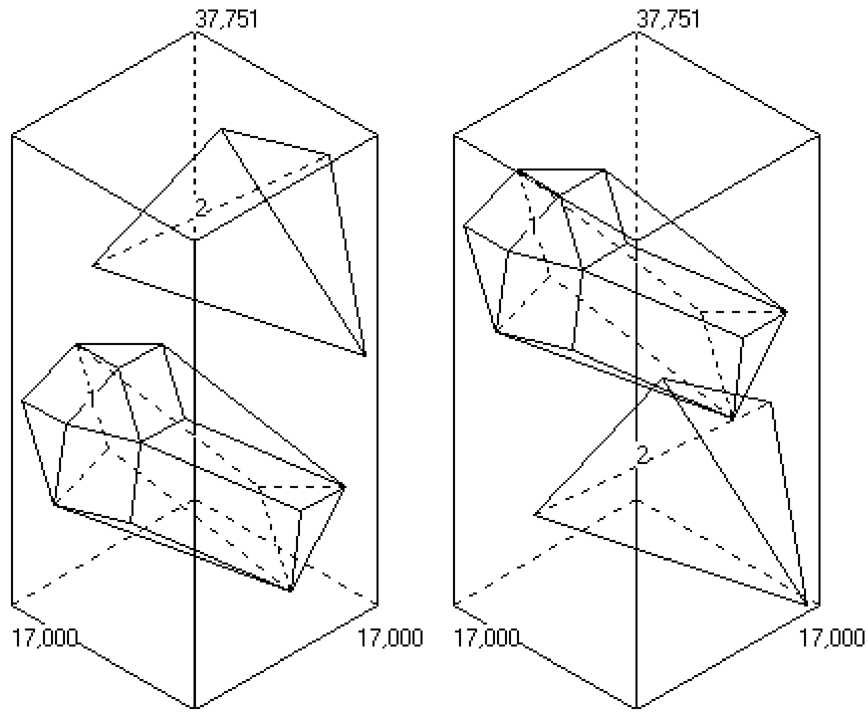

Figure 1: Example for non-connected region of placement points

In detail, let P_i and P_j be such that their sizes along the x- and y-axis are simultaneously greater than $l/2$ and $w/2$, respectively. Let polytopes P_i and P_j be packed as shown in Figs. 1a and 1b, and let these packings correspond to the points $\overline{u}_a, \overline{u}_b \in \Omega$. It is obvious, if the polygons are large enough (or if they are parallelepipeds) then there is no possibility to pass from the packing depicted in Fig. 1a to the packing depicted in Fig. 1b without either going out of the parallelepiped P or without intersecting the polytope P_i by the polytope P_j. This means that points $\overline{u}_a$ and $\overline{u}_b$ belong to different connected components of feasible region Ω. Thus, the number of connected components is at least as large as the number of pairs of polytopes which are sufficiently too large (e. g. whose sizes are greater than $l/2$ and $w/2$ so that they cannot be shifted side by side).

4. Each component of connectedness of Ω is multiply connected.

In order to motivate this statement we consider for the sake of simplicity 2D objects. Let P_1 and P_2 be squares of sizes $a \times a$ and $b \times b$, respectively. Then any of their Φ-functions

Φ_{12} has 0-level surface

$$\tau_{12} = \{(u_1, u_2) \in I\!R^4 : \Phi_{12}(u_1, u_2) = 0\}$$

which is for every fixed u_1 (or u_2) congruent to a square of size $(a+b) \times (a+b)$. Thus, τ_{12} is for $u_1 = (0,0)$ a closed curve which divides $I\!R^2$ into two disjunctive parts Γ and $I\!R^2 \setminus \Gamma$ with $\Gamma = \{u \in I\!R^2 : \Phi_{12}(0,u) < 0\}$. It is evident, the set $I\!R^2 \setminus \Gamma$ is two-connected (cf. [8, 19]). Since situations like set $I\!R^2 \setminus \Gamma$ appear when the feasible region Ω is formed, consequently Ω is multiple-connected. Since the number of Φ-functions forming Ω is C_n^2 therefore connected components of Ω can be C_n^2-connected.

5. The problem is instable in the general case, i. e. an arbitrary small variation in the input data of polytopes P_i, $i = 1, ..., n$, and/or parallelepiped P can lead to a significant variation of the optimal objective function value and of optimal solutions.

For example, let two parallelepipeds P_1 and P_2 be given such that they can be placed exactly within the parallelepiped P as shown in Fig. 2a. If the width is enlarged of at least one of the small parallelepipeds the packing shown in Fig.2b results.

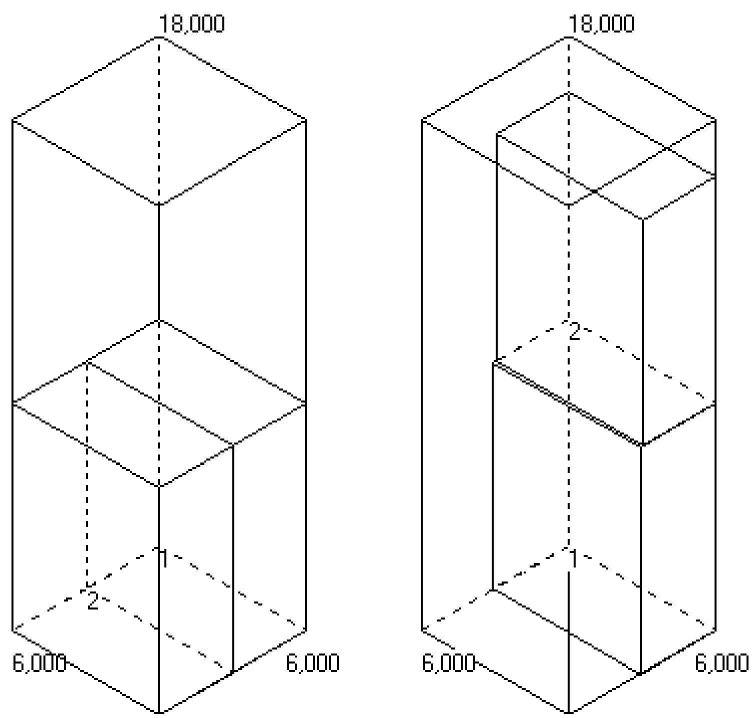

Figure 2: Example for non-stable behavior

6. Because of the definition of Ω (formula (6)) by a finite number of linear restrictions, it follows that Ω can be covered by a finite number of convex (not-necessary disjunctive) polyhedral sets Ω_p, $p = 1, ..., \eta$, i. e.

