

HOPF-GALOIS EXTENSIONS AND AN EXACT SEQUENCE FOR H -PICARD GROUPS

STEFAN CAENEPEEL AND ANDREI MARCUS

ABSTRACT. Let H be a Hopf algebra, and A an H -Galois extension. We investigate H -Morita autoequivalences of A , introduce the concept of H -Picard group, and we establish an exact sequence linking the H -Picard group of A and the Picard group of $A^{\text{co}H}$.

1. INTRODUCTION

The aim of this paper is the following generalization, presented in Section 7 below, of the main result of M. Beattie and A. del Río [4] (see also [14] for an approach based on [13]).

Theorem 1.1. *Assume that H is a cocommutative Hopf algebra over the field k . Let A be a faithfully flat H -Galois extension. There is an exact sequence*

$$1 \rightarrow H^1(H, Z(A^{\text{co}H})) \xrightarrow{g_1} \text{Pic}^H(A) \xrightarrow{g_2} \text{Pic}(A^{\text{co}H})^H \xrightarrow{g_3} H^2(H, Z(A^{\text{co}H})).$$

Here $H^*(H, Z(A^{\text{co}H}))$ are the Sweedler cohomology groups (with respect to the Miyashita-Ulbrich action of H on $Z(A^{\text{co}H})$), $\text{Pic}(A^{\text{co}H})^H$ is the group of H -invariant elements of $\text{Pic}(A^{\text{co}H})$ and $\text{Pic}^H(A)$ is the group of isomorphism classes of invertible relative Hopf bimodules. We shall give later more details about these notations. Moreover, g_1 and g_2 are group-homomorphisms, while g_3 is not.

We give a proof of the theorem by using the ideas of [14] and the results of [6] and [15], obtaining in this way an interesting interpretation of the above theorem in terms of Clifford extendibility to A of $A^{\text{co}H}$ -modules.

The paper is divided as follows. In Section 2 we present our general setting, which involves Hopf-Galois extensions, the Miyashita-Ulbrich action, and most importantly, the concepts of H -Morita context and \square_H -Morita context introduced in [6], and their relationship with Hopf subalgebras. The main result of Section 3 says that if H is cocommutative and A is

2000 *Mathematics Subject Classification.* 16W30, 16D90.

Key words and phrases. Hopf-Galois extension, Morita equivalence, Picard group, cleft extension, Sweedler cohomology.

This research was supported by the bilateral project BWS04/04 “New Techniques in Hopf algebras and graded ring theory” of the Flemish and Romanian governments and by the research project G.0622.06 “Deformation quantization methods for algebras and categories with applications to quantum mechanics” from FWO-Vlaanderen. The second author acknowledges the support of a Bolyai Fellowship of the Hungarian Academy of Science and of the Romanian PN-II-IDEI-PCE-2007-1 project, code ID_532, contract no. 29/28.09.2007.

a faithfully flat H -Galois extension of $B := A^{\text{co}H}$, then the cotensor product $A^{\square_e} := A \square_H A^{\text{op}}$ is a faithfully flat Hopf-Galois extension of the enveloping algebra $B^e := B \otimes B^{\text{op}}$. In the first part of Section 4 we discuss the particular case when A is a cleft extension of the commutative algebra $B := A^{\text{co}H}$, and especially, the characterization of this situation in terms in Sweedler's 1- and 2-cohomology. This is needed in the second part of Section 4, where we review and adapt to our needs the results of Militaru and Ştefan [15] on Clifford extendibility of modules. The cleft extension in discussion is the subalgebra $E := {}_A \text{END}(A \otimes_B M)^{\text{op}}$ of rational elements in ${}_A \text{End}(A \otimes_B M)^{\text{op}}$, where M is an H -invariant B -module, and $E^{\text{co}H} \simeq {}_B \text{End}(M)^{\text{op}}$ is assumed to be commutative. In Section 5 we introduce the H -Picard group $\text{Pic}^H(A)$ and the \square_H -Picard group $\text{Pic}^{\square_H}(A^{\text{co}H})$ of $A^{\text{co}H}$. It is a consequence of the results of [6] that the groups $\text{Pic}^H(A)$ and $\text{Pic}^{\square_H}(A^{\text{co}H})$ are isomorphic. In the situation where H is cocommutative, we can introduce the subgroup $\text{Pic}(A^{\text{co}H})^H$ of $\text{Pic}(A^{\text{co}H})$ consisting of H -stable elements of $\text{Pic}(A^{\text{co}H})$ (Section 6). The definitions of the maps g_1 , g_2 and g_3 , as well as the proof of the main theorem are given in Section 7. The main ingredient here is the application of the Militaru-Stefan lifting theorem to an H -stable invertible (B, B) -bimodule M , by considering the cleft extension $E := {}_{A^{\square_e}} \text{END}(A^{\square_e} \otimes_{B^e} M)^{\text{op}}$ of $E^{\text{co}H} \cong Z(B)$. Note that the action of H on $Z(B)$ coming from E is the same as the Miyashita-Ulbrich action coming from A , hence it is independent of M . Section 8 is concerned with the analysis of the map g_3 . It turns out that the action $\text{Pic}(B)$ on $Z(B)$ induces an action of $\text{Pic}(B)^H$ on $H^n(H, Z(B))$, and that g_3 is a 1-cocycle of the group $\text{Pic}(B)^H$ with values in $H^2(H, Z(B))$.

The exact sequence describing $\text{Pic}^H(A)$ given in Section 7 holds in the case where H is cocommutative; in the general case, we can still give a description of $\text{Pic}^H(A)$, in the case where the coinvariants of A coincide with the groundfield, that is, A is an H -Galois object. This is done in Section 9, and involves Schauenburg's theory of bigalois objects.

Modules will be unital and left, unless otherwise stated. For general results on Hopf algebras the reader is referred to [7], [9] or [16]. For group graded versions of the topics discussed here we also mention [3] and [11].

2. HOPF-GALOIS EXTENSIONS

Throughout this paper, H is a Hopf algebra, with bijective antipode S , over a field k . We use the Sweedler notation for the comultiplication on H : $\Delta(h) = h_{(1)} \otimes h_{(2)}$. \mathcal{M}^H (respectively ${}^H \mathcal{M}$) is the category of right (respectively left) H -comodules. For a right H -coaction ρ (respectively a left H -coaction λ) on a k -module M , we denote

$$\rho(m) = m_{[0]} \otimes m_{[1]} \quad \text{and} \quad \lambda(m) = m_{[-1]} \otimes m_{[0]}.$$

The submodule of coinvariants $M^{\text{co}H}$ of a right (respectively left) H -comodule M consists of the elements $m \in M$ satisfying $\rho(m) = m \otimes 1$ (respectively $\lambda(m) = 1 \otimes m$).

Let A be a right H -comodule algebra. ${}_A \mathcal{M}^H$ and \mathcal{M}_A^H are the categories of left and right relative Hopf modules, and ${}_A \mathcal{M}_A^H$ is the category of relative Hopf bimodules, see [6]. $B = A^{\text{co}H}$ will be the subalgebra of coinvariants of

A. We have two pairs of adjoint functors ($F_1 = A \otimes_B -, G_1 = (-)^{\text{co}H}$) and ($F_2 = - \otimes_B A, G_2 = (-)^{\text{co}H}$) between the categories ${}_B\mathcal{M}$ and ${}_A\mathcal{M}^H$, and between \mathcal{M}_B and \mathcal{M}_A^H . Consider the canonical maps

$$\begin{aligned} \text{can} : A \otimes_B A &\rightarrow A \otimes H, & \text{can}(a \otimes_B b) &= ab_{[0]} \otimes b_{[1]}; \\ \text{can}' : A \otimes_B A &\rightarrow A \otimes H, & \text{can}'(a \otimes_B b) &= a_{[0]}b \otimes a_{[1]}. \end{aligned}$$

We have the following result, due to H.-J. Schneider [18, Theorem I].

Theorem 2.1. *For a right H -comodule algebra A , the following statements are equivalent.*

- (1) (F_2, G_2) is a pair of inverse equivalences;
- (2) (F_2, G_2) is a pair of inverse equivalences and $A \in {}_B\mathcal{M}$ is flat;
- (3) can is an isomorphism and $A \in {}_B\mathcal{M}$ is faithfully flat;
- (4) (F_1, G_1) is a pair of inverse equivalences;
- (5) (F_1, G_1) is a pair of inverse equivalences and $A \in \mathcal{M}_B$ is flat;
- (6) can' is an isomorphism and $A \in \mathcal{M}_B$ is faithfully flat.

If these conditions are satisfied, then we say that A is a faithfully flat H -Galois extension of B .

The Miyashita-Ulbrich action. Let A be a faithfully flat right H -Galois extension, and consider the map

$$\gamma_A = \text{can}^{-1} \circ (\eta_A \otimes H) : H \rightarrow A \otimes_B A, \quad h \mapsto \sum_i l_i(h) \otimes_B r_i(h).$$

Then the element $\gamma_A(h)$ is characterized by the property

$$(1) \quad \sum_i l_i(h)r_i(h)_{[0]} \otimes r_i(h)_{[1]} = 1 \otimes h.$$

For all $h, h' \in H$ and $a \in A$, we have (see [19, 3.4]):

- (2) $\gamma_A(h) \in (A \otimes_B A)^B$;
- (3) $\gamma_A(h_{(1)}) \otimes h_{(2)} = \sum_i l_i(h) \otimes_B r_i(h)_{[0]} \otimes r_i(h)_{[1]}$;
- (4) $\gamma_A(h_{(2)}) \otimes S(h_{(1)}) = \sum_i l_i(h)_{[0]} \otimes_B r_i(h) \otimes l_i(h)_{[1]}$;
- (5) $\sum_i l_i(h)r_i(h) = \varepsilon(h)1_A$;
- (6) $\sum_i a_{[0]}l_i(a_{[1]}) \otimes_B r_i(a_{[1]}) = 1 \otimes_B a$;
- (7) $\gamma_A(hh') = \sum_{i,j} l_i(h')l_j(h) \otimes_B r_j(h)r_i(h')$.

Using the above formulas, it is straightforward to show that $Z(B)$, the center of B , is a right H -module algebra under the Miyashita-Ulbrich action:

$$x \bullet h = \sum_i l_i(h)xr_i(h),$$

for all $x \in Z(B)$, $h \in H$. In what follows, we will view $Z(B)$ as a left H -module algebra via

$$(8) \quad h \cdot x = x \bullet S^{-1}(h) = \sum_i l_i(S^{-1}(h)) x r_i(S^{-1}(h)).$$

We will need the following commutation rule in the sequel.

Lemma 2.2. *For $x \in Z(B)$ and $a \in A$, we have*

$$(9) \quad xa = a_{[0]}(S(a_{[1]}) \cdot x) \quad \text{and} \quad ax = (a_{[1]} \cdot x)a_{[0]}.$$

Proof. From (6), we know that $\sum_i a_{[0]} l_i(a_{[1]}) \otimes_B r_i(a_{[1]}) = 1 \otimes_B a \in B \otimes_B A \subset A \otimes_B A$, and then we can see that

$$x \otimes_B a = \sum_i x a_{[0]} l_i(a_{[1]}) \otimes_B r_i(a_{[1]}) = \sum_i a_{[0]} l_i(a_{[1]}) x \otimes_B r_i(a_{[1]}),$$

hence

$$xa = \sum_i a_{[0]} l_i(a_{[1]}) x r_i(a_{[1]}) = a_{[0]}(S(a_{[1]}) \cdot x).$$

For all $h \in H$, we have that $h \cdot x \in Z(B)$. Apply the first formula of (9) with x replaced by $a_{[1]} \cdot x$; this gives the second formula:

$$(a_{[1]} \cdot x)a_{[0]} = a_{[0]}((S(a_{[1]})a_{[2]}) \cdot x) = ax.$$

□

Morita equivalences. We recall here some concepts and results from [6]. These are the main ingredients in the definition of $\text{Pic}^H(A)$ and of the maps g_1 and g_2 in Theorem 1.1.

Definition 2.3. Let A and A' be right H -comodule algebras. An H -Morita context connecting A and A' is a Morita context $(A, A', M, N, \alpha, \beta)$ such that $M \in {}_A \mathcal{M}_{A'}^H$, $N \in {}_{A'} \mathcal{M}_A^H$, $\alpha : M \otimes_{A'} N \rightarrow A$ is a morphism in ${}_A \mathcal{M}_A^H$ and $\beta : N \otimes_A M \rightarrow A'$ is a morphism in ${}_{A'} \mathcal{M}_{A'}^H$.

Definition 2.4. Assume that A and A' are right faithfully flat H -Galois extensions of $A^{\text{co}H} = B$ and $A'^{\text{co}H} = B'$. A \square_H -Morita context between B and B' is a Morita context $(B, B', M_1, N_1, \alpha_1, \beta_1)$ such that M_1 (resp. N_1) is a left $A \square_H A'^{\text{op}}$ -module (resp. $A' \square_H A^{\text{op}}$ -module) and

- $\alpha_1 : M_1 \otimes_{B'} N_1 \rightarrow B$ is left $A \square_H A'^{\text{op}}$ -linear,
- $\beta_1 : N_1 \otimes_B M_1 \rightarrow B'$ is left $A' \square_H A^{\text{op}}$ -linear.

$\underline{\text{Morita}}(B, B')$ is the category with Morita contexts connecting B and B' as objects. A morphism between the Morita contexts $(B, B', M_1, N_1, \alpha_1, \beta_1)$ and $(B, B', M_2, N_2, \alpha_2, \beta_2)$ is a couple (μ, ν) , with $\mu : M_1 \rightarrow M_2$ and $\nu : N_1 \rightarrow N_2$ bimodule maps such that $\alpha_1 = \alpha_2 \circ (\mu \otimes_{B'} \nu)$ and $\beta_1 = \beta_2 \circ (\nu \otimes_B \mu)$. In a similar way (see [6]), we introduce the categories $\underline{\text{Morita}}^{\square_H}(B, B')$ and $\underline{\text{Morita}}^H(A, A')$.

We recall the following result, see [6, Theorems 5.7 and 5.9].

Theorem 2.5. *Assume that A and A' are right faithfully flat H -Galois extensions of B and B' .*

- (1) *The categories $\underline{\text{Morita}}^H(A, A')$ and $\underline{\text{Morita}}^{\square_H}(B, B')$ are equivalent. The equivalence functors send strict contexts to strict contexts.*

- (2) Let $(B, B', M_1, N_1, \alpha_1, \beta_1)$ be strict Morita context. If M_1 has a left $A \square_H A'^{\text{op}}$ -module structure, then there is a unique left $A' \square_H A^{\text{op}}$ -module structure on N_1 such that $(B, B', M_1, N_1, \alpha_1, \beta_1)$ is a strict \square_H -Morita context. The corresponding strict H -Morita context $(A, B, M, N, \alpha, \beta)$ is given by the following data

$$\begin{aligned} M &= (A \otimes A'^{\text{op}}) \otimes_{A \square_{A'^{\text{op}}}} M_1 \in {}_A \mathcal{M}_{A'}^H; \\ N &= (A' \otimes A^{\text{op}}) \otimes_{A' \square_{A^{\text{op}}}} N_1 \in {}_{A'} \mathcal{M}_A^H; \\ \alpha &= (A \otimes A^{\text{op}}) \otimes_{A \square_{A^{\text{op}}}} \beta_1; \quad \beta = (A' \otimes A'^{\text{op}}) \otimes_{A' \square_{A'^{\text{op}}}} \beta_1. \end{aligned}$$

Hopf subalgebras. Now let K be a Hopf subalgebra of H . We assume that the antipode of K is bijective, and that H is faithfully flat as a left K -module. Let $K^+ = \text{Ker}(\varepsilon_K)$. It is well-known, and easy to prove (see [21, Sec. 1]) that

$$\overline{H} = H/HK^+ \cong H \otimes_K k$$

is a left H -module coalgebra, with operations

$$h \cdot \bar{l} = \overline{hl}, \quad \Delta_{\overline{H}}(\bar{h}) = \bar{h}_{(1)} \otimes \bar{h}_{(2)}, \quad \varepsilon_{\overline{H}}(\bar{h}) = \varepsilon(h).$$

The class in \overline{H} represented by $h \in H$ is denoted by \bar{h} . $\bar{1}$ is a grouplike element of \overline{H} , and we consider coinvariants with respect to this element. A right H -comodule M is also a right \overline{H} -comodule, by corestriction of coscalars:

$$\rho_{\overline{H}}(m) = m_{[0]} \otimes \overline{m}_{[1]}.$$

The \overline{H} -coinvariants of $M \in \mathcal{M}^H$ are then

$$\begin{aligned} M^{\text{co}\overline{H}} &= \{m \in M \mid m_{[0]} \otimes \overline{m}_{[1]} = m \otimes \bar{1}\} \\ &= \{m \in M \mid \rho(m) \in M \otimes K\} \cong M \square_H K. \end{aligned}$$

If A is a right H -comodule algebra, then $A^{\text{co}\overline{H}}$ is a right K -comodule algebra, and $(A^{\text{co}\overline{H}})^{\text{co}K} = A^{\text{co}H}$. In [6, Cor. 7.3], we have seen the following result, based on [19, Remark 1.8].

