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REMARKS ON GENERALIZED BRAUER PAIRS

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Abstract. Let k be an algebraically closed field of characteristic p, G a finite group, N a normal subgroup of G and c a G-stable block of kN. In this case there exists generalized Brauer pairs called (c, G)-Brauer pairs and denoted by (Q, e_Q), where Q is a p-subgroup of G and e_Q a block of $kC_N(Q)$. If $G = N$ the generalized Brauer pairs becomes the usual c-Brauer pairs. If (P, e_P) is a maximal (c, G)-Brauer pair we prove that e_P is a nilpotent block. There is also true a form of Brauer's third main theorem.
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1. PRELIMINARIES

Throughout this paper we consider k an algebraically closed field of characteristic p, G a finite group, N a normal subgroup of G and c a G-stable block of kN, that is c is a primitive idempotent of $Z(kN)$ fixed by conjugation action of G.

Using the approach from [2], in [3, Section 3] R. Kessar and R. Stancu give the definition of a generalized (c, G)-Brauer pair, generalized Brauer category, Brauer homomorphism etc. In the next lines we explicitly, restate definition of generalized (c, G)- Brauer pairs and a few interesting properties. We also include the approach of pointed groups, which is not used in [3].

In section 2 we prove that for a maximal (c, G)-Brauer pair denoted (P, e_P) it is true that e_P is a nilpotent block of $kC_N(P)$.We will make some intuitive connection between maximal (c, G)-Brauer pairs and defect pointed groups.

In section 3 similar to [6, Section 40] we define a normal relation denoted $\trianglelefteq$ between (c, G)-Brauer pairs which has as transitive closure the order relation from [2, Section 1] which we denote $\leq$. This allow us to imitate the proof of Brauer's third main theorem [6, Theorem 40.17].

We will use basic definitions, results and notations regarding block theory from [6].

For any Q a p-subgroup of G the canonical projection from kN to $kC_N(Q)$ induces a surjective homomorphism of algebras from $(kN)^Q$ onto $kC_N(Q)$, the Brauer homomorphism denoted by Br_Q^N (see [1]). Explicitly $\mathrm{Br}_Q^N(x) = x$ if $x \in C_N(Q)$ and $\mathrm{Br}_Q^N(x) = 0$ if $x \notin C_N(Q)$. Since $A = kN$ is a p-permutation

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G-algebra, c is a primitive idempotent of $A^G \subseteq A^N = Z(kN)$ and $cAc = cA = Ac$ remains a p-permutation algebra, we can adopt the approach of [2] for generalized Brauer pairs.

Definition 1. A (c, G)**-Brauer pair** is a pair (Q, e_Q) where Q is a p-subgroup of G such that $\mathrm{Br}_Q^N(c) \neq 0$ and $\mathrm{Br}_Q^N(c)e_Q \neq 0$. When $G = N$ a (c, G)-Brauer pair is also known as a c-Brauer pair.

There is an order relation on the set of generalized Brauer pairs:

Definition 2. Let (R, e_R) and (Q, e_Q) be two (c, G)-Brauer pair.We say that (Q, e_Q) is **contained** in (R, e_R) and we write $(Q, e_Q) \leq (R, e_R)$, if $Q \leq R$ and for any primitive idempotent $i \in (kN)^R$ such that $\mathrm{Br}_R^N(i)e_R \neq 0$ we have that $\mathrm{Br}_Q^N(i)e_Q \neq 0$.

This order relation is compatible with the conjugation of G. By [2, Theorem 1.8] we have the next remark.

Remark 1. If (R, e_R) is a given (c, G)-Brauer pair then for any $Q \leq R$ there is a unique (c, G)-Brauer pair such that $(Q, e_Q) \leq (R, e_R)$.

Remark 2. By [2, Theorem 1.14] we have that G acts transitively on maximal (c, G)-Brauer pairs, equivalently all maximal (c, G)-Brauer pairs are G-conjugate. If (P, e_P) is a maximal (c, G)-Brauer pair then P is called a (c, G)**-defect group**.

If $N = G$ then P is a defect group of c in the usual sense.

2. POINTED GROUPS AND GENERALIZED BRAUER PAIRS

We consider $A = kN$ as p-permutation G-algebra which is not interior, with $Ac = kNc$ a primitive G-algebra. $N_{\{c\}}$ and $G_{\{c\}}$ are pointed groups on A. We remind that P_γ is a defect pointed group of $G_{\{c\}}$ if P_γ is a maximal local pointed group on A included in $G_{\{c\}}$. By [6, Theorem 18.5] this is equivalent with P being a maximal p-subgroup of G such that $Br_P^N(c) \neq 0$.