$$\Omega = \bigcup_{p=1}^{\eta} \Omega_p = \bigcup_{p=1}^{\eta} \left(\bigcap_{j \in T_p} H_{pj} \right) \tag{11}$$

where $H_{pj} = \{\overline{u} \in I\!R^{3n+1} : \Psi_j(\overline{u}) \geq 0\}$, $j \in T_p \subset T$, are some half-spaces.

7. The number η of such convex polyhedral sets depends on the number n of polytopes, the number of facets of the polytopes and on the way of partitioning of Ω. Obviously, $\eta \geq C_\tau^d$ where $\tau = n(n-1)/2$ and $d \geq 6$.

8. Since Ω is a union of convex polyhedral sets (formula (11)) systems of linear inequalities have to be considered. Let

$$\{X \in I\!R^{3n+1} : B_p X + D_p \geq 0\} := \{X \in I\!R^{3n+1} : \Psi_j(\overline{u}) \geq 0,\ j \in T_p\} \tag{12}$$

denote the p-th inequality system, $p = 1, \ldots, \eta$. Each system $B_p X + D_p \geq 0$ consists of $6n$ inequalities corresponding to (8), and C_n^2 inequalities corresponding to (9) and (10) (one inequality for each pair P_i and P_j).

Hence, the number $m_0 = 6n + n(n-1)/2$ of inequalities in each system $B_p X + D_p \geq 0$ is much larger than the number $m_1 = 3n + 1$ of variables. Moreover, matrices B_p, $p = 1, 2, ..., \eta$, are sparse or super-sparse and, in addition, they are close to block diagonal matrices.

9. Among the convex polyhedral sets Ω_p, $p = 1, ..., \eta$, in (11) there can exist pairs Ω_r and Ω_s such that a vertex of one polyhedral set lies on the frontier of the other polyhedral set (cf Fig. 3), i. e.

$$W_r \cap \mathrm{fr}\Omega_s \neq \emptyset \quad \text{or} \quad W_s \cap \mathrm{fr}\Omega_r \neq \emptyset$$

where W_r (W_s) is the set of vertices of Ω_r (Ω_s). It means, at such a vertex there exists more active inequalities as necessary (the basis representation is not unique). Such a covering of Ω is called *singular*. It is easy to see that the singularity of a covering of Ω depends on the choice of convex polyhedral sets Ω_p, $p = 1, ..., \eta$.

It should be noted, that such a singularity is often met in placement problems. To overcome it one of the standard methods can be applied (cf. [20]).

Due to these observations the following conclusions can be drawn.

1. Since the region Ω is disconnected (item 4) and each connected component is multiple-connected and non-convex, problem (5) is multiple extremal. in general.

2. Since objective function $F(X)$ is linear and since feasible region Ω is polyhedral, local minima are contained within the set of vertices of Ω.

3. If feasible region Ω is singular then cycling is possible.

4. Some subregions Ω_{p_j}, $j = 1, ..., \alpha$, may have at least one common vertex. If this vertex is a local minimum it is necessary to examine all these α subregions to prove that this vertex is a local minimum.

5. Because of the representation (11) of Ω by means of a finite number of convex polyhedral sets, an optimal solution X^* of problem (5) can be obtained by solving η linear programming problems, namely among optimal solutions X_p^* where

$$X_p^* = \arg\min\{F(X) : X \in \Omega_p\}, \quad p = 1, ..., \eta, \tag{13}$$

a best one can be selected. Ω_p is specified according to (12) by the inequality system $B_pX + D_p \geq 0$.

6. Each subproblem (13) is a typical linear programming problem. In the following we will consider the following basic problem

$$\min\{CX : X \in W = \{X \in I\!R^{m_1} : BX + D \geq 0\}\} \tag{14}$$

where the matrix B has m_0 rows (cf. item 8).

4 Solution strategy

Because of conclusion 5 in the previous section, a solution of problem (5) can be obtained by solving a sequence of at most η (sub-)problems of form (13). Therefore one can construct a search tree whose terminal nodes represent the η subproblems S_p, $p = 1, ..., \eta$. Then for each subproblem S_p for which fathoming criteria do not work, a problem of form (13) has to be solved. However, the fathoming criteria known so far do not permit to obtain a proved optimal solution X^* for problems of medium size ($n > 5$) in acceptable time.

Furthermore, the problem under consideration possesses such peculiarities that do not allow to utilize modern solution methods of operational research without special adaptation. Moreover, there are not known other exact solution methods for the general case which work if $n > 5$. For that reasons we propose a method for searching of local minima. This method consists of the following four general steps:

1. A vertex V of Ω is computed as some approximation of a local minimum.
2. For this vertex a corresponding convex polyhedral set Ω_p is constructed which contains V.
3. Then problem (13) is solved for set Ω_p.
4. If the point obtained is not a local minimum of problem (5) then a passage to another set is done which is connected with set Ω_p.

In what follows, these steps are described in more detail.

4.1 Construction of approximations

In order to describe the principal approach of constructing a first feasible placement (an initial point for the iteration), we identify polytope P_i with its index i. Then, a permutation $\pi = (j_1, j_2, , ..., _n)$ of $\{1, 2, ..., n\}$ corresponds to a sequence of polytopes

$$P_\pi = (P_{j_1}, P_{j_2}, ..., P_{j_n}).$$

In accordance with sequence π, first polytope P_{j_1} is placed into parallelepiped P so that its placement parameters $x_{j_1}, y_{j_1}, z_{j_1}$ attain minimal values, i.e. $(x_{j_1}, y_{j_1}, z_{j_1}) = (-x^-_{j_1}, -y^-_{j_1}, -z^-_{j_1})$. Now polytope P_{j_1} is considered to be immovable.