Proposition 2.6. *Let H , K and A be as above, and assume that A is a faithfully flat H -Galois extension of B . Then $A^{\text{co}\overline{H}}$ is a faithfully flat K -Galois extension of B .*

Let $i : A^{\text{co}\overline{H}} \rightarrow A$ and $j : K \rightarrow H$ be the inclusion maps. Then we have a commutative diagram

$$\begin{array}{ccc} A^{\text{co}\overline{H}} \otimes_B A^{\text{co}\overline{H}} & \xrightarrow{\text{can}_{A^{\text{co}\overline{H}}}} & A^{\text{co}\overline{H}} \otimes K \\ \downarrow i \otimes_B i & & \downarrow i \otimes j \\ A \otimes_B A & \xrightarrow{\text{can}_A} & A \otimes H \end{array}$$

The map $i \otimes j$ is injective (here we use the fact that we work over a field k). From the fact that $\text{can}_{A^{\text{co}\overline{H}}}$ is an isomorphism, it follows that $i \otimes_B i$ is also injective. For $k \in K$, we then have

$$(\text{can}_A \circ (i \otimes_B i))(\gamma_{A^{\text{co}\overline{H}}}) = ((i \otimes j) \circ \text{can}_{A^{\text{co}\overline{H}}})(\gamma_{A^{\text{co}\overline{H}}}) = 1 \otimes j(k),$$

hence

$$(10) \quad (i \otimes_B i)\gamma_{A^{\text{co}\overline{H}}}(k) = \gamma_A(j(k)).$$

3. COTENSOR PRODUCT OF HOPF-GALOIS EXTENSIONS

Troughout this Section, we assume that H is cocommutative. $\Delta : H \rightarrow H \otimes H$ is a Hopf algebra map, so we can consider H as a Hopf subalgebra of $H \otimes H$. Then $H \otimes H$ is a left H -module by restriction of scalars.

Lemma 3.1. *$H \otimes H$ is faithfully flat as a left H -module.*

Proof. Let $H \otimes \langle H \rangle$ be the vector space $H \otimes H$, but with left H -action $h(k \otimes l) = hk \otimes l$. Then $H \otimes H$ and $H \otimes \langle H \rangle$ are isomorphic as left H -modules, and we have the following natural isomorphisms of functors:

$$- \otimes_H (H \otimes H) \cong - \otimes_H (H \otimes \langle H \rangle) \cong - \otimes_k H,$$

and the result follows from the fact that H is faithfully flat as a k -vector space. \square

In a similar way, we have an isomorphism $(H \otimes H) \otimes_H M \cong H \otimes M$, for every left H -module M . In particular, k is a left H -module via the counit ε , so we have an isomorphism

$$f : (H \otimes H) \otimes_H k \rightarrow H, \quad f(\overline{h \otimes k}) = hS(k)$$

of H -module coalgebras, with left H -action on H given by $h \cdot k = \varepsilon(h)k$.

Lemma 3.2. *Let A and A' be faithfully flat H -Galois extensions of B and B' . Then the following statements hold.*

- (1) $A \otimes A'$ is a faithfully flat $H \otimes H$ -Galois extension of $B \otimes B'$.
- (2) $(A \otimes A')^{\text{co}H \otimes H} \cong A \square_H A'$.
- (3) $(A \square_H A')^{\text{co}H} = B \otimes B'$.

Proof. (1) We first show that $(A \otimes A')^{\text{co}(H \otimes H)} = B \otimes B'$. We have a map

$$f : B \otimes B' \rightarrow (A \otimes A')^{\text{co}(H \otimes H)}, \quad f(b \otimes b') = b \otimes b'.$$

$B \otimes B' = (A \otimes B') \cap (B \otimes A')$ and $(A \otimes A')^{\text{co}(H \otimes H)}$ are both subspaces of $A \otimes A'$, so it suffices to show that f is surjective. Take $\sum_i a_i \otimes a'_i \in (A \otimes A')^{\text{co}(H \otimes H)}$. Then

$$\sum_i a_{i[0]} \otimes a'_{i[0]} \otimes a_{i[1]} \otimes a'_{i[1]} = \sum_i a_i \otimes a'_i \otimes 1 \otimes 1.$$

Applying ε to the fourth tensor factor, we find

$$\sum_i a_{i[0]} \otimes a'_i \otimes a_{i[1]} = \sum_i a_i \otimes a'_i \otimes 1.$$

This means that $\sum_i a_i \otimes a'_i \in B \otimes A'$. In a similar way, we find that $\sum_i a_i \otimes a'_i \in A \otimes B'$.

It is easy to show that $\text{can}_{A \otimes A'}$ is bijective. Finally $A \otimes A'$ is faithfully flat as a right $B \otimes B'$ -module: $B \otimes A'$ is faithfully flat as a right $B \otimes B'$ -module because for every left $B \otimes B'$ -module M there is a natural isomorphism $(B \otimes A') \otimes_{B \otimes B'} M \cong A' \otimes_{B'} M$. Similarly, $A \otimes A'$ is faithfully flat as a right $B \otimes A'$ -module. Then apply the following general property: if $f : A \rightarrow B$ and $g : B \rightarrow C$ are algebra morphisms, and B/A and C/B are faithfully flat, then C/A is faithfully flat.

(2) We can apply Proposition 2.6, with H replaced by $H \otimes H$, K by H and A by $A \otimes A'$. Note that $\sum_i a_i \otimes a'_i \in (A \otimes A')^{\text{co}H \otimes H}$ if and only if

$$\sum_i a_{i[0]} \otimes a'_{i[0]} \otimes a_{i[1]} S(a'_{i[1]}) = \sum_i a_i \otimes a'_i \otimes 1,$$

or

$$\sum_i a_{i[0]} \otimes a'_i \otimes a_{i[1]} = \sum_i a_i \otimes a'_{i[0]} \otimes a'_{i[1]},$$

which means precisely that $\sum_i a_i \otimes a'_i \in A \square_H A'$.

(3) We know that $A \square_H A'$ is a right H -comodule algebra with structure map ρ given by

$$(11) \quad \rho\left(\sum_i a_i \otimes a'_i\right) = \sum_i a_{i[0]} \otimes a'_i \otimes a_{i[1]} = \sum_i a_i \otimes a'_{i[0]} \otimes a'_{i[1]}.$$

Take $x = \sum_i a_i \otimes a'_i \in (A \square_H A')^{\text{co}H}$. It follows from (11) that $x \in (B \otimes A') \cap (A \otimes B') = B \otimes B'$. \square

Combining these observations with Proposition 2.6, we obtain the following result, which is well-known in the situation where $B = B' = k$.

Theorem 3.3. *Let A and A' be faithfully flat H -Galois extensions of B and B' . Then $A \square_H A'$ is a faithfully flat H -Galois extension of $B \otimes B'$.*

We want to apply this theorem in the case when A' is the opposite algebra A^{op} . Since H is cocommutative, A^{op} is a right H -comodule algebra, with coaction ρ given by

$$\rho(a) = a_{[0]} \otimes S(a_{[1]}).$$

Lemma 3.4. *If A is a faithfully flat H -Galois extension of B , then A^{op} is a faithfully flat H -Galois extension of B^{op} .*

Proof. The map $\text{can}_{A^{\text{op}}} : A^{\text{op}} \otimes_{B^{\text{op}}} A^{\text{op}} \rightarrow A^{\text{op}} \otimes H$ is given by

$$\text{can}_{A^{\text{op}}}(a \otimes a') = a'_{[0]} a \otimes S(a'_{[1]}) = (A^{\text{op}} \otimes S) \circ \text{can}'_A.$$

Then $\text{can}_{A^{\text{op}}}$ is bijective since can'_A and S are bijective. We know from Theorem 2.1 that $A \in \mathcal{M}_B$ is faithfully flat, and this implies that $A^{\text{op}} \in {}_{B^{\text{op}}}\mathcal{M}$ is faithfully flat. It then follows from Theorem 2.1 that A^{op} is also a faithfully flat H -Galois extension. \square

Proposition 3.5. *Let A be a faithfully flat H -Galois extension of B . Then $A^{\square e} := A \square_H A^{\text{op}}$ is a faithfully flat H -Galois extension of the enveloping algebra $B^e := B \otimes B^{\text{op}}$. Moreover, the element*

$$(12) \quad \gamma_{A^{\square e}}(h) := \sum_{i,j} (l_i(h_{(1)}) \otimes r_j(h_{(2)})) \otimes_{B \otimes B^{\text{op}}} (r_i(h_{(1)}) \otimes l_j(h_{(2)}))$$

belongs to $A^{\square e} \otimes_{B^e} A^{\square e}$.

Proof. First observe that $\text{can}_{A^e} : A^e \otimes_{B^e} A^e \rightarrow A^e \otimes H \otimes H$ is given by

$$\text{can}_{A \otimes A^{\text{op}}}((a \otimes b) \otimes (a' \otimes b')) = aa'_{[0]} \otimes b'_{[0]} b \otimes a'_{[1]} \otimes S(b'_{[1]}).$$

Recall the notation $\gamma_A(h) := \sum_i l_i(h) \otimes_B r_i(h)$. Then we compute that

$$\begin{aligned} & \text{can}_{A^e} \left(\sum_{i,j} (l_i(h_{(1)}) \otimes r_j(h_{(2)})) \otimes_{B^e} (r_i(h_{(1)}) \otimes l_j(h_{(2)})) \right) \\ &= \sum_{i,j} l_i(h_{(1)}) r_i(h_{(1)})_{[0]} \otimes l_j(h_{(2)})_{[0]} r_j(h_{(2)}) \otimes r_i(h_{(1)})_{[1]} \otimes S(l_j(h_{(2)})_{[1]}) \\ &\stackrel{(1,4)}{=} \sum_j 1 \otimes l_j(h_{(3)}) r_j(h_{(3)}) \otimes h_{(1)} \otimes S(S(h_{(2)})) \stackrel{(5)}{=} 1 \otimes 1 \otimes \Delta(h). \end{aligned}$$

Let $i : A^{\square e} \rightarrow A^e$ be the canonical injection. It follows from (10) that

$$\begin{aligned} (i \otimes_{B \otimes B^{\text{op}}} i)(\gamma_{A^{\square e}}(h)) &= \gamma_{A^e}(\Delta(h)) \\ &= \sum_{i,j} (l_i(h_{(1)}) \otimes r_j(h_{(2)})) \otimes_{B^e} (r_i(h_{(1)}) \otimes l_j(h_{(2)})), \end{aligned}$$

and the statement is proved. \square

4. CLEFT EXTENSIONS AND THE LIFTING THEOREM

In this Section, we adapt and review the results from [15], going back to older results from graded Clifford theory, see [8].

Cleft extensions.

Proposition 4.1. *Let H be a Hopf algebra, A a right H -comodule algebra, and $B = A^{\text{co}H}$. We have a category \mathcal{C}_A , with two objects $\mathbf{1}$ and $\mathbf{2}$, and morphisms*

$$\mathcal{C}_A(\mathbf{1}, \mathbf{1}) = \text{Hom}(H, B) ; \quad \mathcal{C}_A(\mathbf{1}, \mathbf{2}) = \text{Hom}^H(H, A);$$

$$\mathcal{C}_A(\mathbf{2}, \mathbf{1}) = \{u : H \rightarrow A \mid \rho(u(h)) = u(h_{(2)}) \otimes S(h_{(1)}), \text{ for all } h \in H\};$$

$$\mathcal{C}_A(\mathbf{2}, \mathbf{2}) = \{w : H \rightarrow A \mid \rho(w(h)) = w(h_{(2)}) \otimes S(h_{(1)})h_{(3)}, \text{ for all } h \in H\}.$$

The composition of morphisms is given by the convolution product.

Recall that A is called H -cleft if there exists a convolution invertible $t \in \text{Hom}^H(H, A)$, or, equivalently, if $\mathbf{1}$ and $\mathbf{2}$ are isomorphic in \mathcal{C}_A . Then $t(1)^{-1} = u(1)$, and $t' = u(1)t \in \text{Hom}^H(H, A)$ has convolution inverse $ut(1)$, and $t'(1) = 1$. So if A is H -cleft, then there exists a convolution invertible $t \in \text{Hom}^H(H, A)$ with $t(1) = 1$.

If H is cocommutative, then $\mathcal{C}_A(\mathbf{1}, \mathbf{1}) = \mathcal{C}_A(\mathbf{2}, \mathbf{2})$.

If $t \in \text{Hom}^H(H, A)$ is an algebra map, then t is convolution invertible (with convolution inverse $t \circ S$), so A is H -cleft. Consider the space

$$\Omega_A = \{t \in \text{Hom}^H(H, A) \mid t \text{ is an algebra map}\}.$$

We have the following equivalence relation on Ω_A : $t_1 \sim t_2$ if and only if there exists $b \in U(B)$ such that $bt_1(h) = t_2(h)b$, for all $h \in H$. We denote $\bar{\Omega}_A = \Omega_A / \sim$.

Take $t \in \text{Hom}^H(H, A)$ with convolution inverse u such that $t(1_H) = 1_A$, and consider the map

$$\omega_t : H \otimes B \rightarrow B, \quad \omega_t(h \otimes b) = t(h_{(1)})bu(h_{(2)}).$$

Assume that $\Omega_A \neq \emptyset$, and fix $t_0 \in \Omega_A$ with convolution inverse u_0 . Now consider the bijection

$$F : \mathcal{C}_A(\mathbf{1}, \mathbf{1}) = \text{Hom}(H, B) \rightarrow \mathcal{C}_A(\mathbf{1}, \mathbf{2}) = \text{Hom}^H(H, A),$$

$F(v) = v * t_0$, $F^{-1}(t) = t * u_0$. It is then easy to show that $F(v) \in \Omega_A$ if and only if

$$(13) \quad v(hk) = v(h_{(1)})\omega_{t_0}(h_{(2)} \otimes v(k))$$

and $v(1_H) = 1_B$. If (13) holds, then $v(1_H) = 1_B$ if and only if v is convolution invertible. Moreover, $F(v) \sim t_0$ if and only if $v(h) = \omega_{t_0}(h \otimes b)b^{-1}$ for some invertible $b \in B$.

We will now discuss when $F^{-1}(\Omega_A)$ is a subgroup of $\text{Hom}(H, B)$.

Proposition 4.2. *Let H be cocommutative, and let A be an H -cleft right H -comodule algebra. Assume that $B = A^{\text{co}H}$ is commutative. Choose $t \in \text{Hom}^H(H, A)$ with convolution inverse u , such that $t(1) = 1$ and, a fortiori, $u(1_H) = 1_A$. Then we have the following properties.*

- (1) ω_t is independent of the choice of t ;
- (2) $ab = \omega_t(a_{[1]} \otimes b)a_{[0]}$, for all $a \in A$ and $b \in B$.

If $\Omega_A \neq \emptyset$, then we have an algebra map $t \in \text{Hom}^H(H, A)$, and then the map ω_t defines a left H -module algebra structure on B , and we can consider the Sweedler cohomology groups $H^n(H, B)$, see [20]. We then denote $h \cdot b = \omega_t(h \otimes b)$.

