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COMPUTATION AND APPLICATIONS OF SCHUR INDICES

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Abstract. We review some methods for the computation of the Schur indices of Schur algebras. We also give some applications that Schur indices may have in studying other problems, such as providing useful information for the computation of the Wedderburn decomposition of rational group algebras and for the study of the automorphism group of semisimple group algebras.

1. Introduction

The Schur index of a Schur algebra is an invariant of the algebra that results to be a useful tool for the study of different problems. It can be seen as complementary information to be added to data previously obtained. For instance, in the case of the description of the simple components of the Wedderburn decomposition of rational group algebras, the knowledge of the local Schur indices of such algebras provides sometimes the missing information of the description obtained with different methods. We will give an example in the last section to illustrate this situation.

The interest for the computation of the Schur index and related topics can be illustrated by classical references from which we mention only a few of them related with our connection with the subject (see e.g. [2, 4, 5, 16, 17, 20]). We do not pretend to make a complete survey of the results related with the Schur index computation, but rather to review the results from the above references that provide algorithmic methods for the computation of the Schur indices of Schur algebras.

We would like to mention some articles that are related to this topic and some of its applications. Hence, we start with the articles that deal with the characterization of the maximum r-local index of a Schur algebra over an abelian number field K. Since the Schur group $S(K)$ of K is a torsion abelian group, it is enough to compute the maximum of the r-local indices of Schur algebras over K with index a power of p for every prime p dividing the order of the group of roots of unity $W(K)$ of K. In [11] this number was referred as $p^{\beta_p(r)}$ and G.J. Janusz gave a formula for $p^{\beta_p(r)}$ when either

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p is odd or K contains a primitive 4-th root of unity. The remaining cases were studied by J.W. Pendergrass and then completed and unified in [7].

In the study of the automorphism group of a rational group algebra, it has been proved that one important step is the classification of the simple components of the group algebra into isomorphism classes. In [6] it was proved that the ring isomorphism between simple components of the rational group algebras of finite metacyclic groups is determined by the center, the dimension over $\mathbb{Q}$, and the list of local Schur indices at rational primes. More generally, the ring isomorphism between cyclic cyclotomic algebras over cyclotomic number fields is essentially determined by the list of local Schur indices at all rational primes.

The Schur index seems to be very useful also when studying the gap between the groups $S(K)$ and $CC(K)$, that is the Schur group of the field K and the group generated by classes containing cyclic cyclotomic algebras over K, for K an abelian extension of the rationals [8].

Many results related with Schur index computation, including the computation of the Brauer group, were obtained for number fields in connection with the development of class field theory. Therefore, the fields that we consider will be mainly number fields or even abelian number fields.

2. Computation of the Schur index

We start by collecting some notions about Schur algebras and the Brauer group of a field as principal tool for the study of central simple algebras. Then we define the Schur index of a Schur algebra and we present some methods to compute it. Some results of this section are classical, mainly from [17, 16].

For the beginning, we assume that K is a arbitrary field and all algebras are finite dimensional K-algebras. Let A and B be central simple K-algebras. We introduce an equivalence relation on the class of central simple K-algebras. We say that A and B are *Brauer equivalent*, or simply *equivalent* and write $A \sim B$, if there is a division algebra D and positive integers n and m such that $A \simeq \mathcal{M}_n(D)$ and $B \simeq \mathcal{M}_m(D)$ as K-algebras. All the isomorphisms here are K-algebra homomorphisms. The equivalence class of a central simple K-algebra A is denoted by $[A]$.

The *Brauer group* of a field K, denoted by $\mathrm{Br}(K)$, is the set of equivalence classes of central simple K-algebras under the Brauer equivalence, with the tensor product acting as the group operation and the equivalence class of K acting as the identity element. The inverse of $[A]$ is $[A^{\mathrm{op}}]$, where A^{op} is the opposite algebra of A. The Brauer group is torsion, that is, every element of $\mathrm{Br}(K)$ has finite order. The *exponent* of a central simple K-algebra A, denoted $\exp(A)$, is the order of $[A]$ in the Brauer group $\mathrm{Br}(K)$. That is, $\exp(A)$ is the smallest number m such that $A^{\otimes m} \cong \mathcal{M}_r(K)$ for some r, where $A^{\otimes m}$ denotes the tensor product of m copies of A.