Proposition 1. Let P_γ be a defect pointed group of $G_{\{c\}}$ on A. Then there is a unique (c, G)-Brauer pair (P, e_P) such that $\mathrm{Br}_P^N(i)e_P \neq 0$ for any $i \in \gamma$. Moreover (P, e_p) is a maximal (c, G)-Brauer pair, thus P is a (c, G)-defect group.

Proof. For $i \in \gamma$ we have that $i \in (kNc)^P$ is a primitive idempotent with $\mathrm{Br}_P^N(i) \neq 0$. $\mathrm{Br}_P^N(i)$ is a primitive idempotent in $kC_N(P)$ since Br_P^N is surjective. It follows that there is a block $e_P \in Z(kC_N(P))$ such that $\mathrm{Br}_P^N(i)e_P \neq 0$. This block is unique since otherwise by contradiction it follows that $\mathrm{Br}_P^N(i)$ is a primitive idempotent in $kC_N(P)$, which is in the primitive decompositions in $kC_N(P)$ of two blocks.

Since P_γ is a defect pointed group we have that $\mathrm{Br}_P^N(c) \neq 0$ thus $\mathrm{Br}_P^N(c)e_P \neq 0$. By contradiction if $\mathrm{Br}_P^N(c)e_P = 0$ then

$$Br_P^N(i)e_P = Br_P^N(ic)e_P = \mathrm{Br}_P^N(i)\mathrm{Br}_P^N(c)e_P = 0,$$

false. The last part of the proof is obviously. □

For proving the main result of this section we need the following lemma which gives a particular result in group theory.

LEMMA 1. *Let N be a normal subgroup of a finite group G and P a p-subgroup of G such that $P \cap N \neq 1$. Then $Z(P) \cap N \neq 1$.*

Proof. The p-group P acts on the set $P \cap N$ by conjugation. We denote by $\mathcal{O}(n_i), \quad i \in \{1, ..., k\}$ the orbits of this P set, where n_i are chosen representatives. By [5, Theorem 2.97, Proposition 2.98] we have:

$$\mid P \cap N \mid = \sum_{i=1}^{k} \mid \mathcal{O}(n_i) \mid = \sum_{i=1}^{k} [P : P_{n_i}].$$

If $n_i \in Z(P) \cap N$ then his orbit $\mathcal{O}(n_i) = \{n_i\}$ and $P_{n_i} = P$.It follows that:

$$\mid P \cap N \mid = \mid Z(P) \cap N \mid + \sum_{i} [P : P_{n_i}],$$

where the orbit of n_i have more than one element. Since $P \cap N$ is a nontrivial p-group and p divides $\sum_{i} [P : P_{n_i}]$ it follows that p divides $\mid Z(P) \cap N \mid$, which concludes the proof. □

REMARK 3. By [4, Propsition 5.3] applied to our case P_γ is a defect pointed group of $G_{\{c\}}$ on A if and only if $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{G} = G/N$ and there is Q_δ a defect pointed group of $N_{\{c\}}$ on the N-algebra kN such that $Q_\delta \leq P_\gamma$. In this case $Q = P \cap N$, thus $P \cap N \neq 1$.

REMARK 4. Using Lemma 1 and Remark 3 it is not difficult to prove that if P_γ is a defect pointed group of $G_{\{c\}}$ and (P, e_P) is the maximal (c, G)-Brauer pair then $Z(P) \cap N \neq 1$ is included in any defect group of the block e_P in $kC_N(P)$.

Let $B = kNc$ the primitive G-algebra, which is the localization of $G_{\{c\}}$ in A and P_γ a defect of B. We remind that $S(\gamma) = B^P/m_\gamma$ is a simple k-algebra called the **multiplicity algebra**, where $m_\gamma = J(B^P)$ is the unique maximal ideal of B^P such that $\gamma \nsubseteq m_\gamma$. Then $S(\gamma) \simeq \mathrm{End}_k(V(\gamma))$, where $V(\gamma)$ is the simple B^P-module called **the multiplicity module**.

By [6, Lemma 14.5] we can view, slightly differently the multiplicity algebra $S(\gamma)$ as a simple quotient of $kC_N(P)$ and thus $S(\gamma)$ isomorphic with the k-endomorphism algebra of a simple $kC_N(P)$-module. Explicitly this module is $V(\gamma) = kC_N(P)\mathrm{Br}_P^N(i)/J(kC_N(P))\mathrm{Br}_P^N(i)$.

In the next lines we denote by $\overline{N} = N_G(P_\gamma)/P$ and $\overline{C} = C_N(P)/Z(P) \cap N$. Remark that $\overline{C} = C_N(P)/P \cap C_N(P) \simeq PC_N(P)/P$ which is a subgroup of $\overline{N}$.