Proposition 4.3. *Assume that $\Omega_A \neq \emptyset$. Then $\Omega_A \cong Z^1(H, B)$ and $\bar{\Omega}_A \cong H^1(H, B)$.*

Proof. (sketch) If H is cocommutative and B is commutative, then (13) is equivalent to

$$v(hk) = (h_{(1)} \cdot v(k))v(h_{(2)}),$$

which is precisely the condition that v is a Sweedler 1-cocycle. \square

Proposition 4.4. *Now assume that $B = k$; it is not necessary that H is cocommutative. If $\Omega_A \neq \emptyset$, then $\Omega_A \cong \text{Alg}(H, k)$.*

Proof. In this situation, $\omega_t(h \otimes b) = \varepsilon(h)b$, for every choice of t . Then (13) is equivalent to $v(hk) = v(h)v(k)$, and the result follows. \square

Suppose that A is H -cleft. Pick a convolution invertible $t \in \text{Hom}^H(H, A)$ such that $t(1) = 1$. Then consider

$$\sigma : H \otimes H \rightarrow B, \quad \sigma(h \otimes k) = t(h_{(1)})t(k_{(1)})u(h_{(2)}k_{(2)}).$$

Let $B\#_{\sigma}H$ be equal to $B \otimes H$ as a vector space, with right H -coaction $\rho = B \otimes \Delta$, and with multiplication

$$(b\#h)(c\#k) = b(h \cdot c)\sigma(h_{(1)} \otimes k_{(1)})h_{(2)}k_{(2)}.$$

Proposition 4.5. *The map $\phi : B\#_{\sigma}H \rightarrow A$, $\phi(b\#h) = bt(h)$ is an isomorphism of right H -comodule algebras. The inverse of ϕ is given by the formula $\phi^{-1}(a) = a_{[0]}u(a_{[1]})\#a_{[2]}$. Let $\sigma \in Z^2(H, B)$. The following statements are equivalent:*

- (1) $\sigma \in B^2(H, B)$;

- (2) *there exists an algebra map $t' \in \text{Hom}^H(H, A)$;*
(3) $A \cong B \#_{\varepsilon \otimes \varepsilon} H$.

The Militaru-Stefan lifting Theorem. Let A be a faithfully flat H -Galois extension of $B = A^{\text{co}H}$. ${}_A\mathcal{M}^H$ will denote the category of (left-right) relative Hopf modules. Let $P, Q \in {}_A\mathcal{M}^H$. A left A -linear map $f : P \rightarrow Q$ is called *rational* if there exists a (unique) element $f_{[0]} \otimes f_{[1]} \in {}_A\text{Hom}(P, Q) \otimes H$ such that

$$f_{[0]}(p) \otimes f_{[1]} = f(p_{[0]})_{[0]} \otimes S^{-1}(p_{[1]})f(p_{[0]})_{[1]},$$

or, equivalently,

$$(14) \quad \rho(f(p)) = f_{[0]}(p_{[0]}) \otimes p_{[1]}f_{[1]},$$

for all $p \in P$. The subset of ${}_A\text{Hom}(P, Q)$ consisting of rational maps is denoted by ${}_A\text{HOM}(P, Q)$. This is a right H -comodule, and ${}_A\text{END}(P)^{\text{op}}$ is a right H -comodule algebra.

Now take $M \in {}_B\mathcal{M}$. Then $A \otimes_B M \in {}_A\mathcal{M}^H$, and $E = {}_A\text{END}(A \otimes_B M)^{\text{op}}$ is a right H -comodule algebra. From the category equivalence between ${}_B\mathcal{M}$ and ${}_A\mathcal{M}^H$, it follows that

$$F := E^{\text{co}H} = {}_A\text{End}^H(A \otimes_B M)^{\text{op}} \cong {}_B\text{End}(M)^{\text{op}}.$$

B can be viewed as a right H -comodule algebra, with trivial coaction $\rho(b) = b \otimes 1$, for all $b \in B$, so we can consider the category of relative Hopf modules ${}_B\mathcal{M}^H$. If M is a left B -module, then $A \otimes_B M$ and $M \otimes H$ are objects of ${}_B\mathcal{M}^H$. \mathcal{D}_M will be the full subcategory of ${}_B\mathcal{M}^H$, with two objects $A \otimes_B M$ and $M \otimes H$. We then have the following result.

Theorem 4.6. *Let A be a faithfully flat H -Galois extension of $B = A^{\text{co}H}$ and $M \in {}_B\mathcal{M}$. Then the categories \mathcal{C}_E and \mathcal{D}_M are anti-isomorphic.*

Proof. (sketch) We define a contravariant functor $\alpha : \mathcal{C}_E \rightarrow \mathcal{D}_M$ at the objects level in the following obvious way: $\alpha(\mathbf{1}) = M \otimes H$ and $\alpha(\mathbf{2}) = A \otimes_B M$. Before we state the definition at the morphisms level, we observe that we have two natural isomorphisms

$$\beta_1 : {}_B\text{Hom}(A \otimes_B M, M) \rightarrow {}_B\text{Hom}^H(A \otimes_B M, M \otimes H);$$

$$\beta_2 : {}_B\text{Hom}(M \otimes H, M) \rightarrow {}_B\text{End}^H(M \otimes H)$$

defined as follows:

$$\beta_1(\phi)(a \otimes_B m) = \phi(a_{[0]} \otimes_B m) \otimes a_{[1]}; \quad \beta_1^{-1}(\varphi) = (M \otimes \varepsilon) \circ \varphi;$$

$$\beta_2(\Theta)(m \otimes h) = \Theta(m \otimes h_{(1)}) \otimes h_{(2)}; \quad \beta_2^{-1}(\theta) = (M \otimes \varepsilon) \circ \theta.$$

Consider $\eta_M : M \rightarrow (A \otimes_B M)^{\text{co}H}$, the unit of the adjunction (F_2, G_2) (see Section 2) evaluated at M . Since F_2 is an equivalence of categories, η_M is an isomorphism. We have an isomorphism

$$\tilde{\alpha}_{\mathbf{11}} : \mathcal{C}_E(\mathbf{1}, \mathbf{1}) = \text{Hom}(H, E^{\text{co}H}) \rightarrow {}_B\text{Hom}(M \otimes H, M),$$

given by the formulas

$$\tilde{\alpha}_{\mathbf{11}}(v)(m \otimes h) = \eta_M^{-1}(v(h)(1 \otimes_B m));$$

$$\tilde{\alpha}_{\mathbf{11}}^{-1}(\Theta)(h)(a \otimes_B m) = a \otimes_B \Theta(m \otimes h).$$

We then define $\alpha_{11} = \beta_2 \circ \tilde{\alpha}_{11}$. The isomorphism

$$\alpha_{12} : \mathcal{C}_E(\mathbf{1}, \mathbf{1}) = \text{Hom}^H(H, E) \rightarrow {}_B\text{Hom}^H(M \otimes H, A \otimes_B M)$$

is given by the formulas

$$\alpha_{12}(t)(m \otimes h) = t(h)(1 \otimes_B m) ; (\alpha_{12}^{-1}(\psi)(h))(a \otimes_B m) = a\psi(m \otimes h).$$

We have an isomorphism

$$\tilde{\alpha}_{21} : \mathcal{C}_E(\mathbf{2}, \mathbf{1}) \rightarrow {}_B\text{Hom}(A \otimes_B M, M),$$

given by the formulas

$$\begin{aligned} \tilde{\alpha}_{21}(u)(a \otimes_B m) &= \eta_M^{-1}(u(a_{[1]})(a_{[0]} \otimes_B m)); \\ (\tilde{\alpha}_{21}^{-1}(\phi)(h))(a \otimes_B m) &= \sum_i a l_i(h) \otimes_B \phi(r_i(h) \otimes_B m). \end{aligned}$$

We then define $\alpha_{21} = \beta_1 \circ \tilde{\alpha}_{21}$. Finally, the isomorphism

$$\alpha_{22} : \mathcal{C}_E(\mathbf{2}, \mathbf{1}) \rightarrow {}_B\text{End}^H(A \otimes_B M)^{\text{op}}$$

is given by the formulas

$$\begin{aligned} \alpha_{22}(w)(a \otimes_B m) &= w(a_{[1]})(a_{[0]} \otimes_B m); \\ (\alpha_{22}^{-1}(\kappa))(h)(a \otimes_B m) &= \sum_i a l_i(h) \kappa(r_i(h) \otimes_B m). \end{aligned}$$

A long computation shows that α_{22} is a well-defined isomorphism, and that α is a functor. \square

Recall from [19] that $M \in {}_B\mathcal{M}$ is called H -stable if $A \otimes_B M$ and $M \otimes H$ are isomorphic as left B -modules and right H -comodules, or, equivalently, the two objects of \mathcal{D}_M are isomorphic. From Theorem 4.6 we immediately deduce the following result.

Corollary 4.7. *$M \in {}_B\mathcal{M}$ is H -stable if and only if there exists a convolution invertible $t \in \text{Hom}^H(H, E)$.*

Assume that $M \in {}_B\mathcal{M}$ is H -stable. Then there is an isomorphism $\varphi : A \otimes_B M \rightarrow M \otimes H$ in ${}_B\mathcal{M}^H$. Let $\psi = \varphi^{-1}$, $\phi = (M \otimes \varepsilon) \circ \varphi$, $t = \alpha_{12}^{-1}(\psi)$, $u = \alpha_{21}^{-1}(\varphi)$. Then the following assertions are equivalent.

- (1) $t(1) = 1$;
- (2) $u(1) = 1$;
- (3) $\psi(m \otimes 1) = 1 \otimes_B m$, for all $m \in M$;
- (4) $\phi(1 \otimes_B m) = m$, for all $m \in M$.

Indeed, the equivalences 1) \iff 2) and 3) \iff 4) are obvious, and 1) \iff 3) follows immediately from the definition of α_{12} and α_{21}^{-1} .

We have seen (cf. comments following Proposition 4.1) that t' and u' given by $t'(h) = t(h) \circ u(1)$ and $u'(h) = t(1) \circ u(h)$ are convolution inverses, satisfying the additional condition $t'(1) = u'(1) = 1$. Thus $\psi' = \alpha_1(t')$ satisfies (3), and $\phi' = \tilde{\alpha}_2(u')$ satisfies (4). ψ' and ϕ' can be computed explicitly, using the formulas given in the proof of Theorem 4.6:

$$\psi'(m \otimes h) = \psi(\phi(1 \otimes_B m) \otimes h) ; 1 \otimes_B \phi'(a \otimes_B m) = \psi(\phi(a \otimes_B m) \otimes 1).$$

ψ' and φ' are composition inverses. The proof of the following result is now a straightforward exercise.

Proposition 4.8. *Take $\phi \in {}_B\text{Hom}(A \otimes_B M, M)$, and let $u = \tilde{\alpha}_{\mathbf{21}}^{-1}(\phi) \in \mathcal{C}(\mathbf{2}, \mathbf{1})$ and $t = u \circ S^{-1} \in \mathcal{C}(\mathbf{1}, \mathbf{2}) = \text{Hom}^H(H, E)$. Then the following statements are equivalent:*

- (1) $\phi : A \otimes_B M \rightarrow M$, $\phi(a \otimes_B m) = a \cdot m$ is an associative left A -action on M ;
- (2) u is an anti-algebra map;
- (3) t is an algebra map.

Proposition 4.9. *For $i = 1, 2$, take $\phi_i \in {}_B\text{Hom}(A \otimes_B M, M)$, and consider $u_i = \tilde{\alpha}_{\mathbf{21}}^{-1}(\phi_i) \in \text{Hom}^S(H, E)$ and $t_i = u_i \circ S^{-1} \in \text{Hom}^H(H, E)$. Let $M_i = M$ as a left B -module, with left A -action defined by ϕ_i . Then $M_1 \cong M_2$ if and only if $t_1 \sim t_2$.*

Proof. We have that $t_1 \sim t_2$ if and only if there exists an invertible map $f \in {}_B\text{End}(M) \cong E^{\text{co}H}$ such that $t_1(h) \circ (A \otimes_B f) = (A \otimes_B f) \circ t_2(h)$, or, equivalently, $u_1(h) \circ (A \otimes_B f) = (A \otimes_B f) \circ u_2(h)$, for all $h \in H$. This implies that

$$\begin{aligned} 1 \otimes_B \phi_1(a \otimes_B f(m)) &= u_1(a_{[1]})(a_{[0]} \otimes_B f(m)) \\ &= (u_1(a_{[1]}) \circ (A \otimes_B f))(a_{[0]} \otimes_B m) \\ &= ((A \otimes_B f) \circ u_2(a_{[1]}))(a_{[0]} \otimes_B m) \\ &= 1 \otimes_B f(\phi_2(a \otimes_B f(m))), \end{aligned}$$

and $\phi_1(a \otimes_B f(m)) = f(\phi_2(a \otimes_B f(m)))$, for all $a \in A$ and $m \in M$, which means that $f : M_2 \rightarrow M_1$ is an isomorphism of left A -modules.

Conversely, let $f : M_2 \rightarrow M_1$ is an isomorphism of left A -modules. Then $f : M \rightarrow M$ is left B -linear, so $f \in {}_B\text{End}(M)$. Then we have, for all $h \in H$, $a \in A$ and $m \in M$, that

$$\begin{aligned} u_1(h)(a \otimes_B f(m)) &= \sum_i a l_i(h) \otimes_B \phi_1(r_i(h) \otimes_B f(m)) \\ &= \sum_i a l_i(h) \otimes_B f(\phi_2(r_i(h) \otimes_B m)) = (A \otimes_B f)(u_2(h)(a \otimes_B m)), \end{aligned}$$

hence $u_1(h) \circ (A \otimes_B f) = (A \otimes_B f) \circ u_2(h)$, as needed. \square

As an immediate consequence, we obtain the Militaru-Ştefan lifting Theorem.

Corollary 4.10. *Let A be a faithfully flat H -Galois extension of $B = A^{\text{co}H}$ and $M \in {}_B\mathcal{M}$. There is a bijective correspondence between the isomorphism classes of left A -module structures on M extending the B -module structure on M and the elements of $\overline{\Omega}_E$.*

Example 4.11. Let A be an H -Galois object, that is, $A^{\text{co}H} = k$, and $M = k$. Then $E = {}_A\text{End}(A)^{\text{op}} \cong A$ as an H -comodule algebra, and $E^{\text{co}H} = A^{\text{co}H} = k$. The map $\tilde{\alpha}_{\mathbf{21}} : \mathcal{C}_A(\mathbf{1}, \mathbf{2})$ and its inverse are given by the formulas

$$\tilde{\alpha}_{\mathbf{21}}(u)(a) = u(a_{[1]})a_{[0]} \in A^{\text{co}H} = k;$$

$$\tilde{\alpha}_{\mathbf{21}}^{-1}(\phi)(h) = \sum_i l_i(h)\phi(r_i(h)).$$

$\phi \in A^*$ defines an A -action $A \otimes k \rightarrow k$ if and only if ϕ is an algebra map. It follows from Corollary 4.10 that $\overline{\Omega}_A = \text{Alg}(A, k)$. If $\overline{\Omega}_A \neq \emptyset$, then it follows from Proposition 4.4 that $\text{Alg}(A, k) \cong \text{Alg}(H, k)$. The correspondence goes as follows. Fix $\phi_0 \in \text{Alg}(A, k)$. $\phi \in \text{Alg}(A, k)$ corresponding to $v \in \text{Alg}(H, k)$ is given by the formula

$$\phi(a) = v(S(a_{[1]}))l_i(a_{[2]})\phi_0(r_i(a_{[2]}))a_{[0]}.$$

5. PICARD GROUPS

The Picard group of an H -comodule algebra. Consider a Hopf algebra H with bijective antipode and an H -comodule algebra A . Let $\underline{\text{Pic}}^H(A)$ be the category with strict H -Morita contexts of the form $(A, A, P, Q, \alpha, \beta)$ as objects. A morphism between $(A, A, P_1, Q_1, \alpha_1, \beta_1)$ and $(A, A, P_2, Q_2, \alpha_2, \beta_2)$ consists of a couple (f, g) , with $f : P_1 \rightarrow P_2$, $g : Q_1 \rightarrow Q_2$ H -colinear A -bimodule isomorphisms such that $\alpha_1 = \alpha_2 \circ (f \otimes_B g)$ and $\beta_1 = \beta_2 \circ (g \otimes_A f)$. Note that $\underline{\text{Pic}}^H(A)$ has the structure of monoidal category, where the tensor product is given by the formula

$$(A, A, P_1, Q_1, \alpha_1, \beta_1) \otimes (A, A, P_2, Q_2, \alpha_2, \beta_2) = (A, A, P_1 \otimes_A P_2, Q_2 \otimes_A Q_1, \alpha_1 \circ (P_1 \otimes_A \alpha_2 \otimes_A Q_1), \beta_2 \circ (Q_2 \otimes_A \beta_1 \otimes_A P_1)).$$

The unit object is (A, A, A, A, A, A) . Every object $(A, A, P_1, Q_1, \alpha_1, \beta_1)$ of $\underline{\text{Pic}}^H(A)$ has an inverse, namely $(A, A, Q_1, P_1, \beta_1, \alpha_1)$.