For a central simple K-algebra A, the dimension $\dim_K(A)$ of the algebra as a vector space over K is a square and the *degree* is $\deg(A) = (\dim_K A)^{1/2}$. The degree mapping is clearly not invariant under the Brauer equivalence. Because of this fact, and for other reasons, it is useful to define a different numerical function on central simple algebras. This is given by the Schur index. Let A be a central simple K-algebra, so that $A \cong \mathcal{M}_m(D)$ for some unique division K-algebra D. Then the *Schur index* of A, denoted by $\operatorname{ind}(A)$, is the degree of D, that is, the square root of the dimension of D as a vector space over K. If χ is an irreducible complex character of a finite group G and K is a field of characteristic zero, then the *Schur index of* χ over K, denoted by $m_K(\chi)$, is the Schur index of the simple component of the group algebra KG corresponding to the character χ, denoted by $A(\chi, K)$. An alternative way of defining the Schur index of an irreducible complex character with respect to a field K is related to the question:

> "For which fields $K \leq \mathbb{C}$ is the character $\chi \in \operatorname{Irr}(G)$ afforded by a K-representation?"

If $K \leq \mathbb{C}$ is not one of these fields, we wish to measure the extent to which χ fails to be afforded over K. This suggests the following definition from [9]. Let $K \leq L$, where L is any splitting field for the finite group G. Choose an irreducible L-representation ρ which affords χ and an irreducible K-representation φ such that ρ is a constituent of φ^L. Then the multiplicity of ρ as a constituent of φ^L is the *Schur index* of χ over K denoted by $m_K(\chi)$.

Now we would like to define the local index of a central simple algebra. Throughout K denotes a number field. A *prime* of K is an equivalence class of valuations of K. There are the archimedean or *infinite primes*, arising from embeddings of K into the complex field $\mathbb{C}$ and the non-archimedean or *finite primes* of K, arising from discrete P-adic valuations of K, with P ranging over the distinct maximal ideals in the ring of all algebraic integers of K. Let A be a central simple K-algebra and let P range over the primes of K. We use K_P to denote the P-adic completion of K. Let $A_P = K_P \otimes_K A$ the P-adic completion of A. Then A_P is a central simple K_P-algebra and the *local Schur index* of A at P is defined as $m_P(A) = \operatorname{ind}[A_P]$.

Clearly $A_P \sim K_P$ if and only if $m_P(A) = 1$. We say that P is *ramified* in A if $m_P(A) > 1$. An infinite prime P of K corresponds to an archimedean valuation on K which extends the ordinary absolute value on the rational field $\mathbb{Q}$. The P-adic completion K_P is either the real field $\mathbb{R}$ (and P is called a *real prime*), or else the complex field $\mathbb{C}$ (and P is a *complex prime*).

Theorem 2.1. *Let A be a central simple K-algebra, and let m_P be the local index of A at an infinite prime P of K.*

(i) *If P is a complex prime, then $A_P \sim K_P$ and $m_P = 1$.*

(ii) *If P is a real prime, then either $A_P \sim K_P$ and $m_P = 1$, or else $A_P \sim \mathbb{H}$ and $m_P = 2$, where $\mathbb{H}$ is the division algebra of real quaternions.*

If P is any finite prime of K, then K_P is a complete field relative to a discrete valuation, and has a finite residue class field. Considering the Hasse invariant $\mathrm{inv}[A_P]$ of a central simple K_P-algebra, we obtain an isomorphism $\mathrm{inv} : \mathrm{Br}(K_P) \simeq \mathbb{Q}/\mathbb{Z}$. We know that $\mathrm{inv}[A_P] = s_P/m_P$ and $\exp[A_P] = m_P$, where $m_P = \mathrm{ind}[A_P]$ and $(s_P, m_P) = 1$. The following result is also known as the "Local–Global Principle" for algebras.