After that, polytope P_{j_2} is placed into P so that P_{j_2} is tangent to P_{j_1} and to two facets of P and z_{j_2} attains minimal value. Now polytopes P_{j_1} and P_{j_2} are considered to be fixed.

Then polytope P_{j_3} is placed into P so that P_{j_3} is tangent either to P_{j_1} and two facets of P, or to P_{j_2} and two facets of P, or to P_{j_1} and P_{j_2} and one facet of P, such that z_{j_3} gets minimal value. Now polytopes P_{j_1} , P_{j_2} and P_{j_3} are considered to be fixed.

For $k = 4, \ldots, n$ always polytopes P_{j_r}, $r = 1, \ldots, k-1$, are considered to be fixed. Then polytope P_{j_k} is placed into P so that P_{j_k} is tangent either to one already placed polytope P_{j_r} $(r \in \{1, \ldots, k-1\})$ and two facets of P, or is tangent to two already placed polytopes P_{j_r} and P_{j_s} $(r < s \in \{1, \ldots, k-1\})$ and one facet of P, or is tangent to three already placed polytopes P_{j_r}, P_{j_s} and P_{j_t} $(r < s < t \in \{1, \ldots, k-1\})$ such that z_{j_k} gets minimal value. The process is repeated until all polytopes are placed into P.

It is essential that with the placement of P_{j_k} three new restrictions become active.

Generally speaking, the proposed approach realizes a kind of *greedy* strategy – in any step a local optimal decision is made.

More formally, let $P_1, \ldots, P_n$ be a random sequence of the polytopes. The first $k-1$ polytopes are considered to be fixed and $u_i = u_i^0 = (x_i^0, y_i^0, z_i^0)$ denote the placement parameters of P_i, $i = 1, ..., k-1$. Furthermore, let Φ_k be a Φ-function of polytope P_k and set $P_0 = \mathrm{cl}(I\!R^3 \setminus P)$, then

$$\Gamma_k = \{u_k \in I\!R^3 : \Phi_k(u_k) \geq 0\}$$

describes the set of feasible placement positions of P_k within P. Using a Φ-function Φ_{ik} for the pair of polytopes P_i and P_k, $i = 1, 2, ..., k-1$, sets

$$\Gamma_{ik}(u_i^0) = \{u_k \in I\!R^3 : \Phi_{ik}(u_i^0, u_k) \leq 0\}, \quad i = 1, ..., k-1,$$

contain all placement positions of P_k with $P_i(u_i^0) \cap P_k(u_k) \neq \emptyset$. Obviously, a point u is a feasible placement point (of the pole O_k) of P_k if and only if

$$u \in G_k = \mathrm{cl}\left(\Gamma_k \setminus (\bigcup_{i=1}^{k-1} \Gamma_{ik}(u_i^0))\right).$$

In order to find feasible placement points of P_k only the following points (potential extreme points of G_k) are checked whether they yield a feasible placement point of P_k:

a) any vertex v of Γ_k with $v \in G_k$;

b) any intersection point v of an edge of Γ_k with a facet of $\Gamma_{ik}(u_i^0)$ $(i \in \{1, ..., k-1\})$ with $v \in G_k$;

c) any intersection point v of any combination of three facets of Γ_k, $\Gamma_{ik}(u_i^0)$ and $\Gamma_{jk}(u_j^0)$, $i < j \in \{1, ..., k-1\}$, with $v \in G_k$;

d) any intersection point v of any combination of three facets of $\Gamma_{ik}(u_i^0)$, $\Gamma_{jk}(u_j^0)$ and $\Gamma_{lk}(u_l^0)$, $i < j < l \in \{1, ..., k-1\}$, with $v \in G_k$;

Among the feasible placement points found, a point with minimal z-coordinate is chosen as u_k^0.

It is easy to verify that the point $X = (u_1^0, u_2^0, ..., u_n^0, h^0)$, where $h^0 = \max\{z_i^0 + z_i^+ : i = 1, \ldots, n\}$, is some vertex of Ω.

Notice, this approach requires a high computational amount if the polytopes possess a large number of facets and/or if the number of polytopes being placed is not small (i. e. $n \geq 20$). Therefore, the approach described above should only be applied when the number of already placed polytopes is small, e. g. $k \leq 7$. Otherwise some modification of the approach above is applied.

At first, for any polytope P_i a parallelepiped $\overline{P}_i$ is constructed which enclosures P_i and has minimal volume ($i = 1, 2, ..., n$). Clearly, it holds that

$$\overline{P}_i = \{(x, y, z) \in I\!R^3 : x_i^- \leq x \leq x_i^+,\ y_i^- \leq y \leq y_i^+,\ z_i^- \leq z \leq z_i^+\}.$$

Again we assume that polytopes $P_1, P_2, ..., P_{k-1}$ are already placed with their placement parameters $u_i^0 = (x_i^0, y_i^0, z_i^0)$, $i = 1, 2, ..., k-1$, respectively. Then the placement parameters $u_k = (x_k, y_k, z_k)$ of polytope P_k are calculated as follows:
First, all parallelepipeds $\overline{P}_i$ are placed into parallelepiped P with placement parameters u_i^0, $i = 1, ..., k-1$. Note, the parallelepipeds may intersect each other. Then, the parallelepiped $\overline{P}_k$ is placed into parallelepiped P so that $\overline{P}_k$ do not overlap with parallelepipeds $\overline{P}_i$, $i = 1, ..., k-1$, and the z-coordinate is minimum.

Let u_k^* denote this placement point of $\overline{P}_k$. Since, the pole of $\overline{P}_k$ coincides with the pole of polytope P_k, it is obvious that $\text{int}P_k(u_k^*) \cap \overline{P}_i = \emptyset$ for $i = 1, 2, ..., k-1$. Without loss of generality, let point u_k^* be determined by parallelepipeds $\overline{P}_{i_1}$, $\overline{P}_{i_2}$ and $\overline{P}_{i_3}$. Then a parallelepiped $\Lambda_k(u_k^*)$ is constructed with minimal volume such that it contains the polytopes Γ_{i_1k}, Γ_{i_2k} and Γ_{i_3k} and whose facets are parallel to coordinate planes.