Up to isomorphism, a strict H -Morita context is completely determined by one of its underlying bimodules; therefore, we use the shorter notation $\underline{P}_1 = (A, A, P_1, Q_1, \alpha_1, \beta_1)$. $\text{Pic}^H(A) = K_0 \underline{\text{Pic}}^H(A)$, the set of isomorphism classes in $\underline{\text{Pic}}^H(A)$, is a group under the operation induced by the tensor product, and is called the H -Picard group of A . If $H = k$, and B is a k -algebra, then $\text{Pic}^k(B) = \text{Pic}(B)$ is the classical Picard group of B .

The \square -Picard group of B . Let $M, N \in {}_{A^{\square e}}\mathcal{M}$. In [6], it is shown that $M \otimes_B N \in {}_{A^{\square e}}\mathcal{M}$. We will need an explicit formula for the $A^{\square e}$ -action on $M \otimes_B N$, given in Proposition 5.1 below.

In the proof [6, Theorem 2.4], it is shown that we have an isomorphism

$$\alpha_N : A \otimes_B N \rightarrow A^e \otimes_{A^{\square e}} N, \quad \alpha_N(a \otimes_B n) = (a \otimes 1) \otimes_{A^{\square e}} n.$$

We claim that the inverse α_N^{-1} of α_N is given by the formula

$$\alpha_N^{-1}((d \otimes e) \otimes_{A^{\square e}} n) = \sum_i dl_i(S(e_{[1]})) \otimes_B (r_i(S(e_{[1]})) \otimes e_{[0]}) \cdot n.$$

It follows from Lemma 6.4 that α_N^{-1} is well-defined. Using the property that $\gamma_A(1_H) = 1_A \otimes_B 1_A$, we find that

$$(\alpha_N^{-1} \circ \alpha_N)(a \otimes_B n) = \alpha_N^{-1}((a \otimes 1) \otimes_{A^{\square e}} n) = a \otimes_B n.$$

We also compute that

$$\begin{aligned}
& (\alpha_N \circ \alpha_N^{-1})((d \otimes e) \otimes_{A^{\square e}} n) \\
&= \alpha_N \left(\sum_i dl_i(S(e_{[1]})) \otimes_B (r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n) \right) \\
&= \sum_i (dl_i(S(e_{[1]})) \otimes 1) \otimes_{A^{\square e}} (r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n) \\
&= \left(\sum_i dl_i(S(e_{[1]})) r_i(S(e_{[1]})) \otimes_{e_{[0]}} \right) \otimes_{A^{\square e}} n \stackrel{(5)}{=} (d \otimes e) \otimes_{A^{\square e}} n.
\end{aligned}$$

Using α_N , the left A^e -action on $A^e \otimes_{A^{\square e}} N$ can be transported to a left A^e -action on $A \otimes_B N$:

$$\begin{aligned}
& (d \otimes e)(a \otimes_B n) = \alpha_N^{-1}((d \otimes e)\alpha_N(a \otimes_B n)) \\
&= \alpha_N^{-1}((da \otimes e) \otimes_{A^{\square e}} n) = \sum_i dal_i(S(e_{[1]})) \otimes_B (r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n).
\end{aligned}$$

If $M, N \in {}_A \mathcal{M}^H$, then $A^e \otimes_{A^{\square e}} M, A^e \otimes_{A^{\square e}} N \in {}_A \mathcal{M}_A^H$, hence

$$(A^e \otimes_{A^{\square e}} M) \otimes_A (A^e \otimes_{A^{\square e}} N) \cong (A \otimes_B M) \otimes_A (A \otimes_B N) \cong A \otimes_B M \otimes_B N$$

in the category ${}_A \mathcal{M}_A^H$. On $(A \otimes_B M) \otimes_A (A \otimes_B N)$, the A -bimodule structure (or left A^e -module structure) is given by the formula

$$\begin{aligned}
& (d \otimes e) \cdot ((a \otimes_B m) \otimes_A (a' \otimes_B n)) \\
&= (d \otimes 1) \cdot (a \otimes_B m) \otimes_A (a \otimes e) \cdot (a' \otimes_B n) \\
&= \sum_i (da \otimes_B m) \otimes_A (a' l_i(S(e_{[1]})) \otimes_B (r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n)).
\end{aligned}$$

We transport this left A^e -module structure to $A \otimes_B M \otimes_B N$:

$$\begin{aligned}
& (d \otimes e) \cdot (a \otimes_B m \otimes_B n) \\
&= \sum_i (1 \otimes l_i(S(e_{[1]}))) \cdot (da \otimes_B m) \otimes_B (r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n) \\
&= \sum_{i,j} dal_j \left(S(l_i(S(e_{[1]}))_{[1]}) \right) \\
&\quad \otimes_B \left(r_j \left(S(l_i(S(e_{[1]}))_{[1]}) \right) \otimes r_i(S(e_{[1]}))_{[0]} \right) \cdot m \\
&\quad \otimes_B \left(r_i(S(e_{[1]})) \otimes_{e_{[0]}} \cdot n \right) \\
&\stackrel{(4)}{=} \sum_{i,j} dal_j(S(e_{[1]})) \otimes_B (r_j(S(e_{[1]})) \otimes l_i(S(e_{[2]}))) \cdot m \\
&\quad \otimes_B (r_i(S(e_{[2]}) \otimes_{e_{[0]}} \cdot n).
\end{aligned}$$

Now take $\sum_k a_k \otimes a'_k \in A^{\square e}$. Using the above formula, we compute that

$$\begin{aligned}
& \left(\sum_k a_k \otimes a'_k \right) \cdot (1 \otimes_B m \otimes_B n) = \sum_{i,j,k} a_k l_j(S(a'_{k[1]})) \\
&\quad \otimes_B (r_j(S(a'_{k[1]})) \otimes l_i(S(a'_{k[2]}))) \cdot m \otimes_B (r_i(S(a'_{k[2]})) \otimes a'_{k[0]}) \cdot n
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j,k} a_{k[0]} l_j(S(a_{k[1]})) \otimes_B (r_j(S(a_{k[1]})) \otimes l_i(S(a_{k[2]}))) \cdot m \\
 &\quad \otimes_B (r_i(S(a_{k[2]})) \otimes a'_k) \cdot n \\
 &\stackrel{(6)}{=} 1 \otimes_B (a_{k[0]} \otimes l_i(a_{k[1]})) \cdot m \otimes_B (r_i(a_{k[1]}) \otimes a'_k) \cdot n
 \end{aligned}$$

The map

$$M \otimes_B N \rightarrow (A \otimes_B M \otimes_B N)^{\text{co}H}, \quad m \otimes_B n \mapsto 1 \otimes_B m \otimes_B n$$

is an isomorphism. Hence the left $A^{\square e}$ -action on $A \otimes_B M \otimes_B N$ restricts to an action on $(A \otimes_B M \otimes_B N)^{\text{co}H}$, and defines an action on $M \otimes_B N$. We can summarize this as follows.

Proposition 5.1. *Let $M, N \in {}_{A^{\square e}}\mathcal{M}$. Then we have the following action on $M \otimes_B N$:*

$$(15) \quad \left(\sum_k a_k \otimes a'_k \right) \cdot (m \otimes_B n) = (a_{k[0]} \otimes l_i(a_{k[1]})) \cdot m \otimes_B (r_i(a_{k[1]}) \otimes a'_k) \cdot n.$$

Now let $\underline{\text{Pic}}^{\square H}(B)$ be the category with strict \square_H -Morita contexts of the form $(B, \bar{B}, M, N, \gamma, \delta)$ as objects. A morphism between the \square_H -Morita contexts $(B, B, M_1, N_1, \gamma_1, \delta_1)$ and $(B, B, M_2, N_2, \gamma_2, \delta_2)$ consists of a couple (f, g) with $f: M_1 \rightarrow M_2$ and $g: N_1 \rightarrow N_2$ left $A^{\square e}$ -module isomorphisms such that $\gamma_1 = \gamma_2 \circ (f \otimes_B g)$ and $\delta_1 = \delta_2 \circ (g \otimes_B f)$.

It follows from Proposition 5.1 that $\underline{\text{Pic}}^{\square H}(B)$ is a monoidal category, with tensor product induced by the tensor product over B , and unit object (B, B, B, B, B, B) . Every object in $\underline{\text{Pic}}^{\square H}(B)$ has an inverse, and we call $K_0 \underline{\text{Pic}}^{\square H}(B) = \text{Pic}^{\square H}(B)$ the \square_H -Picard group of B . From Theorem 2.5 and the construction preceding Proposition 5.1, it follows that $\underline{\text{Pic}}^H(A)$ and $\underline{\text{Pic}}^{\square H}(B)$ are equivalent monoidal categories, so we conclude that $\text{Pic}^H(A) \cong \text{Pic}^{\square H}(B)$.

6. THE H -STABLE PART OF THE PICARD GROUP

Throughout this Section, we assume that H is cocommutative. Now let A be a right H -Galois extension of B . Our next aim is to introduce the H -invariant subgroup $\text{Pic}(B)^H$ of $\text{Pic}(B)$; roughly spoken, an object of $\underline{\text{Pic}}(B)$ represents an element of $\text{Pic}(B)^H$ if its connecting modules M and N are H -stable. First we need to fix some technical details.

We consider the category ${}_B\mathcal{M}_B^H$. Its objects are B -bimodules and right H -comodules M , such that the right H -coaction ρ is left and right B -linear, that is, $\rho(bmb') = bm_{[0]}b' \otimes m_{[1]}$, for all $b, b' \in B$ and $m \in M$. The morphisms are the H -colinear B -bimodule maps. For $M, N \in {}_B\mathcal{M}_B^H$, we consider the *generalized cotensor product*

$$\begin{aligned}
 M \otimes_B^H N &= \left\{ \sum_i m_i \otimes_B n_i \in M \otimes_B N \mid \right. \\
 &\quad \left. \sum_i m_{i[0]} \otimes_B n_i \otimes m_{i[1]} = \sum_i m_i \otimes_B n_{i[0]} \otimes n_{i[1]} \right\}.
 \end{aligned}$$

Then $M \otimes_B^H N$ is an object of ${}_B\mathcal{M}_B^H$, with right H -coaction

$$\rho\left(\sum_i m_i \otimes n_i\right) = \sum_i m_{i[0]} \otimes_B n_i \otimes m_{i[1]} = \sum_i m_i \otimes_B n_{i[0]} \otimes n_{i[1]}.$$

We have a functor $- \otimes H : {}_B\mathcal{M}_B \rightarrow {}_B\mathcal{M}_B^H$. For $M \in {}_B\mathcal{M}_B$, the structure on $M \otimes H$ is given by the formulas

$$\rho(m \otimes h) = m \otimes \Delta(h), \quad b(m \otimes h)b' = bmb' \otimes h.$$

In particular, $B \otimes H \in {}_B\mathcal{M}_B^H$. The functor $- \otimes H$ is monoidal in the sense of our next Lemma.

Lemma 6.1. *For $M, M' \in {}_B\mathcal{M}_B$, we have a natural isomorphism*

$$(M \otimes H) \otimes_B^H (M' \otimes H) \cong (M \otimes_B M') \otimes H$$

in ${}_B\mathcal{M}_B^H$.

Proof. It is easy to see that the map

$$\begin{aligned} \kappa : (M \otimes_B M') \otimes H &\rightarrow (M \otimes H) \otimes_B^H (M' \otimes H), \\ m \otimes_B m' \otimes h &\mapsto (m \otimes h_{(1)}) \otimes_B (m' \otimes h_{(2)}) \end{aligned}$$

is well-defined and right H -colinear. We claim that κ is bijective, with inverse given by the formula

$$\kappa^{-1}\left(\sum_j (m_j \otimes h_j) \otimes_B (m'_j \otimes h'_j)\right) = \sum_j m_j \otimes m'_j \otimes h_j \varepsilon(h'_j).$$

It is clear that $\kappa^{-1} \circ \kappa = M \otimes_B M' \otimes H$. If

$$x := \sum_j (m_j \otimes h_j) \otimes_B (m'_j \otimes h'_j) \in (M \otimes H) \otimes_B^H (M' \otimes H),$$

then

$$\sum_j (m_j \otimes h_{j(1)}) \otimes_B (m'_j \otimes h'_j) \otimes h_{j(2)} = \sum_j (m_j \otimes h_j) \otimes_B (m'_j \otimes h'_{j(1)}) \otimes h'_{j(2)}.$$

Applying ε to the third tensor factor, we find

$$(\kappa \circ \kappa^{-1})(x) = \sum_j (m_j \otimes h_{j(1)}) \otimes_B (m'_j \otimes \varepsilon(h'_j) h_{j(2)}) = x,$$

hence the claim is verified. \square

Lemma 6.2. *For all $P \in {}_B\mathcal{M}_B^H$, we have that*

$$P \otimes_B^H (B \otimes H) \cong (B \otimes H) \otimes_B^H P \cong P$$

in ${}_B\mathcal{M}_B^H$.

Proof. We have a well-defined morphism

$$\alpha : P \rightarrow P \otimes_B^H (B \otimes H), \quad \alpha(p) = p_{[0]} \otimes_B (1 \otimes p_{[1]})$$

in ${}_B\mathcal{M}_B^H$. The inverse of α is given by the formula

$$\alpha^{-1}\left(\sum_i p_i \otimes_B (b_i \otimes h_i)\right) = \sum_i p_i b_i \varepsilon(h_i).$$

It is clear that $\alpha^{-1} \circ \alpha = P$. If $\sum_i p_i \otimes_B (b_i \otimes h_i) \in P \otimes_B^H (B \otimes H)$, then

$$\sum_i p_{i[0]} \otimes_B (b_i \otimes h_i) \otimes p_{i[1]} = \sum_i p_i \otimes_B (b_i \otimes h_{i(1)}) \otimes h_{i(2)}.$$

Then we find

$$\begin{aligned} (\alpha \circ \alpha^{-1}) \left(\sum_i p_i \otimes_B (b_i \otimes h_i) \right) &= \sum_i p_{i[0]} b_i \varepsilon(h_i) \otimes_B p_{i[1]} \\ &= \sum_i p_i b_i \varepsilon(h_{i(1)}) \otimes_B (1 \otimes h_{i(2)}) = \sum_i p_i \otimes_B (b_i \otimes h_i). \end{aligned}$$

□

Observe that $A^{\square e} = A \square_H A^{\text{op}} \in {}_{B^e} \mathcal{M}_{B^e}^H$, with left and right B^e -action given by the formula

$$(b \otimes b') \left(\sum_i a_i \otimes a'_i \right) (c \otimes c') = \sum_i b a_i c \otimes c' a'_i b'.$$

Hence we have a second functor

$$A^{\square e} \otimes_{B^e} - : {}_B \mathcal{M}_B \rightarrow {}_B \mathcal{M}_B^H.$$

Take $M \in {}_B \mathcal{M}_B$. $A^{\square e} \otimes_{B^e} M$ is a left B^e -module, and, a fortiori, a B -bimodule. The right H -coaction on $A^{\square e} \otimes_{B^e} M$ is given by the formula

$$\begin{aligned} \rho \left(\left(\sum_k a_k \otimes a'_k \right) \otimes_{B^e} m \right) &= \left(\sum_k a_{k[0]} \otimes a'_k \right) \otimes_{B^e} m \otimes a_{k[1]} \\ &= \left(\sum_k a_k \otimes a'_{k[0]} \right) \otimes_{B^e} m \otimes a'_{k[1]}. \end{aligned}$$

Our next aim is to show that the functor $A^{\square e} \otimes_{B^e} -$ is also monoidal. Before we can show this, we need a few technical Lemmas. Let $M \in {}_B \mathcal{M}_B$. Then $A^{\text{op}} \otimes_B M \in {}_B \mathcal{M}_B^H$, with the right H -coaction induced by the coaction on A^{op} .