Theorem 2.2 (Hasse–Brauer–Noether–Albert)**.** *Let A be a central simple K-algebra. Then $A \sim K \iff A_P \sim K_P$ for each prime P of K.*

Remark 2.3. (i) For each prime P of K, there is a homomorphism $\mathrm{Br}(K) \to \mathrm{Br}(K_P)$ defined by $K_P \otimes_K -$. Let $[A] \in \mathrm{Br}(K)$ and m_P be the local index of A at P. Then $m_P = 1$ almost everywhere, which means that $[A_P] = 1$ almost everywhere. Hence there is a well defined homomorphism

$$\mathrm{Br}(K) \to \bigoplus_P \mathrm{Br}(K_P).$$

The Hasse–Brauer–Noether–Albert Theorem is precisely the assertion that this map is monic.

(ii) A stronger result, due to Hasse, describes the image of $\mathrm{Br}(K)$ in $\bigoplus_P \mathrm{Br}(K_P)$ by means of Hasse invariants. Hence, the following sequence is exact:

$$1 \to \mathrm{Br}(K) \to \bigoplus_P \mathrm{Br}(K_P) \stackrel{\mathrm{inv}}{\to} \mathbb{Q}/\mathbb{Z} \to 0, \tag{1}$$

where inv denotes the Hasse invariant map, computed locally on each component: $\mathrm{inv} = \bigoplus \mathrm{inv}_{K_p}$. From the exactness of the previous sequence (1) it follows the next relation which is considered many times a formulation of the Hasse–Brauer–Noether–Albert Theorem in terms of Hasse invariants:

$$\sum_P \mathrm{inv}[A_P] = 0,\ [A] \in \mathrm{Br}(K). \tag{2}$$

Of course, $\mathrm{inv}[A_P] = 0$ if P is a complex prime, while $\mathrm{inv}[A_P] = 0$ or $1/2$ if P is a real prime. The exactness of (1) also tells us that, other than (2), these are the *only* conditions which the set of local invariants $\{\mathrm{inv}[A_P]\}$ must satisfy. In other words, suppose that we are given in advance any set of fractions $\{x_P\}$ from $\mathbb{Q}/\mathbb{Z}$, such that $x_P = 0$ almost everywhere, $\sum x_P = 0$, $x_P = 0$ if P is complex, $x_P = 0$ or $1/2$ if P is real. Then there is a unique $[A] \in \mathrm{Br}(K)$ such that

$$\mathrm{inv}[A_P] = x_P \text{ for all } P.$$

Two of the major consequences of the Brauer–Hasse–Noether–Albert Theorem are the following theorems.

Theorem 2.4. *Let $[A] \in \mathrm{Br}(K)$ having local indices $\{m_P\}$. Then $\mathrm{ind}[A] = \exp[A] = \mathrm{lcm}\ \{m_P\}$, the least common multiple of the m_P's.*

Theorem 2.5. *Every central simple K-algebra is isomorphic to a cyclic algebra.*

The theorem asserts that every central division algebra over a number field K is isomorphic to $(L/K, \sigma, a)$ for a suitable cyclic extension L/K with generating automorphism σ and suitable $a \in K^*$. Equivalently, A contains a maximal commutative subfield L which is a cyclic field extension of K.

The Schur group of a field K is the answer to the following question:

> "What are the classes in the Brauer group $\mathrm{Br}(K)$ occurring in the Wedderburn decomposition of the group algebra KG, for G a finite group?"

Let A be a central simple algebra over K. If A is spanned as a K-vector space by a finite subgroup of its group of units, then A is called a *Schur algebra* over K. Equivalently, A is a Schur algebra over K if and only if A is a simple component central over K of the group algebra KG for some finite group G. The *Schur subgroup* of the Brauer group $\mathrm{Br}(K)$, denoted by $S(K)$, consists of those classes that contain a Schur algebra over K. The fact that $S(K)$ is a subgroup of $\mathrm{Br}(K)$ is a direct consequence of the isomorphism $KG \otimes_K KH \cong K(G \times H)$.

The Brauer–Witt Theorem has been the corner stone result for the study of the Schur group of a field. It asserts that in order to calculate $S(K)$, one may restrict to the classes in $\mathrm{Br}(K)$ containing cyclotomic algebras $A = (K(\zeta)/K, \tau)$, that is, crossed products

$$K(\zeta) *_{\tau}^{\alpha} \mathrm{Gal}(K(\zeta)/K) = \bigoplus_{\sigma \in \mathrm{Gal}(K(\zeta)/K)} K(\zeta)\overline{\sigma},$$

where ζ is a root of unity, the action α is the natural action of $\mathrm{Gal}(K(\zeta)/K)$ on $K(\zeta)$ and all the values of the 2-cocycle τ are roots of unity in $K(\zeta)$.