Then the search for feasible positions of P_k within P is now restricted on the set

$$D_k := G_k \cap \Lambda_k(u_k^*).$$

Hence, in order to find possible placement points of the pole O_k of P_k in D_k it is necessary to select all Γ_{ik}, $i \in \{1, 2, ..., k-1\}$, for which

$$\Lambda_k(u_k^*) \cap \Gamma_{ik}(u_i^0) \neq \emptyset.$$

Let $\Gamma_{i_jk}(u_{i_j}^0)$, $j = 1, 2, ..., r \leq k-1$, be these sets. After that, the algorithm described above (beginning from step 3) is applied only for sets $\Gamma_{i_jk}(u_{i_j}^0)$, $j = 1, 2, ..., r$. It is important to note, that on each step of the algorithm any point which does not belong to parallelepiped $\Lambda_k(u_k^*)$ is rejected.

4.2 Construction of convex sets

Due to the algorithm described above, this procedure of placing the polytopes into P yields a point $X^0 = (u_1^0, u_2^0, ..., u_n^0, h) \in \Omega$. Point X^0 may be either an interior or a frontier point of Ω. X^0 belongs to one or several convex subregions Ω_p, say $p \in P_0 \subset \{1, 2, \ldots, \eta\}$.

So it is necessary to construct one of the convex subsets Ω_ρ ($\rho \in P_0$).

We represent Ω_ρ as intersection of two convex sets Ω^* and Ω^*_ρ, i. e. $\Omega_\rho = \Omega^* \cap \Omega^*_\rho$, where

$$\Omega^* = \{\overline{u} \in I\!R^{m_1} : \Phi_i(u_i, h) \geq 0,\ i = 1, 2, ..., n\},$$

results from formula (7), and set Ω^*_ρ contains one inequality of (10) for each pair $i \neq j$, say $\varphi_{ij}^{t_{ij}}(u) \geq 0$ with $t_{ij} \in K_{ij}$, which is fulfilled at point X^0, i. e.

$$\Omega^*_\rho = \{\overline{u} \in I\!R^{m_1} : \varphi_{ij}^{t_{ij}}(u) = B_{ij}(u_j - u_i) + D_{ij} \geq 0,\ i < j \in \{1, 2, ..., n\}\}.$$

Since $X^0 \in \Omega$ (because of construction) there exists at least one such inequality for each pair $i \neq j$. If for some pair $i \neq j$ more than one inequality exists which are fulfilled at point X^0, any of them can be taken.

As computational experience show, a good way to construct Ω^*_ρ for the point $X = (u_1, \ldots, u_n, h) \in \Omega$ is to extract such a function φ^t_{ij} with

$$\varphi^t_{ij}(u_i, u_j) = \Phi_{ij}(u_i, u_j) \geq 0$$

for all $i < j \in \{1, 2, ..., n\}$. As result we obtain some inequality system which consists of $C_n^2 + 6n$ inequalities, namely $\frac{n}{2}(n-1)$ of kind $\varphi^t_{ij}(u_i, u_j) \geq 0$ and $6n$ of kind $f_{ik}(u, h) \geq 0$ ($k = 1, \ldots, 6$). We denote the resulting system of inequalities as

$$B_\rho X + D_\rho \geq 0.$$

Solving the linear programming problem

$$F(X) = F(u, h) \to \min \qquad \text{s.t. } B_\rho X + D_\rho \geq 0$$

a point X^1 is obtained which is either a local minimum of the initial problem (5) or not (some vertex of Ω or some interior point).

4.3 Search for local minimum of problem (5)

Within a connected component of Ω a sequence of path-wise connected convex subsets Ω_{i_j} will be constructed until a proved local minimum is found. In order to do this, an exact solution method is applied that permits to find a local optimal solution $X^* = (u^*, h^*)$ of problem (5). Thereby, solution X^* is obtained as result of a descent procedure which starts at point $X^0 \in \Omega_{i_1} \subset \Omega$. Within the descent process a simultaneous (and successive) search and construction of new subregions is realized together with the solution of corresponding LP problems. Without loss of generality, let Ω_1, Ω_2, ..., Ω_μ ($\mu \leq \eta$) denote the resulting sequence of subregions which form some path-wise connected component of Ω. It is evident, a local minimum X^* is found if it is a local minimum for all subregions which contain X^*.

In order to illustrate this procedure we assume that Ω consists of four chained (convex) subregions Ω_i, $i = 1, 2, 3, 4$ as depicted in Fig. 3. In this case the point X^0 is an inner point of Ω_1. Point X^1 which is the minimum solution of

$$F(X) \to \min \qquad \text{s. t. } X \in \Omega_1$$

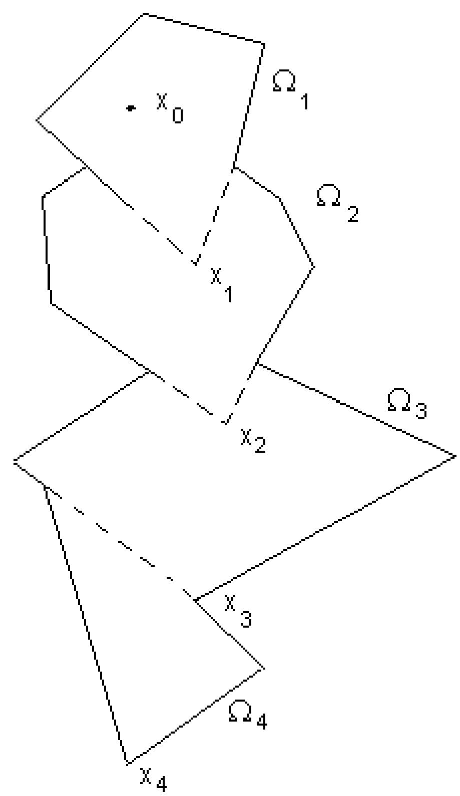

Figure 3: Connected convex sets of a connected component

is an inner point of Ω_2. Point X^2, solution of the next minimum problem, is inner point of Ω_3, and point X^3 is a frontier point of Ω_4. Finally, X^4 is a local minimum.