Lemma 6.3. *Suppose that $M \in {}_B \mathcal{M}_B$ is flat as a left B -module. Then the map*

$$f : A^{\square e} \otimes_B M \rightarrow A \square_H (A^{\text{op}} \otimes_B M), \quad \sum_k (a_k \otimes a'_k) \otimes_B m \mapsto \sum_k a_k \otimes (a'_k \otimes_B m)$$

is an isomorphism. In a similar way, if M is flat as a right B -module, then

$$M \otimes_B A^{\square e} \cong (M \otimes_B A) \square_H A^{\text{op}}.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A^{\square e} \otimes_B M & \longrightarrow & A^e \otimes_B M \rightrightarrows A^e \otimes_B M \otimes H \\ & & \downarrow f & & \cong \downarrow \\ 0 & \longrightarrow & A \square_H (A^{\text{op}} \otimes_B M) & \longrightarrow & A \otimes A^{\text{op}} \otimes_B M \rightrightarrows A^e \otimes_B M \otimes H \end{array}$$

The top row is exact because M is left B -flat, and because of the definition of the generalized cotensor product. The exactness of the bottom row also follows from the definition of the generalized cotensor product. It follows from the Five Lemma that f is an isomorphism. □

Lemma 6.4. *For all $a \in A$, the element*

$$x := \sum_i l_i(S(a_{[1]})) \otimes_B r_i(S(a_{[1]})) \otimes a_{[0]} \in A \otimes_B A^{\square e}.$$

Proof. By Lemma 6.3, it suffices to show that $x \in (A \otimes_B A) \square_H A^{\text{op}}$. Indeed,

$$\begin{aligned} & \sum_i l_i(S(a_{[1]})) \otimes_B r_i(S(a_{[1]}))_{[0]} \otimes a_{[0]} \otimes r_i(S(a_{[1]}))_{[1]} \\ & \stackrel{(3)}{=} \sum_i l_i(S(a_{[2]})) \otimes_B r_i(S(a_{[2]})) \otimes a_{[0]} \otimes S(a_{[1]}). \end{aligned}$$

□

Lemma 6.5. *If $\sum_k a_k \otimes a'_k \in A^{\square e}$, then the element*

$$\begin{aligned} x &= \sum_{i,k} a_{k[0]} \otimes l_i(a_{k[1]}) \otimes_B r_i(a_{k[1]}) \otimes a'_k \\ &= \sum_{i,k} a_k \otimes l_i(S(a'_{k[1]})) \otimes_B r_i(S(a'_{k[1]})) \otimes a'_{k[0]} \in A^{\square e} \otimes_B^H A^{\square e}. \end{aligned}$$

Proof. It follows from Proposition 3.5 that $A^{\square e}$ is flat as a left B^e -module. Since B^e is flat as a left B -module, we have that $A^{\square e}$ is flat as a left B -module. We have shown in Lemma 6.4 that $x \in A^e \otimes_B A^{\square e}$. Now

$$\begin{aligned} & \sum_{i,k} a_{k[0]} \otimes l_i(a_{k[2]}) \otimes_B r_i(a_{k[2]}) \otimes a'_k \otimes a_{k[1]} \\ & \stackrel{(4)}{=} \sum_{i,k} a_{k[0]} \otimes l_i(a_{k[1]})_{[0]} \otimes_B r_i(a_{k[1]}) \otimes a'_k \otimes S(l_i(a_{k[1]}))_{[1]}, \end{aligned}$$

so $x \in A \square_H (A^{\text{op}} \otimes_B A^{\square e}) = A^{\square e} \otimes_B A^{\square e}$, by Lemma 6.3. It then follows immediately that $x \in A^{\square e} \otimes_B^H A^{\square e}$. □

Lemma 6.6. *We have an isomorphism of vector spaces $f : A^{\square e} \otimes B \rightarrow A^{\square e} \otimes_B^H A^{\square e}$, given by the formula*

$$f\left(\sum_k a_k \otimes a'_k \otimes b\right) = \sum_{i,k} a_{k[0]} \otimes b l_i(a_{k[1]}) \otimes_B r_i(a_{k[1]}) \otimes a'_k.$$

Proof. It follows from Lemma 6.5 that f is well-defined. The inverse of f is defined as follows. For $y = \sum_k a_k \otimes a'_k \otimes_B a''_k \otimes a'''_k \in A^{\square e} \otimes_B^H A^{\square e}$, we let

$$f^{-1}(y) = \sum_k a_k \otimes a'''_k \otimes a'_k a''_k.$$

Let us show that f^{-1} is well-defined. First we show that $f^{-1}(y) \in A^e \otimes B$. Since $y \in A^{\square e} \otimes_B^H A^{\square e}$, we have that

$$\begin{aligned} & \sum_k a_k \otimes a'''_k \otimes a'_{k[0]} a''_{k[0]} \otimes a'_{k[1]} a''_{k[1]} \\ &= \sum_k a_k \otimes a'''_k \otimes a'_k a''_{k[0]} \otimes S(a''_{k[2]}) a''_{k[1]} \\ &= \sum_k a_k \otimes a'''_k \otimes a'_k a''_k \otimes 1. \end{aligned}$$

For any vector space V , we have that $(V \otimes A)^{\text{co}H} = V \otimes B$ (B is flat over k), so the above computation shows that $f^{-1}(y) \in A^e \otimes B$.

Let us next show that $f^{-1}(y) \in A^{\square e} \otimes B$: since $y \in A^{\square e} \otimes_B^H A^{\square e}$, we have that

$$\sum_k a_{k[0]} \otimes a'_k \otimes_B a''_k \otimes a'''_k \otimes a_{k[1]} = \sum_k a_k \otimes a'_k \otimes_B a''_k \otimes a'''_{k[0]} \otimes S(a'''_{k[1]}),$$

hence

$$\sum_k a_{k[0]} \otimes a'''_k \otimes a'_k a''_k \otimes a_{k[1]} = \sum_k a_k \otimes a'''_{k[0]} \otimes a'_k a''_k \otimes S(a'''_{k[1]}).$$

Let us finally verify that f and f^{-1} are inverses.

$$\begin{aligned} (f^{-1} \circ f)\left(\sum_k a_k \otimes a'_k \otimes b\right) &= f^{-1}\left(\sum_{i,k} a_{k[0]} \otimes b l_i(a_{k[1]}) \otimes_B r_i(a_{k[1]}) \otimes a'_k\right) \\ &= \sum_{i,k} a_{k[0]} \otimes a'_k \otimes b l_i(a_{k[1]}) r_i(a_{k[1]}) \stackrel{(5)}{=} \sum_k a_k \otimes a'_k \otimes b; \end{aligned}$$

$$\begin{aligned} (f \circ f^{-1})\left(\sum_k a_k \otimes a'_k \otimes_B a''_k \otimes a'''_k\right) &= f\left(\sum_k a_k \otimes a'''_k \otimes a'_k a''_k\right) \\ &= \sum_{i,k} a_{k[0]} \otimes a'_k a''_k l_i(a_{k[1]}) \otimes_B r_i(a_{k[1]}) \otimes a'''_k \\ &= \sum_{i,k} a_k \otimes a'_k a''_{k[0]} l_i(a'''_{k[1]}) \otimes_B r_i(a'''_{k[1]}) \otimes a'''_k \\ &\stackrel{(6)}{=} \sum_k a_k \otimes a'_k \otimes_B a''_k \otimes a'''_k. \end{aligned}$$

□

Take $M, M' \in {}_B\mathcal{M}_B$ and consider the composition $\tilde{g} = (\text{id} \otimes \text{can}^{-1} \otimes \text{id}) \circ (\rho_A \otimes \text{id})$:

$$\begin{aligned} A^e \otimes_{B^e} (M \otimes_B M') &\cong A \otimes_B M \otimes_B B \otimes_B M' \otimes_B A \\ &\longrightarrow A \otimes_B M \otimes_B B \otimes_B H \otimes_B M' \otimes_B A \\ &\longrightarrow A \otimes_B M \otimes_B A \otimes_B A \otimes_B M' \otimes_B A \\ &\cong A^e \otimes_{B^e} M \otimes_B A^e \otimes_{B^e} M'. \end{aligned}$$

We compute that

$$\begin{aligned} \tilde{g}\left(\sum_k a_k \otimes a'_k \otimes_{B^e} (m \otimes_B m')\right) \\ = \sum_{i,k} (a_{k[0]} \otimes l_i(a_{k[1]})) \otimes_{B^e} m \otimes_B (r_i(a_{k[1]}) \otimes a'_k) \otimes_{B^e} m'. \end{aligned}$$

It follows from Lemma 6.6 that \tilde{g} restricts to a map

$$g : A^{\square e} \otimes_{B^e} (M \otimes_B M') \rightarrow (A^{\square e} \otimes_{B^e} M) \otimes_B^H (A^{\square e} \otimes_{B^e} M').$$

It is obvious that $g \in {}_B\mathcal{M}_B^H$, and that g is bijective with inverse

$$\begin{aligned}
(16) \quad g^{-1} \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \otimes_B (a''_k \otimes a'''_k) \otimes_{B^e} m' \right) \\
= \sum_k (a_k \otimes a'''_k) \otimes_{B^e} (ma'_k a''_k \otimes_{B^e} m') \\
= \sum_k (a_k \otimes a'''_k) \otimes_{B^e} (m \otimes_{B^e} a'_k a''_k m').
\end{aligned}$$

As a conclusion, we obtain the following Lemma.

Lemma 6.7. *For $M, M' \in {}_B\mathcal{M}_B$, we have an isomorphism*

$$g : A^{\square e} \otimes_{B^e} (M \otimes_B M') \rightarrow (A^{\square e} \otimes_{B^e} M) \otimes_B^H (A^{\square e} \otimes_{B^e} M').$$

Remark 6.8. It follows from Lemmas 6.1 and 6.7 that, for $M, M', M'' \in {}_B\mathcal{M}_B$, we have isomorphisms

$$\begin{aligned}
& ((M \otimes H) \otimes_B^H (M' \otimes H)) \otimes_B^H (M'' \otimes H) \\
& \cong (M \otimes H) \otimes_B^H ((M' \otimes H) \otimes_B^H (M'' \otimes H)), \\
& ((A^{\square e} \otimes_{B^e} M) \otimes_B^H (A^{\square e} \otimes_{B^e} M')) \otimes_B^H (A^{\square e} \otimes_{B^e} M'') \\
& \cong (A^{\square e} \otimes_{B^e} M) \otimes_B^H ((A^{\square e} \otimes_{B^e} M') \otimes_B^H (A^{\square e} \otimes_{B^e} M''))
\end{aligned}$$

in ${}_B\mathcal{M}_B^H$ that are natural in M, M', M'' .

We now consider the notion of H -stability, as introduced before Corollary 4.7, but with B replaced by B^e and A by $A^{\square e}$. The (B, B) -bimodule M is H -stable if there exists an isomorphism

$$\varphi_M : A^{\square e} \otimes_{B^e} M \rightarrow M \otimes H$$

in the category ${}_B\mathcal{M}_B^H$.

Proposition 6.9. *If $M, M' \in {}_B\mathcal{M}_B$ are H -stable, then $M \otimes_B M'$ is also H -stable.*

Proof. We define $\varphi_{M \otimes_B M'}$ by the commutativity of the following diagram:

$$(17) \quad \begin{array}{ccc}
A^{\square e} \otimes_{B^e} (M \otimes_B M') & \xrightarrow{g} & (A^{\square e} \otimes_{B^e} M) \otimes_B^H (A^{\square e} \otimes_{B^e} M') \\
\varphi_{M \otimes_B M'} \downarrow & & \downarrow \varphi_M \otimes_B^H \varphi_{M'} \\
M \otimes_B M' \otimes H & \xrightarrow{\kappa} & (M \otimes H) \otimes_B^H (M' \otimes H)
\end{array}$$

□

Suppose that $M, M' \in {}_B\mathcal{M}_B$ are H -stable, and let $\psi_M = \varphi_M^{-1}$, $\psi_{M'} = \varphi_{M'}^{-1}$, $t_M = \alpha_{\mathbf{12}}^{-1}(\psi_M)$, $t_{M'} = \alpha_{\mathbf{12}}^{-1}(\psi_{M'})$. For later use, we compute $t_{M \otimes_B M'} = \alpha_{\mathbf{12}}^{-1}(\psi_{M \otimes_B M'})$ in terms of t_M and $t_{M'}$. To this end, we first introduce the following Sweedler-type notation for the map t_M :

$$t_M(h)(1_{A^{\square e} \otimes_{B^e} m}) = (m(h)^+ \otimes m(h)^-) \otimes_{B^e} m(h)^0.$$

Summation is implicitly understood. Using the definition of α_{12} and the commutativity of (17), we compute

$$\begin{aligned}
t_{M \otimes_B M'}(h)(1_{A^{\square e}} \otimes_{B^e} (m \otimes_B m')) &= \psi_{M \otimes_B M'}(m \otimes_B m' \otimes h) \\
&= (g^{-1} \circ (\psi_M \otimes_B^H \psi_{M'})) \circ k(m \otimes_B m' \otimes h) \\
&= (g^{-1} \circ (\psi_M \otimes_B^H \psi_{M'}))((m \otimes h_{(1)}) \otimes_B (m' \otimes h_{(2)})) \\
&= g^{-1}(t(h_{(1)})(1_{A^{\square e}} \otimes_{B^e} m) \otimes_B t(h_{(2)})(1_{A^{\square e}} \otimes_{B^e} m')) \\
&= g^{-1}\left(\left((m(h_{(1)})^+ \otimes m(h_{(1)}^-) \otimes_{B^e} m(h_{(1)}^0)\right)\right. \\
&\quad \left. \otimes_B \left((m'(h_{(2)})^+ \otimes m'(h_{(2)}^-) \otimes_{B^e} m'(h_{(2)}^0)\right)\right) \\
&= (m(h_{(1)})^+ \otimes m'(h_{(2)}^-) \otimes_{B^e} (m(h_{(1)}^0 m(h_{(1)}^- m'(h_{(2)}^+ \otimes_B m'(h_{(1)}^0)^0) \\
&= (m(h_{(1)})^+ \otimes m'(h_{(2)}^-) \otimes_{B^e} (m(h_{(1)}^0 \otimes_B m(h_{(1)}^- m'(h_{(2)}^+ m'(h_{(1)}^0)^0).
\end{aligned}$$

We will need a slight improvement of this formula. For $b \in B$, we have

$$\begin{aligned}
t_M(h)(1_{A^{\square e}} \otimes_{B^e} mb) &= t_M(h)((1 \otimes b) \otimes_{B^e} m) \\
&= (1 \otimes b)t_M(h)(1_{A^{\square e}} \otimes_{B^e} m) = (m(h)^+ \otimes m(h)^- b) \otimes_{B^e} m(h)^0,
\end{aligned}$$

hence

$$\begin{aligned}
(18) \quad t_{M \otimes_B M'}(h)(1_{A^{\square e}} \otimes_{B^e} (mb \otimes_B m')) &= (m(h_{(1)})^+ \otimes m'(h_{(2)}^-) \\
&\quad \otimes_{B^e} (m(h_{(1)}^0 m(h_{(1)}^- b m'(h_{(2)}^+ \otimes_B m'(h_{(1)}^0)^0).
\end{aligned}$$

We have that B is a left $A^{\square e}$ -module, with action $\phi_B((\sum_k a_k \otimes a'_k) \otimes_{B^e} b) = \sum_k a_k b a'_k$. The corresponding map $\varphi_B = \beta_1(\phi_B) : A^{\square e} \otimes_{B^e} B \rightarrow B \otimes H$ is given by

$$\varphi_B\left(\left(\sum_k a_k \otimes a'_k\right) \otimes_{B^e} b\right) = \sum_k a_{k[0]} b a'_k \otimes a_{k[1]} = \sum_k a_k b a'_{k[0]} \otimes S(a_{k[1]}).$$

It follows from Corollary 4.7 and Proposition 4.8 that φ_B is an isomorphism in ${}_B \mathcal{M}_B^H$.