Theorem 2.6 (Brauer–Witt)**.** *A Schur algebra over K, that is, a simple component of a group algebra KG with center K, is Brauer equivalent to a cyclotomic algebra over K.*

The elements of the Brauer group are characterized by invariants, hence it is reasonable to ask whether the elements of $S(K)$ are distinguished in $\mathrm{Br}(K)$ by behavior of invariants. M. Benard [1] had shown the following.

Theorem 2.7. *If $[A] \in S(K)$, for K an abelian number field, p is a rational prime and P_1, P_2 are primes of K over the prime p, then $A \otimes_K K_{P_1}$ and $A \otimes_K K_{P_2}$ have the same index.*

Furthermore, M. Benard and M. Schacher [2] have shown the following.

Theorem 2.8. *Let $[A] \in S(K)$. Then:*

(1) *If the index of A is m then ζ_m is in K, where ζ_m is a primitive m-th root of unity.*
(2) *If P is a prime of K lying over the rational prime p and $\sigma \in$ Gal$(K/\mathbb{Q})$ with $\zeta_m^{\sigma} = \zeta_m^{b}$ then the p-invariant of A satisfies the relation* $\mathrm{inv}_P(A) \equiv b\, \mathrm{inv}_{P^{\sigma}}(A) \mod 1$.

If a central simple algebra A over K satisfies (1) and (2) above then A is said to have *uniformly distributed invariants*. Based on this result, R.A. Mollin defined the group $U(K)$ as the subgroup of $\mathrm{Br}(K)$ consisting of those algebra classes which contain an algebra with uniformly distributed invariants [12]. It follows from the Benard-Schacher result that $S(K)$ is a subgroup of $U(K)$. General properties of $U(K)$ and the relationship between $S(K)$ and $U(K)$ are investigated in [12].

There are additional restrictions on the collection of local indices of central simple algebras that lie in the Schur subgroup of an abelian number field. The following is a useful consequence of results of Witt ([19], Satz 10 and 11). It also holds in the more general setting of central simple algebras over K that have uniformly distributed invariants [12].

Theorem 2.9. *Let K be an abelian number field and $A \in S(K)$. If p is an odd prime, then $p \equiv 1 \mod m_p(A)$. If $p = 2$ then $m_2(A) \leq 2$.*

The previous result is also a consequence of a result from [10] and [20] describing the Schur group of a subextension of a cyclotomic extension of the local field $\mathbb{Q}_p$, for p an odd prime number.

Theorem 2.10. *Let k be a subfield of the cyclotomic extension $\mathbb{Q}_p(\zeta_m)$, $e = e(k/\mathbb{Q}_p)$ the ramification index and e_0 the largest factor of e coprime to p. Then $S(k)$ is a cyclic group of order $(p-1)/e_0$ and it is generated by the class of the cyclic algebra $(k(\zeta_p)/k, \sigma, \zeta)$, where ζ is a generator of the group of roots of unity in k with order coprime to p.*

Let q be a prime integer, $\mathbb{Q}_q$ the complete q-adic rationals, and k a subfield of $\mathbb{Q}_q(\zeta_m)$ for some positive integer m. The following lemma from [10] is helpful to compute the index of cyclic algebras over the local field k.

Proposition 2.11. *Let E/k be a Galois extension with ramification index $e = e(E/k)$ and ζ be a root of unity in k having order relatively prime to q. Then $\zeta = N_{E/k}(x)$ for some $x \in E \Longleftrightarrow \zeta = \xi^e$ for ξ a root of unity in k.*

Notice that having $A = (E/k, \sigma, a)$ a cyclic algebra, $\exp[A]$ is the least positive integer t such that $a^t \in N_{E/k}(E^*)$, where $N_{E/k} : E \to k$ denotes the norm map of the extension E/k. Moreover, if $\exp[A] = [E : k]$, then A is a division algebra. This is a consequence of the property of cyclic algebras saying that $(E/k, \sigma, a) \sim k$ if and only if $a \in N_{E/k}(E^*)$. Proposition 2.11 gives then a criterion to decide when $a^t \in N_{E/k}(E^*)$, for a a root of 1, that is, exactly when $a^t = \xi^{e(E/k)}$, for ξ a root of 1 in k.