The search for a local minimum uses the following two iterations:

(I) If the problem is non-singular at point X_j then problem

$$X_{j+1} = \arg\min\{F(X) : X \in \Omega_j\} \tag{15}$$

yields the next iterate X_{j+1} and a corresponding subregion $\Omega_{j+1} \neq \Omega_j$ with $X_{j+1} \in \Omega_{j+1}$, or the verification that X_j is a local minimum, $j = 0, 1, ...$

(II) If the problem is singular at X_j then it is necessary to construct successively further (all) subregions $\Omega_{j,k}$, $k = 1, 2, \ldots$, which contain X_j and to solve the problem

$$X_{j,k} = \arg\min\{F(X) : X \in \Omega_{j,k}\} \tag{16}$$

until either an improved solution $X_{j,k}$ is obtained (a descent is reached and $X_{j+1} := X_{j,k}$), or X_j is a proved local minimum.

Next, we will describe how we can pass from a previous linear programming problem (according to (15) or (16)) to the successive one.

One opportunity consists in an appropriate modification of the simplex method. The main difference of this modification to the standard method lies in the pivoting procedure (detection of input and output of columns with respect to a current basis). Here a possibility of simultaneous change of inequalities, and therefore a modification of the feasible region, appears.

To explain the procedure in more detail, we use the representation of Ω as given in formula (11). Let $\Omega_\rho = \bigcap_{j \in T_\rho} H_{\rho j}$ be the current subregion, and let $\overline{x}$ denote a feasible point of Ω_ρ obtained in a certain step of the simplex method. Then for every active constraint $\varphi_{\rho j}$ i. e. $\varphi_{\rho j}(\overline{x}) = 0$, a half-space $H_{\rho k}$ $(k \in \{1, 2, ..., \gamma_\rho,\ k \neq j)$ with $\overline{x} \in \mathrm{int} H_{\rho k}$ is searched $(\varphi_{\rho k}(\overline{x}) \geq 0)$. If such a half-space is found then constraint $\varphi_{\rho j} \geq 0$ is replaced by $\varphi_{\rho k} \geq 0$ in order to obtain a new subregion $\Omega_{\rho+1}$. Such exchange of constraints can be done when an optimal solution with respect to Ω_ρ is found, or in each iteration of the simplex method. If the new subregion is found then the constraint $\varphi_{\rho k} \geq 0$ is announced as artificial and $\varphi_{\rho k}$ is excluded from the matrix of problem conditions. All constraints specifying $\Omega_{\rho+1}$ which are not included in the matrix are added in it.

This solution strategy with a simultaneous exchange of constraints can be realized by a modification of the standard simplex method. Such a substitution can be inserted into the process of pivoting (choice of pivot column). When looking at pivot columns each candidate is analyzed whether the candidate can be exchanged. If such substitution is possible it is immediately realized. The sequence of actions are repeated until a decision column will be chosen so that any substitution becomes impossible. A way of detection and output of columns from the basis for which appear possibility of substitution is a compliment to a procedure of standard simplex method.

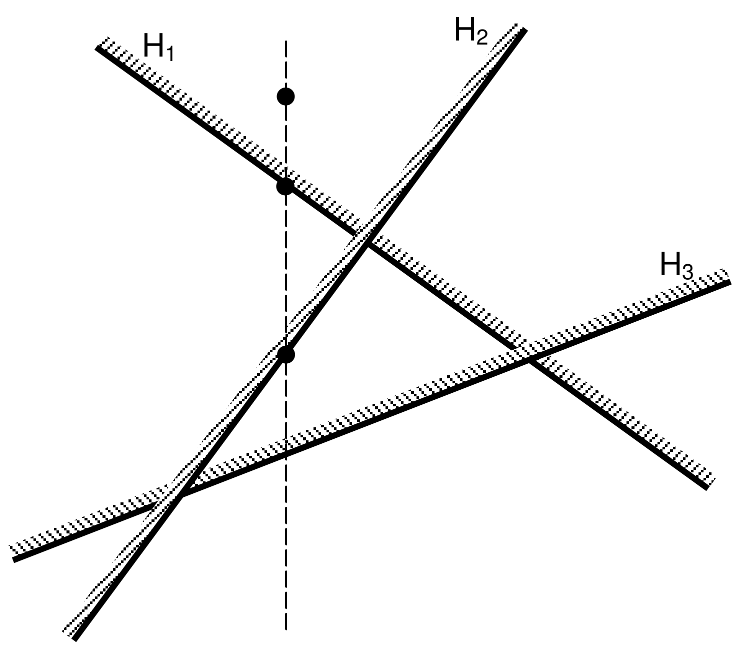

Figure 4: Exchange of constraints

The search of decision columns is illustrated in Fig. 4 for the case $\Omega = \Omega_1 \cup \Omega_2 = (H_1 \cap H_3) \cup (H_2 \cap H_3)$. Let $X_0 \in int\Omega_1$. A constraint of the artificial basis is drawn by dotted line. Since the starting point $X_0 \in int\Omega_1$ and the first constraint (which is met a constraint of the artificial basis) is $\varphi_1 \geq 0$, the constraint $\varphi_1 \geq 0$ is included in the basis as result of the choice of decision columns by the standard procedure. Bearing in mind that the new (intermediate) point $X' \in \mathrm{int}H_2$, a substitution of $\varphi_1 \geq 0$ (H_1) by $\varphi_2 \geq 0$ (H_2) is effected in the matrix of problem conditions, i. e. after the standard procedure of choosing a decision column we form the new convex region Ω_2. Then we repeat the standard procedure of choosing decision columns. As result the constraint $\varphi_2 \geq 0$ (H_2), for which it is impossible to realize a similar substitution, is inserted into the new basis.