LEMMA 2. *In the above conditions it is true that the multiplicity module $V(\gamma)$ is simple and projective as $k\overline{C}$-module.*

Proof. $V(\gamma)$ is a simple $kC_N(P)$-module and $Z(P) \cap N$ a normal p-subgroup of $C_N(P)$. By [6, Corollary 21.2] $Z(P) \cap N$ acts trivially on every simple $kC_N(P)$-module, thus $V(\gamma)$ is simple as $k\overline{C}$-module.

The multiplicity algebra $S(\gamma)$ has a $\overline{N}$-algebra structure, which is not necessarily interior on restriction to the subgroup $PC_G(P)/P$ but is interior on restriction to the subgroup $\overline{C}$. By [6, Example 10.9] the multiplicity module $V(\gamma)$ of P_γ is endowed with a $k_\sharp\widehat{\overline{N}}$-module structure which extends the structure of $V(\gamma)$ as $k\overline{C}$-module. Since B is a primitive G-algebra by [6, Theorem 19.2] we have that $V(\gamma)$ is projective as $k_\sharp\widehat{\overline{N}}$-module. By [6, Corollary 17.8] this is equivalent with the fact that the $\overline{N}$-algebra $S(\gamma) = \mathrm{End}_k(V(\gamma))$ is projective algebra relative to $\{1\}$. Further this is equivalent to the surjectivity of the relative transfer map $t_1^{\overline{N}} : S(\gamma) \longrightarrow S(\gamma)^{\overline{N}}$.

Since $V(\gamma)$ is simple on restriction to $k\overline{C}$ it follows that $S(\gamma)^{\overline{C}} \cong k$ by Schur's lemma, and a fortiori $S(\gamma)^{\overline{N}} \cong k$. Therefore the relative trace map $t_1^{\overline{N}}$ factorizes as:

$$S(\gamma) \xrightarrow{t_1^{\overline{C}}} k \xrightarrow{t_{\overline{C}}^{\overline{N}}} k\ .$$

$\overline{N}/\overline{C}$ acts trivially on k thus by definition of relative trace map $t_{\overline{C}}^{\overline{N}}$ is multiplication by $[\overline{N} : \overline{C}]$, which is either 0 or an isomorphism.

We conclude that $t_1^{\overline{N}}$ is surjective if and only if $t_1^{\overline{C}}$ is surjective and $[\overline{N} : \overline{C}]1_k \neq 0$. By [6, Corollary 17.4] it follows that $V(\gamma)$ is projective on restriction to $k\overline{C}$. □

We conclude with the main result of this section:

PROPOSITION 2. Let P_γ be a defect pointed group of $G_{\{c\}}$ and (P, e_P) the unique maximal (c, G)-Brauer pair with the property that $\mathrm{Br}_P^N(i)e_P \neq 0$. Then it follows that $Z(P) \cap N$ is a defect group of e_P. In particularly e_P is a nilpotent block of $kC_N(P)$.

Proof. From Lemma 2 we know that $V(\gamma)$ is simple and projective as $k\overline{C}$-module, thus by [6, Theorem 39.1] $V(\gamma)$ belongs to a block $\overline{e}$ of $k\overline{C}$ with defect 0. By [6, Proposition 39.2] $\overline{e}$ lifts to a block e of $kC_N(P)$ with defect group $Z(P) \cap N$ since $Z(P) \cap N$ is a central p-subgroup of $C_N(P)$. Moreover there is a unique simple $kC_N(P)e$ module up to isomorphism which is $V(\gamma)$. Since e_P belongs to $V(\gamma)$ it follows that $e_P = e$ and the defect group is $Z(P) \cap N$.

Since e_P has $Z(P) \cap N$ as defect group which is central in $kC_N(P)$ by [6, Corllary 49.11] it follows that e_P is a nilpotent block. $\square$

If $G = N$ we obtain the well known result that $Z(P)$ is the defect group of e_P as a block of $kC_G(P)$ and e_P is a nilpotent block.

3. BRAUER'S THIRD MAIN THEOREM

If Q and P are two p-subgroups such that Q is normal in P then $kC_N(Q)$ is a P-algebra by conjugation on which Q acts trivially and we view as a P/Q-algebra. Moreover is a p-permutation algebra. Thus there is the Brauer homomorphism, which we denote by $\mathrm{Br}^N_{P/Q}$ and which appear in [2, Proposition 1.5]

$$\mathrm{Br}^N_{P/Q} : (kC_N(Q))^{P/Q} \longrightarrow kC_N(P).$$

$\mathrm{Br}^N_{P/Q}$ is in fact restriction of the Brauer homomorphism Br^N_P for kN to $(kC_N(Q))^P$.