Now take $\underline{M} = (B, B, M, N, \alpha, \beta) \in \underline{\text{Pic}}(B)$. We call \underline{M} H -stable if there exist isomorphisms

$$\varphi_M : A^{\square e} \otimes_{B^e} M \rightarrow M \otimes H \quad \text{and} \quad \varphi_N : A^{\square e} \otimes_{B^e} N \rightarrow N \otimes H$$

such that the following diagrams commute:

$$(19) \quad \begin{array}{ccc}
A^{\square e} \otimes_{B^e} (M \otimes_B N) & \xrightarrow{A^{\square e} \otimes_{B^e} \alpha} & A^{\square e} \otimes_{B^e} B \\
\varphi_{M \otimes_B N} \downarrow & & \downarrow \varphi_B \\
M \otimes_B N \otimes H & \xrightarrow{\alpha \otimes H} & B \otimes H
\end{array}$$

$$(20) \quad \begin{array}{ccc}
A^{\square e} \otimes_{B^e} (N \otimes_B M) & \xrightarrow{A^{\square e} \otimes_{B^e} \beta} & A^{\square e} \otimes_{B^e} B \\
\varphi_{N \otimes_B M} \downarrow & & \downarrow \varphi_B \\
N \otimes_B M \otimes H & \xrightarrow{\beta \otimes H} & B \otimes H
\end{array}$$

Theorem 6.10. *Let H be a cocommutative Hopf algebra, and let A be a faithfully flat Hopf-Galois extension of $A^{\text{co}H} = B$. Then*

$$\text{Pic}(B)^H = \{[\underline{M}] \in \text{Pic}(B) \mid \underline{M} \text{ is } H\text{-stable}\}$$

is a subgroup of $\text{Pic}(B)$, called the H -stable part of $\text{Pic}(B)$.

Proof. Assume that \underline{M}_1 and \underline{M}_2 are H -stable. It follows from Proposition 6.9 that $M_1 \otimes_B M_2$ and $N_2 \otimes_B N_1$ are H -stable. A commutative diagram argument taking Remark 6.8 into account shows that the diagrams (19-20), with M replaced by $M_1 \otimes_B M_2$ and N by $N_2 \otimes_B N_1$, commute. This implies that $\underline{M}_1 \otimes_B \underline{M}_2$ is H -stable. Finally, if \underline{M} is H -stable, then it is clear from the definition that $\underline{M}^{-1} = (B, B, N, M, \beta, \alpha)$ is also H -stable. \square

7. A HOPF ALGEBRA VERSION OF THE BEATTIE-DEL RÍO EXACT SEQUENCE

As in the previous Section, let H be a cocommutative Hopf algebra, and A a faithfully flat H -Galois extension of B . Take $\underline{M} \in \underline{\text{Pic}}(B)^H$. Then we have an isomorphism $\varphi : A^{\square e} \otimes_{B^e} M \rightarrow M \otimes H$ in ${}_B\mathcal{M}_B^H$. We have that $E = {}_{A^{\square e}}\text{END}(A^{\square e} \otimes_{B^e} M)^{\text{op}}$ is an H -comodule algebra.

Lemma 7.1. $E^{\text{co}H} \cong Z(B)$.

Proof. We first observe that

$$E^{\text{co}H} = {}_{A^{\square e}}\text{End}^H(A^{\square e} \otimes_{B^e} M) \cong {}_{B^e}\text{End}(M) = {}_B\text{End}_B(M).$$

The second isomorphism is due to the fact that $A^{\square e} \otimes_{B^e} - : {}_{B^e}\mathcal{M} \rightarrow {}_{A^{\square e}}\mathcal{M}^H$ is a category equivalence, by Theorem 2.1 and Proposition 3.5. Since \underline{M} is a strict Morita context, we have that $- \otimes_B M$ is an autoequivalence of \mathcal{M}_B . $- \otimes_B M$ and its adjoint send B -bimodules to B -bimodules, so $- \otimes_B M$ also defines an autoequivalence of ${}_B\mathcal{M}_B$. Consequently ${}_B\text{End}_B(M) \cong {}_B\text{End}_B(B) \cong Z(B)$. \square

For later use, we give an explicit description of the isomorphism

$$\lambda : Z(B) \rightarrow E^{\text{co}H} = {}_{A^{\square e}}\text{End}^H(A^{\square e} \otimes_{B^e} M), \quad x \mapsto \lambda_x :$$

$$(21) \quad \lambda_x \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \right) = \sum_k (a_k \otimes a'_k) \otimes_{B^e} xm.$$

We have seen in Theorem 4.6 that there are isomorphisms

$$\alpha_{12} : \text{Hom}^H(H, E) \rightarrow {}_B\text{Hom}_B^H(M \otimes H, A^{\square e} \otimes_{B^e} M),$$

$$\alpha_{21} : \mathcal{C}(\mathbf{2}, \mathbf{1}) \rightarrow {}_B\text{Hom}_B^H(A^{\square e} \otimes_{B^e} M, M \otimes H).$$

Using Proposition 3.5, we compute $u = \alpha_{21}^{-1}(\varphi)$ and $t = \alpha_{12}^{-1}(\varphi^{-1})$:

$$(22) \quad t(h) \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \right) = \sum_k (a_k \otimes a'_k) \varphi^{-1}(m \otimes h);$$

$$(23) \quad u(h) \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \right) = \sum_{i,j,k} (a_k l_i(h_{(1)}) \otimes r_j(h_{(2)}) a'_k) \otimes_{B^e} \phi((r_i(h_{(1)}) \otimes l_j(h_{(2)})) \otimes_{B^e} m).$$

Since $E^{\text{co}H} \cong Z(B)$ is commutative, we can apply Proposition 4.2, and we find that $Z(B)$ is a left H -module algebra. We will show in Proposition 7.3 that the left H -action on $Z(B)$ is independent of the choice of $\underline{M} \in \underline{\text{Pic}}(B)^H$, and is given by the Miyashita-Ulbrich action (8).

Lemma 7.2. *For $x \in Z(B)$, $m \in M$ and $h \in H$, we have that*

$$(24) \quad \lambda_x(\varphi^{-1}(m \otimes h)) = \varphi^{-1}((h_{(2)} \cdot x)m \otimes h_{(1)}).$$

Proof. Write

$$(25) \quad \varphi^{-1}(m \otimes h) = \sum_k (s_k \otimes s'_k) \otimes_{B^e} m_k \in A^{\square e} \otimes_{B^e} M.$$

Since φ^{-1} is right H -colinear, we have that

$$(26) \quad \varphi^{-1}(m \otimes h_{(1)}) \otimes h_{(2)} = \sum_k (s_{k[0]} \otimes s'_k) \otimes_{B^e} m_k \otimes s_{k[1]}.$$

Then we compute

$$\begin{aligned} \varphi^{-1}(xm \otimes h) &= x\varphi^{-1}(m \otimes h) \\ &\stackrel{(25)}{=} \sum_k (xs_k \otimes s'_k) \otimes_{B^e} m_k \stackrel{(9)}{=} \sum_k (s_{k[0]}(S(s_{k[1]}) \cdot x) \otimes s'_k) \otimes_{B^e} m_k \\ &= \sum_k (s_{k[0]} \otimes s'_k) \otimes_{B^e} (S(s_{k[1]}) \cdot x)m_k \stackrel{(21,26)}{=} \lambda_{S(h_{(2)}) \cdot x}(\varphi^{-1}(m \otimes h_{(1)})). \end{aligned}$$

and it follows that

$$\varphi^{-1}((h_{(2)} \cdot x)m \otimes h_{(1)}) = \lambda_{S(h_{(2)}) \cdot (h_{(3)} \cdot x)}(\varphi^{-1}(m \otimes h_{(1)})) = \lambda_x(\varphi^{-1}(m \otimes h)).$$

□

Proposition 7.3. *Assume that $\underline{M} \in \underline{\text{Pic}}(B)$ is H -stable. The corresponding left H -action on $E^{\text{co}H}$ is given by the formula $h \bullet \lambda_x = \lambda_{h \cdot x}$, for all $x \in Z(B)$. This means that the transported action on $Z(B)$ is the Miyashita-Ulbrich action given by (9).*

Proof. Take $x \in Z(B)$ and the corresponding $\lambda_x \in E^{\text{co}H}$. The action of $h \in H$ on λ_x is given by (see Proposition 4.2)

$$h \bullet \lambda_x = u(h_{(1)}) \circ \lambda_x \circ t(h_{(2)}),$$

and we have

$$\begin{aligned} \diamond &:= (h \bullet \lambda_x) \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \right) \\ &\stackrel{(22)}{=} (u(h_{(1)}) \circ \lambda_x) \left(\sum_k (a_k \otimes a'_k) \varphi^{-1}((h_{(3)} \cdot x)m \otimes h_{(2)}) \right) \\ &\stackrel{(24)}{=} u(h_{(1)}) \left(\sum_k (a_k \otimes a'_k) \varphi^{-1}((h_{(3)} \cdot x)m \otimes h_{(2)}) \right). \end{aligned}$$

Now write

$$(27) \quad \varphi^{-1}((h_{(2)} \cdot x)m \otimes h_{(1)}) = \sum_q (s_q \otimes s'_q) \otimes_{B^e} m_q.$$

Since φ^{-1} is right H -colinear, we have that

$$\varphi^{-1}((h_{(3)} \cdot x)m \otimes h_{(2)}) \otimes h_{(1)} = \sum_q (s_{q[0]} \otimes s'_q) \otimes_{B^e} m_q \otimes s_{q[1]},$$

hence

$$\begin{aligned} \diamond &= u(s_{q[1]}) \left(\sum_{q,k} (a_k s_{q[0]} \otimes s'_q a'_k) \otimes_{B^e} m_q \right) \\ &\stackrel{(23)}{=} \sum_{i,j,k,q} (q_k s_{q[0]} l_i(s_{q[1]}) \otimes r_j(s_{q[2]}) s'_q a'_k) \otimes_{B^e} \phi((r_i(s_{q[1]}) \otimes l_j(s_{q[2]})) \otimes_{B^e} m_q). \end{aligned}$$

Using (6,12), we find

$$\begin{aligned} &\sum_{i,q} (s_{q[0]} \otimes s'_q) (l_i(s_{q[1]}) \otimes r_j(s_{q[2]})) \otimes_{B^e} (r_i(s_{q[1]}) \otimes l_j(s_{q[2]})) \\ &= \sum_q 1_{A^{\square e}} \otimes_{B^e} (s_q \otimes s'_q) \in B^e \otimes_{B^e} A^{\square e}. \end{aligned}$$

Since ϕ is left B^e -linear, we find

$$\begin{aligned} \diamond &= \sum_{i,j,k,q} (a_k \otimes a'_k) \otimes_{B^e} \\ &\quad \phi((s_{q[0]} l_i(s_{q[1]}) r_j(s_{q[1]}) \otimes l_j(s_{q[2]}) r_j(s_{q[2]}) s'_q) \otimes_{B^e} m_q) \\ &= \sum_{k,q} (a_k \otimes a'_k) \otimes_{B^e} \phi(s_q \otimes s'_q) \otimes_{B^e} m_q \\ &\stackrel{(27)}{=} \sum_{k,q} (a_k \otimes a'_k) \otimes_{B^e} ((M \otimes \varepsilon) \circ \varphi \circ \varphi^{-1})((h_{(2)} \cdot x)m \otimes h_{(1)}) \\ &= \sum_{k,q} (a_k \otimes a'_k) \otimes_{B^e} (h \cdot x)m = \lambda_{h \cdot x} \left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} m \right). \end{aligned}$$

This shows that $h \bullet \lambda_x = \lambda_{h \cdot x}$, for all $x \in Z(B)$. \square

It follows from the discussion in Section 5 that the functor $\underline{\text{Pic}}^{\square H}(B) \rightarrow \underline{\text{Pic}}(B)$ restricting the $A^{\square e}$ -module structure on the connecting bimodules to the B -bimodule structure is strongly monoidal. This implies that we have a group homomorphism

$$g_2 : \text{Pic}^{\square H}(B) \rightarrow \text{Pic}(B).$$

Proposition 7.4. *The groups $\text{Ker}(g_2)$ and $H^1(H, Z(B))$ are isomorphic.*

Proof. Take $[\underline{M}] = [(B, B, M, N, \alpha, \beta)] \in \text{Ker}(g_2)$. Then M and N are isomorphic to B as B -bimodules. \underline{M} is described completely once we know the left $A^{\square e}$ -module structure on $M = B$, by Theorem 2.5 (2). Isomorphism classes of left $A^{\square e}$ -module structures on B are in bijective correspondence to the elements of $\overline{\Omega}_E$, cf. Corollary 4.10. It follows from Proposition 4.3 that $\overline{\Omega}_E \cong H^1(H, Z(B))$, hence we have a bijection between $H^1(H, Z(B))$ and $\text{Ker}(g_2)$, and an injection

$$g_1 : H^1(H, Z(B)) \rightarrow \text{Pic}^{\square H}(B).$$

We will now describe this injection explicitly, and show that it preserves multiplication.