4.4 Searching for local minimum of problem (14)

Let the starting point $X^0 \in \Omega$ be a point of some subregion $\Omega_\rho \subset \Omega$. So, first of all it is probably necessary to pass from point X^0 to some vertex of Ω_ρ. For that purpose there are known several various methods. However, taking into account that starting point X^0 is constructed as described above, the following approach can be applied. In the initialization step of solving problem (14) it is necessary to construct an artificial basis system of inequalities which is based on the initial point $X^0 = (x_1^0, x_2^0, ..., x_{m_1}^0, h^0) \in \Omega_\rho$. In detail, let $X = (x_1, x_2, ..., x_{m_1}, h)$ denote the vector of variables, then we construct the following additional inequality system

$$X * X^0 = \begin{cases} u_1 *_1 u_1^0 \\ u_2 *_2 u_2^0 \\ .... \\ u_{m_1} *_{m_1} u_{m_1}^0, \end{cases} \tag{17}$$

where $*_i$ denotes one of relation signs $\leq$ or $\geq$. Symbol $*_i$ can arbitrarily be chosen either $\leq$ or $\geq$. For the sake of definiteness let $*_i$ be $\geq, i = 1, 2, ...m_1$ in the first step. Hence, we formulate the following problem

$$\min CX \tag{18}$$

$$B_{l_\rho} X + D_{l_\rho} \geq 0 \tag{19}$$

$$X * X^0 \tag{20}$$

In order to solve such a linear programming problem, again a modification of the simplex method can be applied to save computational amount (cf. e. g. [20]). to solve problem. Moreover, such a modification allows the immediate exchange of problem constraints in the solution process as desired.

5 Examples

In this section we present some examples which have been solved by the proposed method.

Example 1 There are given polytopes P_i, $i = 1, 2, ..., 7$, depicted in Fig. 5. Corresponding coordinates of their vertices are summarized in Table 1. The task is to pack exactly each polytope (for short: $b = (1, 1, 1, 1, 1, 1, 1)^T$) into a parallelepiped P of width $w = 12$ and length $l = 10$. The approximation (the initial heuristic solution) to the local minimum

Table 1: Coordinates of example polytopes

$P_1 \backslash n_i$	1	2	3	4	5	6	7	8	9
x	3	3	3	3	0	0	0	0	5
y	6	6	0	0	6	6	0	0	3
z	0	8	8	0	0	8	8	0	4

$P_2 \backslash n_i$	1	2	3	4
x	8	-3	6	0
y	0	4	2	0
z	-4	-4	10	-4

$P_3 \backslash n_i$	1	2	3	4	5	6	7
x	3	3	3	0	0	0	0
y	0	4	0	4	4	0	0
z	-4	-4	8	-4	8	8	-4

$P_4 \backslash n_i$	1	2	3	4	5	6	7	8	9	10
x	2	1	2	-1	-1	2	2	1	-1	-1
y	0	2	4	4	0	0	4	2	4	0
z	0	-4	0	0	0	7	7	12	8	8

$P_5 \backslash n_i$	1	2	3	4	5	6	7	8	9	10	11
x	2	2	1	1	0	-2	-1	0	2	0	2
y	-4	4	2	-2	4	0	2	-4	0	0	4
z	0	0	6	6	0	0	6	0	-4	-4	-4

$P_6 \backslash n_i$	1	2	3	4	5	6
x	4	4	6	6	0	0
y	7	7	0	0	0	0
z	0	7	7	0	0	7

$P_7 \backslash n_i$	1	2	3	4	5	6	7	8	9	10
x	4	4	2	1	1	3	0	0	-2	-2
y	-4	-4	0	5	5	0	0	0	-5	-5
z	2	-1	-4	-4	5	5	5	-4	-1	2

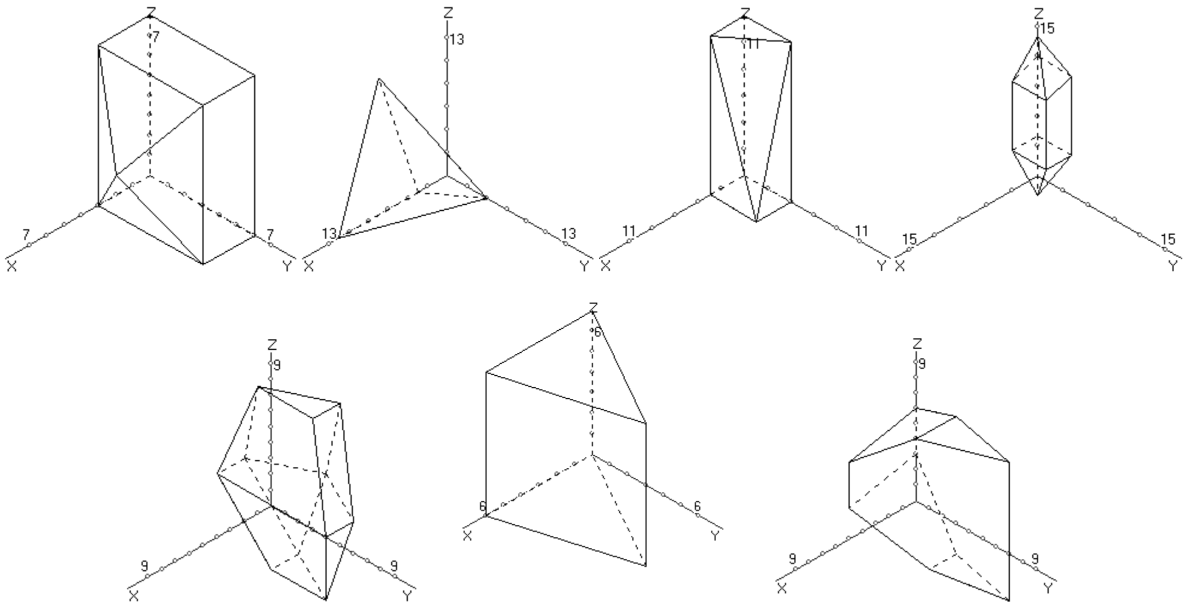

Figure 5: Objects 1-7

is shown in Fig. 6 and the resulting local minimum is shown in Fig. 7. Coordinates of poles of polytopes P_i, $i = 1, 2, ..., 7$, corresponding to the local minimum are summarized in Table 2. Notice, the height h is reduced from 28.5 to 27.0 which is an improvement of 5.3 percent.