By [2, Proposition 1.5, Theorem 1.8] we have the next remark:

Remark 5. If (P, e) is a (c, G)-Brauer pair and Q is normal in P there is a unique (c, G)-Brauer pair $(Q, f) \leq (P, e)$ such that $\mathrm{Br}^N_{P/Q}(f)e = e$. We define a new relation by saying that (Q, f) is normal in (P, e) if and only if $Q \trianglelefteq P$ and f is the unique block of $kC_N(Q)$ invariant under P such that $\mathrm{Br}^N_{P/Q}(f)e = e$. We write this $(Q, f) \trianglelefteq (P, e)$.

Remark 6. Similar to [6, Corollary 40.10] we have that the order relation $\leq$ on (c, G)-Brauer pairs is the transitive closure of the relation $\trianglelefteq$.

Now we can imitate the proof of [6, Theorem 40.17] which is known in the literature "Brauer's third main theorem".

Theorem 1. *Let c be the principal block of kN, where N is normal in G and Q any p-subgroup of G. Then we have that:*

a) *The principal block c is G-stable.*
b) $\mathrm{Br}^N_Q(c)$ *is a primitive idempotent in $Z(kC_N(Q))$ and is the principal block of $kC_N(Q)$.*
c) (Q, e) *is a (c, G)-Brauer pair if and only if e is the principal block of $kC_N(Q)$.*
d) *The (c, G)-defect groups of c are the Sylow p-subgroups of G.*

Proof. a) Let $\mathcal{S}X = \sum_{x \in X} x$ where X is a subset of G. For all $g \in G$ we know that ${}^g c$ is a primitive idempotent in $Z(kN)$. Using [6, Lemma 40.16] we prove that ${}^c g$ is the principal block, which concludes a). We have that:

$$gcg^{-1}\mathcal{S}N = gc^{-1} \sum_{n \in N} g^{-1}n = gc \sum_{n_1 \in N} n_1 g^{-1} \neq 0,$$

since N normal in G and c is the principal block.

b) For any R a p-subgroup of G we denote by e_R the principal block of $kC_N(R)$. Firs note that by definition of Br_Q^N we have that $\mathrm{Br}_Q^N(\mathcal{S}N) = \mathcal{S}C_N(Q)$. It follows that:

$$\mathrm{Br}_Q^N(c)\mathcal{S}C_N(Q) = \mathrm{Br}_Q^N(c\mathcal{S}N) = \mathrm{Br}_Q^N(\mathcal{S}N) = \mathcal{S}C_N(Q),$$

so that e_Q appears in a decomposition of $\mathrm{Br}_Q^N(c)$ in $ZkC_N(Q)$. Particularly in the case that P is a Sylow p-subgroup of G e_P appears in a decomposition of $\mathrm{Br}_Q^N(c)$ thus (P, e_P) is a maximal (c, G)-Brauer pair. If (P, f) is any (c, G)-Brauer pair (which is maximal since P is Sylow), by Remark 2 there is $g \in N_G(P)$ such that $f =^g e_P$. Since ${}^gC_N(P) = C_N(P)$ we have that:

$${}^g e_P\mathcal{S}C_N(P) =^g \mathcal{S}(e_P C_N(P)) =^g \mathcal{S}C_N(P) = \mathcal{S}C_N(P).$$

So ${}^g e_P$ is the principal block. It follows that $f = e_P$, the principal block is the only block which appears in the decomposition of $\mathrm{Br}_P^N(c)$. Thus $\mathrm{Br}_P^N(c) = e_P$ for all Sylow p-subgroups of G, which prove b) in the Sylow case.

We prove b) by descending induction and using Remarks 5 and 6 it suffices to prove that if $(R, f) \trianglelefteq (Q, e_Q)$ then $f = e_R$. Now $\mathrm{Br}_{Q/R}^N(f)e_Q = e_Q$ by definition of $\trianglelefteq$ and since $\mathrm{Br}_{Q/R}^N(\mathcal{S}C_N(R)) = \mathcal{S}C_N(Q)$ we have:

$$\begin{aligned}\mathrm{Br}_{Q/R}^N(f\mathcal{S}C_N(R))e_Q &= \mathrm{Br}_{Q/R}^N(f)\mathcal{S}C_N(Q)e_Q = \mathrm{Br}_{Q/R}^N(f)e_Q\mathcal{S}C_N(Q) \\ &= e_Q\mathcal{S}C_N(Q) = \mathcal{S}C_N(Q) \neq 0.\end{aligned}$$

By contradiction it follows that $f\mathcal{S}C_N(R) \neq 0$, thus f is the principal block

c) This follows by b).

d) If P is a Sylow p-subgroup then by b) $\mathrm{Br}_P^N(c) \neq 0$, thus P is maximal with this property. This gives that P is a (c, G)-defect group.

□

If $G = N$ we obtain in the above theorem statements a) and b) from [6, Theorem 40.17] and [6, Corollary 40.18].

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