Let ϕ_0 be the left $A^{\square e}$ -action on B corresponding to the trivial element in $\text{Pic}^{\square H}(B)$:

$$\phi_0\left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} b\right) = \sum_k a_k b a'_k.$$

Let $u_0 = \tilde{\alpha}_{\mathbf{21}}^{-1}$ be the corresponding element in $\mathcal{C}_E(\mathbf{2}, \mathbf{1})$. Using the formulas in the proof of Theorem 4.6 we obtain that

$$u_0(h)\left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} b\right) = \sum_{i,j,k} (a_k l_i(h_{(1)}) \otimes r_j(h_{(2)}) a'_k) \otimes_{B^e} r_i(h_{(1)}) b l_j(h_{(2)}).$$

Let $\alpha \in Z^1(H, Z(B))$, and take $G(\alpha) = t = \alpha * t_0 \in \Omega_E$ (see Proposition 4.3). Then $t(h) = t_0(h_{(1)}) \circ \alpha(h_{(2)})$, and $u(h) = t(S(h)) = u_0(h_{(1)}) \circ \alpha(S(h_{(2)}))$. We compute $\phi_\alpha = \tilde{\alpha}_{\mathbf{21}}(u)$, using the formulas given in the proof of Theorem 4.6:

$$\begin{aligned} & 1 \otimes_{B^e} \phi_\alpha\left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} b\right) \\ &= \sum_k u(a_{k[1]})\left((a_{k[0]} \otimes a'_k) \otimes_{B^e} b\right) \\ &= \sum_k u_0(a_{k[1]})\left((a_{k[0]} \otimes a'_k) \otimes_{B^e} \alpha(S(a_{k[2]}))b\right) \\ &= \sum_{i,j,k} (a_{k[0]} l_i(a_{k[1]}) \otimes r_j(a_{k[2]}) a'_k) \otimes_{B^e} r_i(a_{k[1]}) \alpha(S(a_{k[3]})) b l_j(a_{k[2]}) \\ &= \sum_{i,j,k} 1_{A^{\square e}} \otimes_{B^e} a_{k[0]} l_i(a_{k[1]}) r_i(a_{k[1]}) \alpha(S(a_{k[3]})) b l_j(a_{k[2]}) r_j(a_{k[2]}) a'_k \\ &= \sum_k 1_{A^{\square e}} \otimes_{B^e} a_{k[0]} \alpha(S(a_{k[1]})) b a'_k \end{aligned}$$

This means that $g_1(\alpha)$ is represented by B , with left $A^{\square e}$ -action given by

$$(28) \quad \sum_k (a_k \otimes a'_k) \cdot_\alpha b = \phi_\alpha\left(\sum_k (a_k \otimes a'_k) \otimes_{B^e} b\right) = a_{k[0]} \alpha(S(a_{k[1]})) b a'_k.$$

Let $\beta \in Z^1(H, Z(B))$ be another cocycle. Then $g_1(\alpha) \otimes_B g_1(\beta) = B \otimes_B B \cong B$ as a (B, B) -bimodule, with left $A^{\square e}$ -action

$$\begin{aligned}
& \sum_k (a_k \otimes a'_k) \cdot b \cong \sum_k (a_k \otimes a'_k) \cdot (1 \otimes_B b) \\
& \stackrel{(15)}{=} \sum_{i,k} (a_{k[0]} \otimes l_i(a_{k[1]})) \cdot_\alpha 1 \otimes_B (r_i(a_{k[1]}) \otimes a'_k) \cdot_\beta b \\
& \stackrel{(28)}{=} \sum_{i,k} a_{k[0]} \alpha(S(a_{k[1]})) l_i(a_{k[2]}) \otimes_B r_i(a_{k[2]})_{[0]} \beta(S(r_i(a_{k[2]})_{[1]})) b a'_k \\
& \stackrel{(3)}{=} \sum_{i,k} a_{k[0]} \alpha(S(a_{k[1]})) l_i(a_{k[2]}) \otimes_B r_i(a_{k[2]}) \beta(S(a_{k[3]})) b a'_k \\
& \cong \sum_{i,k} a_{k[0]} \alpha(S(a_{k[1]})) l_i(a_{k[2]}) r_i(a_{k[2]}) \beta(S(a_{k[3]})) b a'_k \\
& \stackrel{(5)}{=} \sum_k a_{k[0]} \alpha(S(a_{k[1]})) \beta(S(a_{k[2]})) b a'_k \\
& = \sum_k a_{k[0]} (\alpha * \beta)(S(a_{k[1]})) b a'_k \\
& = \sum_k (a_k \otimes a'_k) \cdot_{\alpha * \beta} b.
\end{aligned}$$

This shows that $g_1(\alpha) \otimes_B g_1(\beta) = g_1(\alpha * \beta)$, that is, g_1 is a group monomorphism. \square

Let $\underline{M} \in \underline{\text{Pic}}(B)$ be H -stable. Then there exists an isomorphism

$$\psi : M \otimes H \rightarrow A^{\square e} \otimes_{B^e} M$$

in ${}_B \mathcal{M}_B^H$ such that $\psi(m \otimes 1) = 1_{A^{\square e}} \otimes_{B^e} m$, for all $m \in M$ (see the arguments given after Corollary 4.7). Then $t := \alpha_1^{-1}(\psi) \in \text{Hom}^H(H, E)$ is convolution invertible and satisfies the condition $t(1) = 1$. In Proposition 4.5, we constructed a cocycle $\sigma \in Z^2(H, Z(B))$. Now let $g_3([\underline{M}]) = [\sigma] \in H^2(H, Z(B))$. This defines a map

$$g_3 : \text{Pic}(B)^H \rightarrow H^2(H, Z(B)).$$

It follows from Proposition 4.5 that $g_3([\underline{M}]) = 1$ if and only if there exists an algebra map $t' \in \text{Hom}^H(H, E)$. By Proposition 4.8, this is equivalent to the existence of an associative left $A^{\square e}$ -action $\phi : A^{\square e} \otimes_{B^e} M \rightarrow M$, which is equivalent to $[\underline{M}] \in \text{Im}(g_2)$. We conclude that $\text{Im}(g_2) = \text{Ker}(g_3)$. Our observations can be summarized as follows.

Theorem 7.5. *Let H be a cocommutative Hopf algebra over a field k , and A a faithfully flat Hopf-Galois extension of $B = A^{\text{co}H}$. Then we have an exact sequence*

$$1 \rightarrow H^1(H, Z(B)) \xrightarrow{g_1} \text{Pic}^{\square H}(B) \cong \text{Pic}^H(A) \xrightarrow{g_2} \text{Pic}(B)^H \xrightarrow{g_3} H^2(H, Z(B)).$$

Observe that $\text{Pic}^{\square H}(B) \cong \text{Pic}^H(A)$ and $\text{Pic}(B)^H$ are non-abelian groups. The category of groups is not an abelian category, so it makes no sense to talk about exact sequences of groups. In the statement in Theorem 7.5, exactness means that g_1 is an injective map, and that $\text{Im}(g_i) = \{x \mid g_{i+1}(x) = 1\}$,

for $i = 1, 2$. The maps g_1 and g_2 are group homomorphisms. An example given in [4] shows that g_3 is not a group homomorphism in general, even in the case of group graded algebras. We will discuss in Section 8 the property satisfied by g_3 .

8. g_3 IS A 1-COCYCLE

We recall from [10] that $\text{Pic}(B)$ acts on $Z(B)$ as follows. For $[\underline{M}] \in \text{Pic}(B)$, we have a map $\xi_M : Z(B) \rightarrow Z(B)$ characterized by the property

$$(29) \quad \xi_M(x) = y \iff mx = ym, \quad \text{for all } m \in M.$$

It is easy to show that $\xi_M(xy) = \xi_M(x)\xi_M(y)$. We will show that this action defines an action of $\text{Pic}(B)^H$ on $H^n(H, Z(B))$, so that we can consider the group of cocycles $Z^1(\text{Pic}(B)^H, H^2(H, Z(B)))$. We will then show that g_3 is such a 1-cocycle.

Our first aim is to show that the action $\text{Pic}(B)^H$ on $Z(B)$ commutes with the action of H on $Z(B)$. First, we need some Lemmas.

Lemma 8.1. *Take $[\underline{M}] \in \text{Pic}(B)^H$. For all $x \in Z(B)$, $m \in M$ and $\sum_i a_i \otimes a'_i \in A^{\square e}$, we have that*

$$(30) \quad \left(\sum_i a_i \otimes a'_i x \right) \otimes_{B^e} m = \left(\sum_i \xi_M(x) a_i \otimes a'_i \right) \otimes_{B^e} m$$

in $A^{\square e} \otimes_{B^e} M$.

Proof. This follows immediately from the fact that $mx \otimes h = \xi_M(x)m \otimes h$ in $M \otimes H$, for all $m \in M$, $x \in Z(B)$ and $h \in H$, and the fact that we a (B, B) -bimodule isomorphism $\psi_M : M \otimes H \rightarrow A^{\square e} \otimes_{B^e} M$. \square

Lemma 8.2. *The map*

$$l : A^{\square e} \otimes_{B^e} B \rightarrow A \otimes_B^H A^{\text{op}}, \quad \left(\sum_i a_i \otimes a'_i \right) \otimes_{B^e} b \mapsto \sum_i a_i b \otimes_B a'_i$$

is an isomorphism.

Proof. Observe first that $A^{\square e} \otimes_{B^e} B$ and $A \otimes_B^H A^{\text{op}}$ are objects of the category ${}_{A^{\square e}} \mathcal{M}^H$. It follows from Theorem 2.1 and Proposition 3.5 that it suffices to show that

$$(A \otimes_B^H A^{\text{op}})^{\text{co}H} \cong B \cong (A^{\square e} \otimes_{B^e} B)^{\text{co}H}.$$

Take

$$\sum_i a_i \otimes_B a'_i \in (A \otimes_B^H A^{\text{op}})^{\text{co}H} \subset A \otimes_B^H A^{\text{op}}.$$

Then

$$\sum_i a_{i[0]} \otimes_B a'_i \otimes a_{i[1]} = \sum_i (a_i b \otimes_B a'_i) \otimes 1.$$

From the fact that $A \in {}_B \mathcal{M}$ is faithfully flat, we deduce that $\sum_i a_i \otimes_B a'_i \in A^{\text{co}H} \otimes_B A = B \otimes_B A$, hence

$$\sum_i a_i \otimes_B a'_i = 1 \otimes_B \sum_i a_i a'_i = 1 \otimes_B a.$$

Since $\sum_i a_i \otimes_B a'_i \in A \otimes_B^H A^{\text{op}}$, we also have that

$$1 \otimes_B a_{[0]} \otimes S(a_{[1]}) = 1 \otimes_B a \otimes 1.$$

Apply ρ_A to the second tensor factor (ρ_A is left B -linear), and then multiply the second and third tensor factor. This gives $1 \otimes_B a_{[0]} \otimes a_{[1]} = 1 \otimes_B a \otimes_B 1$, and it follows that $a \in B$. This shows that the map

$$f : B \rightarrow (A \otimes_B^H A^{\text{op}})^{\text{co}H}, \quad f(b) = 1 \otimes_B b$$

is an isomorphism. \square

Lemma 8.2 tells us that the map $A^{\square e} \rightarrow A \otimes_B^H A^{\text{op}}$ induced by the canonical surjection $A^e \rightarrow A \otimes_B A^{\text{op}}$ is surjective.

Proposition 8.3. *Let $\underline{M} = (B, B, M, N, \alpha, \beta)$ represent an H -stable element of $\text{Pic}(B)$. Then*

$$\xi_M(h \cdot x) = h \cdot (\xi_M(x)),$$

for all $h \in H$ and $x \in Z(B)$.

Proof. For $\sum_k a_k \otimes a'_k \in A^{\square e}$, $x \in Z(B)$ and $m \in M$, we compute that

$$\begin{aligned} & \left(\sum_k \xi_M(x) a_k \otimes a'_k \right) \otimes_{B^e} m \stackrel{(30)}{=} \left(\sum_k a_k \otimes a'_k x \right) \otimes_{B^e} m \\ & \stackrel{(9)}{=} \left(\sum_k a_k \otimes (a'_{k[1]} \cdot x) a'_{k[0]} \right) \otimes_{B^e} m \\ & = \left(\sum_k a_k \otimes a'_{k[0]} \right) \otimes_{B^e} m (a'_{k[1]} \cdot x) \\ & \stackrel{(29)}{=} \left(\sum_k a_k \otimes a'_{k[0]} \right) \otimes_{B^e} \xi_M(a'_{k[1]} \cdot x) m \\ & = \left(\sum_k a_k \xi_M(a'_{k[1]} \cdot x) \otimes a'_{k[0]} \right) \otimes_{B^e} m \\ & \stackrel{(9)}{=} \left(\sum_k a_{k[1]} \cdot \xi_M(a'_{k[1]} \cdot x) a_{k[0]} \otimes a'_{k[0]} \right) \otimes_{B^e} m \\ & = \left(\sum_k a_{k[1]} \cdot \xi_M(S(a_{k[2]}) \cdot x) a_{k[0]} \otimes a'_k \right) \otimes_{B^e} m. \end{aligned}$$

Now take an arbitrary $n \in N$. Applying Lemma 6.5, we find

$$\begin{aligned} & \sum_{i,k} \left((\xi_M(x) a_{k[0]} \otimes l_i(a_{k[1]})) \otimes_{B^e} m \right) \otimes_B \left((r_i(a_{k[1]}) \otimes a'_k) \otimes_{B^e} n \right) \\ & = \sum_{i,k} \left(((a_{k[1]} \cdot (\xi_M(S(a_{k[2]}) \cdot x)) a_{k[0]} \otimes l_i(a_{k[3]})) \otimes_{B^e} m) \right. \\ & \quad \left. \otimes_B \left((r_i(a_{k[3]}) \otimes a'_k) \otimes_{B^e} n \right) \right). \end{aligned}$$

Now we apply

$$g^{-1} : (A^{\square e} \otimes_{B^e} M) \otimes_B^H (A^{\square e} \otimes_{B^e} N) \rightarrow A^{\square e} \otimes_{B^e} (M \otimes_B N)$$

to both sides (see (16)). Using (5), we obtain

$$\begin{aligned} & \sum_k (\xi_M(x) a_k \otimes a'_k) \otimes_{B^e} (m \otimes_B n) \\ & = \left(\sum_k a_{k[1]} \cdot \xi_M(S(a_{k[2]}) \cdot x) a_{k[0]} \otimes a'_k \right) \otimes_{B^e} (m \otimes_B n). \end{aligned}$$

Now $M \otimes_B N \cong B$. It follows that

$$\sum_k (\xi_M(x) a_k \otimes a'_k) \otimes_{B^e} b = \left(\sum_k a_{k[1]} \cdot \xi_M(S(a_{k[2]}) \cdot x) a_{k[0]} \otimes a'_k \right) \otimes_{B^e} b,$$

for all $\sum_k a_k \otimes a'_k \in A^{\square e}$, $x \in Z(B)$ and $b \in B$. Using Lemma 8.2, we find that

$$\begin{aligned} \sum_k \xi_M(x) a_k \otimes_B a'_k &= \sum_k a_{k[1]} \cdot \xi_M(S(a_{k[2]}) \cdot x) a_{k[0]} \otimes_B a'_k \\ &= \sum_k S(a'_{k[1]}) \cdot \xi_M(a'_{k[2]} \cdot x) a_k \otimes_B a'_{k[0]} \end{aligned}$$

for all $\sum_k a_k \otimes_B a'_k \in A \otimes_B^H A^{\text{op}}$ and $x \in Z(B)$.

Now take $h \in H$. It follows from (3-4) that $\gamma_A(h) = \sum_i l_i(h) \otimes_B r_i(h) \in A \otimes_B^H A^{\text{op}}$. Therefore

$$\begin{aligned} \sum_i \xi_M(x) l_i(h) \otimes_B r_i(h) &= \sum_i (S(r_i(h)_{[1]}) \cdot \xi_M(r_i(h)_{[1]} \cdot x)) l_i(h) \otimes_B r_i(h)_{[0]} \\ &\stackrel{(3)}{=} \sum_i S(h_{(2)}) \cdot \xi_M(h_{(3)} \cdot x) \otimes l_i(h_{(1)}) \otimes_B r_i(h_{(1)}). \end{aligned}$$

We apply $(A \otimes \varepsilon) \circ \gamma_A$ to both sides; this gives

$$\xi_M(x) \varepsilon(h) = S(h_{(1)}) \cdot \xi_M(h_{(2)} \cdot x),$$

and, finally,

$$h \cdot \xi_M(x) = h_{(1)} \cdot \xi_M(x) \varepsilon(h_{(2)}) = (h_{(1)} S(h_{(2)})) \cdot \xi_M(h_{(3)} \cdot x) = \xi_M(h \cdot x),$$

which gives the desired formula. \square

Proposition 8.4. *The action of $\text{Pic}(B)$ on $Z(B)$ induces an action of $\text{Pic}(B)^H$ on $Z^n(H, Z(B))$, $B^n(H, Z(B))$ and $H^n(H, Z(B))$. More precisely, if $f : H^{\otimes n} \rightarrow Z(B)$ is a cocycle (resp. a coboundary), then $\xi_M \circ f$ is also a cocycle (resp. a coboundary).*

Proof. This follows immediately from Proposition 8.3 and the definition of Sweedler cohomology, see [20] or [5, Sec. 9.1]. \square

Since $\text{Pic}(B)^H$ acts on $H^2(H, Z(B))$, we can consider the cohomology group $H^1(\text{Pic}(B)^H, H^2(H, Z(B)))$.

Theorem 8.5. $g_3 \in Z^1(\text{Pic}(B)^H, H^2(H, Z(B)))$.