Table 2: Coordinates of poles of the local optimum in Example 1

P_i	1	2	3	4	5	6	7
x	0	3	5	10	2	3.71	2
y	0	6	0	0.7	4	0	5
z	0	4	4	4	12	12.8	22

Example 2 The same 7 polytopes now with supply $b = (1, 1, 1, 1, 3, 2, 3)^T$ (i. e. 12 polytopes) have to be placed into a parallelepiped P with width $w = 15$ and length $l = 14$. The approximation to the local minimum is shown in Fig. 8 and the resulting local minimum is depicted in Fig. 9. Coordinates of the poles of the local minimum are summarized in Table 3. Notice, the height h is reduced from 32.208 to 30.917 which is an improvement of 4.0 percent.

Example 3 The same 7 polytopes now with supply $b = (2, 4, 6, 4, 4, 3, 2)^T$ (i. e. 25 polytopes) have to be placed into a parallelepiped P with width $w = 15$ and length $l = 16$. The approximation to the local minimum is shown in Fig. 10 and the resulting local minimum is depicted in Fig. 11. Coordinates of the poles of the local minimum are summarized in Table 4. Notice, the height h is reduced from 53.958 to 45.860 which is an improvement of 15.0 percent.

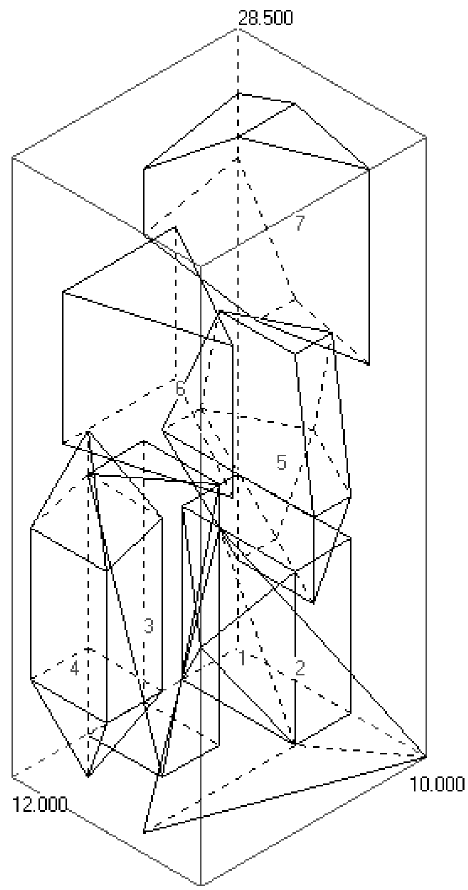

Figure 6: Heuristic solution

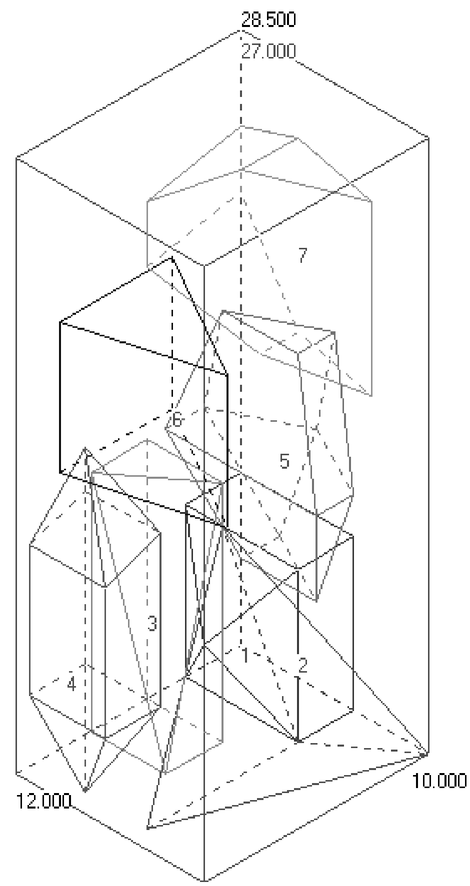

Figure 7: Proved local minimum

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Table 3: Coordinates of poles in Example 2

P_i	P_5	P_7	P_3	P_6	P_6	P_7	P_7	P_1	P_5	P_2	P_5	P_4
x	2	7	11	9	9	8.6	4.38	8.38	13	3.57	2	1
y	4	5	0	5	5	7.8	7	0	7.77	6	4	8
z	4	4	4	0	7	4	11.91	14	18	20.91	24.91	4