By the Brauer–Witt Theorem, every Schur algebra is Brauer equivalent to a cyclotomic algebra and, if the center is a number field, then it is isomorphic to a cyclic algebra. Algebras with these two features were called cyclic cyclotomic and were studied in [8]. Hence, *cyclic cyclotomic algebra* over a number field K is a cyclic algebra $(K(\zeta)/K, \sigma, \xi)$, where ζ and ξ are roots of unity.

A Schur algebra over K is cyclic cyclotomic if and only if it is generated over K by a metacyclic group if and only if it is a simple component of a group algebra KG for G a metacyclic group (see [13]). The next proposition gives useful information on the local indices of cyclic cyclotomic algebras. First we need the following lemma from [17].

Lemma 2.12. *Let K be a local field, W an unramified extension of K of degree f and v be the P-adic valuation on K. Given any $\alpha \in K$, the equation $N_{W/K}(x) = \alpha$, with $x \in W$ is solvable for x if and only if f divides $v(\alpha)$.*

Proposition 2.13. *Let $A = (K(\zeta_n)/K, \sigma, \zeta_m)$ for K a number field and ζ_n, ζ_m roots of unity of orders n and m respectively. If p is a prime of K, then $m_p(A)$ divides m. If $m_P(A) \neq 1$ and p is a finite prime then p divides n.*

Proof. $[A]^m = [(K(\zeta_n)/K, \sigma, 1)] = 1$, hence $m_p(A)$ divides $m(A)$ which divides m. Furthermore, if $p \nmid n$, then $K(\zeta_n)/K$ is unramified at p and $v_p(\zeta_m) = 0$ since ζ_m is a unit in the ring of integers of K. By Lemma 2.12, the equation $N_{K_p(\zeta_n)/K_p}(x) = \zeta_m$ has a solution in $K(\zeta_n)$ and so $m_p(A) = 1$. □

The results presented in this section provide the algorithmic methods that one could follow in order to compute the (local) Schur indices of a Schur algebra. In the next section we give an example of such a computation.

3. Applications: An example

We present an example where the knowledge of the (local) Schur indices of a Wedderburn component of a rational group algebra is essential in order to obtain a precise description of it as Brauer equivalent to a cyclotomic algebra by completing the information obtained with other methods.

In [14] a theoretical algorithm was given for the computation of the Wedderburn decomposition of semisimple group algebras KG, for finite groups G and fields K of characteristic zero, based on a computational approach of the Brauer–Witt Theorem. In [15], the theoretical algorithm was improved and a working algorithm was presented, which was the support for the implementation of this method in a computer package called `wedderga` for the computer system GAP. The theoretical algorithm has as input a group algebra KG and as output the Wedderburn components A_χ, parameterized by representatives of the K-equivalence classes of the irreducible characters of the finite group G. The components A_χ are described as $M_{d_1/d_2}(B)$, where d_1 is the degree of the character χ, d_2 is the degree of a computed cyclotomic algebra B. Notice that the size of the matrix A_χ is a rational number rather than an integer. Although this does not make literal sense, still the algorithm provides a lot of information on the Wedderburn decomposition.

The group of smallest order for which this phenomenon occurs is the group $[240, 89]$ in the library of the GAP system. In an example presented in the last remark from [14], it was shown a limitation of the method proposed to describe the Wedderburn components as matrices over divison algebras when wanting to precise determine these division algebras, using this group.

In the following example we compute the local Schur indices of the simple component from that example and we provide the desired description of the simple algebra.