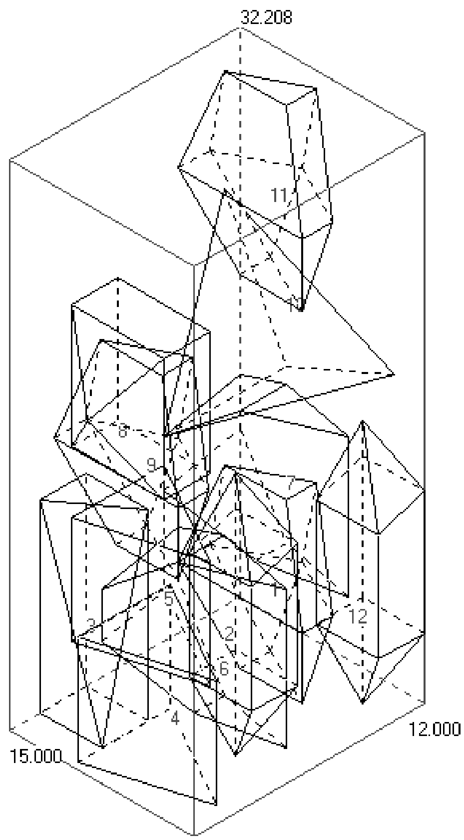

Figure 8: Heuristic solution

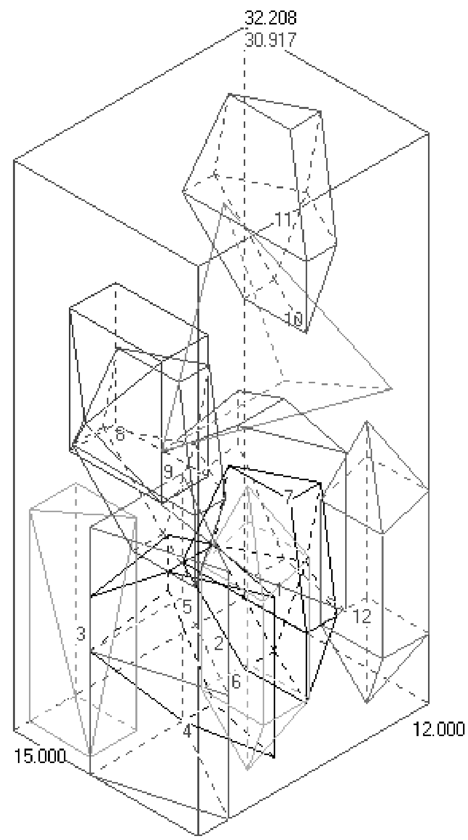

Figure 9: Proved local minimum

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Table 4: Coordinates of poles in Example 3

P_i	P_3	P_5	P_1	P_2	P_3	P_6	P_1	P_2	P_3	$P_7$0	P_5	P_3	P_2		
x	0	4.33	6.33	3.45	4	4.9	0	6.11	2.48	11	6.59	0	3.92		
y	0	4	0	8	3.5	0	5.79	7.69	1.01	7	4.19	5.01	8		
z	4	4	0	4	4	8	4.66	15.95	15.02	17.16	19.45	16.66	26.16		

P_i	P_2	P_4	P_5	P_6	P_4	P_4	P_4	P_6	P_3	P_3	P_5	P_7	
x	3	13	13	0	10	8.5	13	0	10.5	0.46	5.5	8.42	
y	0	0	8	4.12	1.55	5.67	2.68	0	8	6.06	8	5	
z	27.02	21.73	26.48	26.86	24.09	33.86	33.86	33.86	36.84	37.86	39.86	39.22	

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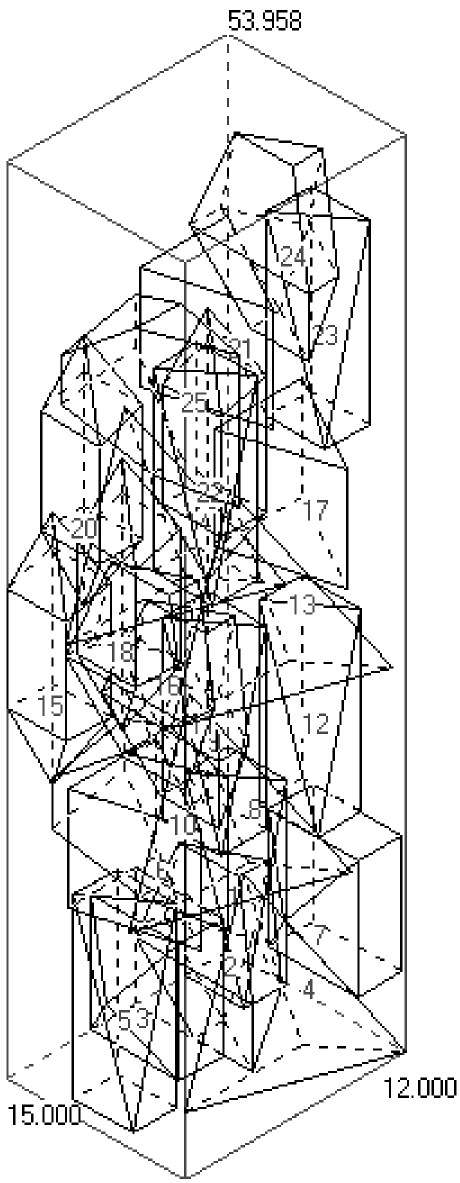

Figure 10: Heuristic solution

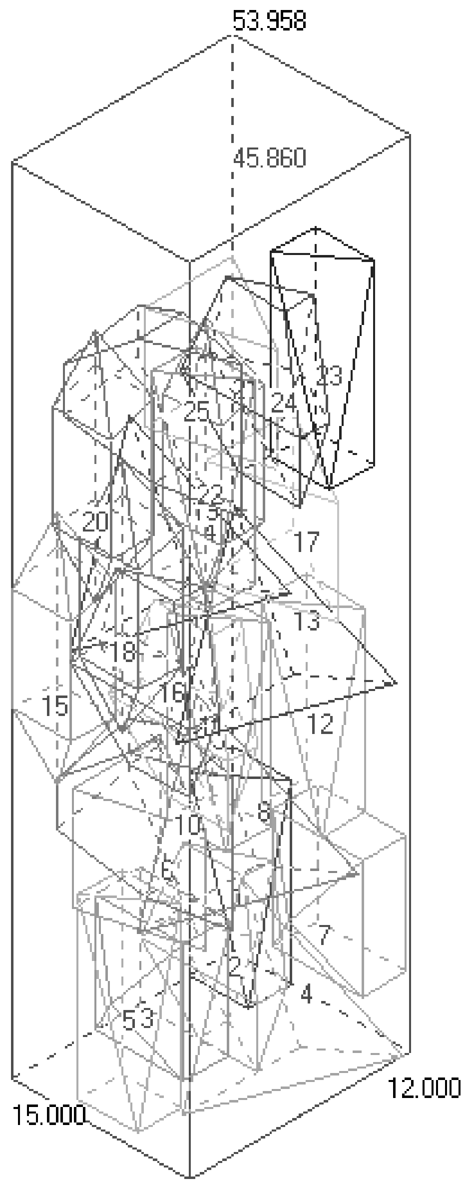

Figure 11: Proved local minimum