Example 3.1. Let G be the group $[240, 89]$ in the library of the GAP system. Then the output of one of the implemented functions of a previous version of `wedderga` applied to $\mathbb{Q}G$ provided the following numerical information for one of the simple factors of $\mathbb{Q}G$:

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[ 3/4, 40, [ [ 4, 17, 20 ] , [ 2, 31, 0 ] ] ].
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Notice that the first entry of this 4-tuple which gives us the size of the matrix is not an integer and a formal presentation of the corresponding simple algebra is given by

$$A \simeq \mathcal{M}_{3/4}\left(\mathbb{Q}(\zeta_{40})(g, h|\zeta_4^g = \zeta_{40}^{17}, \zeta_4^h = \zeta_{40}^{31}, g^4 = -1, h^2 = 1, gh = hg)\right).$$

Denote $A = \mathcal{M}_{3/4}(B)$. The center of the algebra B is $\mathbb{Q}(\sqrt{2})$ and the algebras $\mathbb{Q}(\zeta_8)(h|\zeta_8^h = \zeta_8^{-1}, h^2 = 1)$ and $\mathbb{Q}(\zeta_5)(g|\zeta_5^g = \zeta_5^2, g^4 = -1)$ are simple algebras in B. Furthermore, $\mathbb{Q}(\zeta_8)(h|\zeta_8^h = \zeta_8^{-1}, h^2 = 1) \simeq M_2(\mathbb{Q}(\sqrt{2}))$ and

$$\begin{aligned} B &= M_2(\mathbb{Q}(\sqrt{2})) \otimes_{\mathbb{Q}(\sqrt{2})} (\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_5)(g|\zeta_5^g = \zeta_5^2, g^4 = -1)) \\ &= M_2(\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_5)(g|\zeta_5^g = \zeta_5^2, g^4 = -1)) \end{aligned}$$

Hence, we can formally describe the algebra A as isomorphic to

$$M_{3/2}(\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_5)(g|\zeta_5^g = \zeta_5^2, g^4 = -1))$$

and we conclude that the algebra A is isomorphic to either $M_3(D)$ for some division quaternion algebra over $\mathbb{Q}(\sqrt{2})$ or to $M_6(\mathbb{Q}(\sqrt{2}))$. In fact, in order to decide which one of these options is the correct one, one should compute the local Schur indices of the cyclic algebra

$$C = \mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_5)(g|\zeta_5^g = \zeta_5^2, g^4 = -1) = (\mathbb{Q}(\sqrt{2}, \zeta_5)/\mathbb{Q}(\sqrt{2}), -1).$$

The algebra C has local index 2 at ∞, because $\mathbb{R} \otimes_{\mathbb{Q}(\sqrt{2})} (\mathbb{Q}(\sqrt{2}, \zeta_5)/\mathbb{Q}(\sqrt{2}), -1) \simeq (\mathbb{C}/\mathbb{R}, -1) \simeq \mathbb{H}(\mathbb{R})$. Thus $A \simeq M_3(D)$, for D a division algebra of index 2 and center $\mathbb{Q}(\sqrt{2})$. Notice that D is determined by its Hasse invariants by the Hasse–Brauer–Noether–Albert Theorem. Now we prove that the local indices of A at the finite primes are all 1. By Proposition 2.13, $m_p(A) = 1$ for every finite prime p not dividing 5. Thus, we only have to compute $m_5(A)$. Note that $\zeta_4 \in \mathbb{Q}_5$, so $\mathbb{Q}_5(\sqrt{2}) = \mathbb{Q}_5(\zeta_8)$ is the unique unramified extension of $\mathbb{Q}_5$ of degree 2. Thus $\zeta_8 \in Z(\mathbb{Q}(\sqrt{2})_5 \otimes_{\mathbb{Q}(\sqrt{2})} C)$ and $N_{\mathbb{Q}_5(\sqrt{2},\zeta_5)/\mathbb{Q}_5(\sqrt{2})}(\zeta_8) = -1$. By Proposition 2.11, $m_5(A) = 1$.

Thus, the Hasse invariants of A at the finite primes are all 0 and they are $1/2$ at the two infinite primes, since the algebra has real completion isomorphic to $\mathbb{H}(\mathbb{R})$ at both infinite primes of $\mathbb{Q}(\sqrt{2})$. Using these calculations, one deduces that D is the quaternion algebra $\mathbb{H}(\mathbb{Q}(\sqrt{2}))$ and $A \simeq M_3(\mathbb{H}(\mathbb{Q}(\sqrt{2})))$.

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