Proof. Let $[M], [M'] \in \text{Pic}(B)^H$, and consider the corresponding total integrals

$$t_M : H \rightarrow E := {}_{A^{\square e}} \text{END}(A^{\square e} \otimes_{B^e} M), \quad t_{M'} : H \rightarrow E'.$$

We recall from Section 4 that $[\sigma_M] = g_3[M]$ is defined by the formula

$$t_M(k) \circ t_M(h) = \sigma_M(h_{(1)} \otimes k_{(1)}) t_M(h_{(2)} k_{(2)}).$$

This means that

$$\begin{aligned}
& (t_M(k) \circ t_M(h))(1_{A^{\square e}} \otimes_{B^e} m) \\
&= t_M(k)((m(h)^+ \otimes m(h)^-) \otimes_{B^e} m(h)^0) \\
&= (m(h)^+ m(h)^0 (k)^+ \otimes m(h)^0 (k)^- m(h)^-) \otimes_{B^e} m(h)^0 (k)^0
\end{aligned}$$

equals

$$\begin{aligned}
& \sigma_M(h_{(1)} \otimes k_{(1)}) t_M(h_{(2)} k_{(2)})(1_{A^{\square e}} \otimes_{B^e} m) \\
&= (\sigma_M(h_{(1)} \otimes k_{(1)}) m(h_{(2)} k_{(2)})^+ \otimes m(h_{(2)} k_{(2)})^-) \otimes_{B^e} m(h_{(2)} k_{(2)})^0.
\end{aligned}$$

Then we compute

$$\begin{aligned}
& (t_{M \otimes_B M'}(k) \circ t_{M \otimes_B M'}(h))(1_{A^{\square e}} \otimes_B (m \otimes_B m')) \\
&\stackrel{(18)}{=} t_{M \otimes_B M'}(k)((m(h_{(1)})^+ \otimes m'(h_{(2)})^-) \\
&\quad \otimes_{B^e} (m(h_{(1)})^0 m(h_{(1)})^- m'(h_{(2)})^+ \otimes_B m'(h_{(2)})^0)) \\
&\stackrel{(18)}{=} (m(h_{(1)})^+ m(h_{(1)})^0 (k_{(1)})^+ \otimes m'(h_{(2)})^0 (k_{(2)})^- m'(h_{(2)})^-) \\
&\quad \otimes_{B^e} (m(h_{(1)})^0 (k_{(1)})^0 m(h_{(1)})^0 (k_{(1)})^- m(h_{(1)})^- \\
&\quad m'(h_{(2)})^+ m'(h_{(2)})^0 (k_{(2)})^+ \otimes_B m'(h_{(2)})^0 (k_{(2)})^0),
\end{aligned}$$

hence

$$\begin{aligned}
& g\left((t_{M \otimes_B M'}(k) \circ t_{M \otimes_B M'}(h))(1_{A^{\square e}} \otimes_B (m \otimes_B m'))\right) \\
&= ((m(h_{(1)})^+ m(h_{(1)})^0 (k_{(1)})^+ \otimes m(h_{(1)})^0 (k_{(1)})^- m(h_{(1)})^-) \\
&\quad \otimes_{B^e} m(h_{(1)})^0 (k_{(1)})^0) \\
&\quad \otimes_B ((m'(h_{(2)})^+ m'(h_{(2)})^0 (k_{(2)})^+ \otimes m'(h_{(2)})^0 (k_{(2)})^- m'(h_{(2)})^-) \\
&\quad \otimes_{B^e} m'(h_{(2)})^0 (k_{(2)})^0) \\
&= ((\sigma(h_{(1)} \otimes k_{(1)}) m(h_{(2)} k_{(2)})^+ \otimes m(h_{(2)} k_{(2)})^-) \otimes_{B^e} m(h_{(2)} k_{(2)})^0) \\
&\quad \otimes_B ((\sigma'(h_{(3)} \otimes k_{(3)}) m'(h_{(4)} k_{(4)})^+ \otimes m'(h_{(4)} k_{(4)})^-) \\
&\quad \otimes_{B^e} m'(h_{(4)} k_{(4)})^0) \\
&\stackrel{(30)}{=} ((\sigma(h_{(1)} \otimes k_{(1)}) \xi_M(\sigma'(h_{(2)} \otimes k_{(2)})) m(h_{(3)} k_{(3)})^+ \otimes m(h_{(3)} k_{(3)})^-) \\
&\quad \otimes_{B^e} m(h_{(3)} k_{(3)})^0) \otimes_B ((m'(h_{(4)} k_{(4)})^+ \otimes m'(h_{(4)} k_{(4)})^-) \\
&\quad \otimes_{B^e} m'(h_{(4)} k_{(4)})^0) \\
&= \sigma(h_{(1)} \otimes k_{(1)}) \xi_M(\sigma'(h_{(2)} \otimes k_{(2)})) g(t_{M \otimes_B M'}(hk)(1_{A^{\square e}} \otimes_B (m \otimes_B m'))).
\end{aligned}$$

This shows that

$$t_{M \otimes_B M'}(k) \circ t_{M \otimes_B M'}(h) = \sigma(h_{(1)} \otimes k_{(1)}) \xi_M(\sigma'(h_{(2)} \otimes k_{(2)})) t_{M \otimes_B M'}(hk).$$

Consequently,

$$\sigma_{M \otimes_B M'} = \sigma_M * (\xi_M \circ \sigma_{M'}),$$

which proves the Theorem. \square

9. GALOIS OBJECTS OVER NONCOCOMMUTATIVE HOPF ALGEBRAS

Let H be a (possibly non-cocommutative) Hopf algebra with bijective antipode, and A an H -Galois extension of $B = A^{\text{co}H}$. We can still define the Picard groups $\text{Pic}^H(A)$, $\text{Pic}(B)$ and $\text{Pic}^{\square H}(B)$, and we still have that $\text{Pic}^H(A) \cong \text{Pic}^{\square H}(B)$, cf. Section 5. We can therefore ask whether the exact sequence from Theorem 7.5 can be generalized to non-cocommutative Hopf algebras. The obstructions are the following.

- (1) We need the property that $A \square_H A^{\text{op}}$ is an H -Galois extension (see Theorem 3.3 and Proposition 3.5) in order to apply Corollary 4.10 (with H replaced by $A \square_H A^{\text{op}}$);
- (2) We used the fact that H is cocommutative when we defined the H -stable part of $\text{Pic}(B)$ (see Section 6);
- (3) We want to have a group structure on $\Omega_{A \square_H A^{\text{op}}}$.

These problems can be fixed in the case where the algebra of coinvariants B coincides with the groundfield k , that is, when A is a Galois object. Examples of Galois objects are for example classical Galois field extensions (then $H = (kG)^*$, with G a finite group); other examples of Galois objects over noncocommutative algebras have been studied in [1, 2].

In this case, $\Omega_{A \square_H A^{\text{op}}} \cong \text{Alg}(H, k)$ is a group, by Proposition 4.4, and problem 3) is fixed. To handle problem 1), we invoke the theory of Hopf-Bigalois objects, as developed by Schauenburg [17]. If A is a right H -Galois object, then there exists another Hopf algebra $L = L(A, H)$, unique up to isomorphism, such that A is an (L, H) -Bigalois object, that is, A is left L -Galois object, a right H -Galois object, and an (L, H) -bicomodule. For the construction of L , we refer to [17, Sec. 3]. If H is cocommutative, then $L = H$. We can then introduce the Harrison groupoid [17, Sec. 4]. Objects are Hopf algebras with bijective antipode, morphisms are Hopf-Bigalois objects, and the composition of morphisms is given by the cotensor product. The inverse of a morphism A between L and H (that is, an (L, H) -Bigalois object) is A^{op} , with left H -coaction λ given by the formula $\lambda(a) = S^{-1}(a_{[1]}) \otimes a_{[0]}$. In particular, $(A \square_H A^{\text{op}})$ is an (L, L) -Bigalois object, and, in particular, a right H -Galois object. Applying Proposition 4.4 and Corollary 4.10, we obtain

$$\overline{\Omega}_{A \square_H A^{\text{op}}} \cong \Omega_{A \square_H A^{\text{op}}} \cong \text{Alg}(A \square_H A^{\text{op}}, k) \cong \text{Alg}(L, k).$$

The isomorphism $\text{Alg}(A \square_H A^{\text{op}}, k) \cong \text{Alg}(L, k)$ can also be obtained as follows. Since A^{op} is the inverse of A in the Harrison groupoid, we have that $A \square_H A^{\text{op}} \cong L$ as bicomodule algebras.

Since $\text{Pic}(B) = 1$ (k is a field), the map $\text{Pic}^{\square H}(B) \rightarrow \text{Pic}(B)$ is trivial. Its kernel is $\overline{\Omega}_{A \square_H A^{\text{op}}}$, so we obtain the following result.

Proposition 9.1. *Let H be a Hopf algebra with bijective antipode, A a right H -Galois object, and $L = L(A, H)$. Then $\text{Pic}^H(A) \cong \text{Pic}^{\square H}(k) \cong \text{Alg}(L, k)$.*

If H is cocommutative, then $L = H$, so $\text{Pic}^H(A) \cong \text{Alg}(H, k)$. This isomorphism can be described explicitly. The isomorphism $\text{Alg}(H, k) \rightarrow \text{Alg}(A \square_H A^{\text{op}}, k)$ is a particular case of (28). For an algebra morphism

$\alpha : H \rightarrow k$, the corresponding $\phi_\alpha : A \square_H A^{\text{op}} \rightarrow k$ is given by

$$\phi_\alpha\left(\sum_j a_j \otimes a'_j\right) = \sum_j a_j a'_{j[0]} \alpha(a'_{j[1]}),$$

and the corresponding $A \square_H A^{\text{op}}$ -action on k is induced by α .

Let us now compute the corresponding A -bimodule structure on A . It is shown in [6, Prop. 2.3] that we have a right H -colinear isomorphism

$$f : A \otimes (A \square_H A^{\text{op}}) \rightarrow A \otimes A^{\text{op}}, \quad f\left(a \otimes \left(\sum_j a_j \otimes a'_j\right)\right) = \sum_j a a_j \otimes a'_j.$$

The inverse of f is given by the formula

$$f^{-1}(a \otimes a') = \sum_i l_i(S(a'_{[1]})) \otimes r_i(S(a'_{[1]})) \otimes a'_{[0]}.$$

For $N \in A \square_H A^{\text{op}} \mathcal{M}$, we have an isomorphism

$$g : A \otimes N \xrightarrow{\psi} A \otimes (A \square_H A^{\text{op}}) \otimes_{A \square_H A^{\text{op}}} N \xrightarrow{f \otimes N} (A \square_H A^{\text{op}}) \otimes_{A \square_H A^{\text{op}}} N.$$

Here ψ is the natural isomorphism. The A -bimodule structure on $A \otimes N$ is obtained by transporting the A -bimodule structure on $(A \square_H A^{\text{op}}) \otimes_{A \square_H A^{\text{op}}} N$ to $A \otimes N$ using g . Take $a, a', a'' \in A$ and $n \in N$. Then

$$a' g(a \otimes n) a'' = a' ((a \otimes 1) \otimes_{A \square_H A^{\text{op}}} n) a'' = (a' a \otimes a'') \otimes_{A \square_H A^{\text{op}}} n.$$

Now

$$\begin{aligned} a' \cdot (a \otimes n) \cdot a'' &= g^{-1}(a' g(a \otimes n) a'') = \psi^{-1}(f^{-1}(a' a \otimes a'') \otimes_{A \square_H A^{\text{op}}} n) \\ &= \sum_i a' a l_i(S(a''_{[1]})) \otimes (r_i(S(a''_{[1]})) \otimes a''_{[0]}) n \in A \otimes N. \end{aligned}$$

Now let $N = k$, with left $A \square_H A^{\text{op}}$ -action given by ϕ_α , and identify $A \otimes N \cong A$ using the natural isomorphism. The corresponding A -bimodule structure on $A \otimes N \cong A$ is then given by the formula

$$\begin{aligned} a' \cdot a \cdot a'' &= \sum_i a' a l_i(S(a''_{[1]})) \phi_\alpha(r_i(S(a''_{[1]})) \otimes a''_{[0]}) \\ &= \sum_i a' a l_i(S(a''_{[2]})) r_i(S(a''_{[2]})) a''_{[0]} \alpha(a''_{[1]}) \stackrel{(5)}{=} a' a a''_{[0]} \alpha(a''_{[1]}). \end{aligned}$$

We conclude that the $(A \otimes A^{\text{op}}, H)$ -Hopf module P representing the element in $\text{Pic}^H(A)$ corresponding to α is equal to A as a left A -module and a right H -comodule, and with right A -module action given by the formula

$$(31) \quad a \cdot a' = a a'_{[0]} \alpha(a'_{[1]}).$$

Example 9.2. Let $q = p^d$, and k a field of characteristic p . Consider the Hopf algebra $H = k[x]/(x^q - x)$, with x primitive and $S(x) = -x$. If $d = 1$, then H is the dual of the group algebra over the cyclic group of order p . The H -Galois are known, we refer to [5, Sec. 11.3] for detail. More precisely, the group of Galois objects $\text{Gal}(k, H) \cong k/\{a^q - a \mid a \in k\}$. The Galois object corresponding to $a \in k$ is the Artin-Schreier extension

$$S = k[y]/(y^q - y - a)$$

with coaction $\rho_S(y) = y \otimes 1 + 1 \otimes x$. Furthermore

$$\text{Alg}(H, k) \cong \{b \in k \mid b^q = b\}.$$

The algebra morphism α corresponding to $b \in k$ is determined by the formula $\alpha(x) = b$. Now fix $a \in k$, and consider $S = k[y]/(y^q - y - a)$. It follows from Proposition 9.1 that

$$\text{Pic}^H(S) \cong \{b \in k \mid b^q = b\}.$$

The $(S \otimes S^{\text{op}}, H)$ -Hopf module P representing the element of $\text{Pic}^H(S)$ corresponding to b satisfying $b^q = b$ is equal to S as a left S -module and a right H -comodule. The right S -action on P is completely determined by the right action of y on $p \in P = S$. Since $y_{[0]}\alpha(y_{[1]}) = y + b$, formula (31) takes the form

$$p \cdot y = p(y + b).$$

Example 9.3. We keep the notation of Example 9.2. Let B be a k -algebra, and $A = B \otimes S$, $\rho_A = B \otimes \rho : B \otimes S \rightarrow B \otimes S \otimes H$. Then

$$\text{can}_A = B \otimes \text{can}_S : A \otimes_B A = (B \otimes S) \otimes_B (B \otimes S) \cong B \otimes S \otimes S \rightarrow B \otimes S \otimes H = A \otimes H$$

is an isomorphism, hence A is an H -Galois extension of B .

We claim that the Miyashita-Ulbrich action on $Z(B)$ is trivial. Let $\gamma_S(h) = \sum_i l_i(h) \otimes r_i(h) \in S \otimes S$, for all $h \in H$. It is easy to see that

$$\text{can}_A\left(\sum_i 1_B \otimes l_i(h) \otimes r_i(h)\right) = 1_B \otimes 1_S \otimes h = 1_A \otimes h,$$

hence

$$\gamma_A(h) = \sum_i (1_B \otimes l_i(h)) \otimes_B (1_B \otimes r_i(h)),$$

and, for $x \in Z(B) \cong Z(B) \otimes k$,

$$h \cdot x = \sum_i (1_B \otimes l_i(h))(1_B \otimes 1_k)(1_B \otimes r_i(h)) = \varepsilon(h)x.$$

Now it follows that

$$H^1(H, Z(B)) \cong \text{Alg}(H, B) = \{b \in B \mid b^q = b\}.$$

Our next aim is to show that every element of $\text{Pic}(B)$ is H -stable. First observe that $A^{\text{op}} = B^{\text{op}} \otimes S^{\text{op}}$, with $S^{\text{op}} = S$ as an algebra, and with H -coaction given by $\rho(y) = y \otimes 1 - 1 \otimes x$. Then

$$A \square_H A^{\text{op}} = B \otimes B^{\text{op}} \otimes (S \square_H S^{\text{op}}) = B^e \otimes S^{\square e}.$$

Now let $M \in \underline{\text{Pic}}(B)$. Then $A^{\square e} \otimes_{B^e} M = M \otimes S^{\square e} \cong M \otimes H$, since $S^{\square e} \cong H$. This shows that M is H -stable, and it follows that $\text{Pic}(B) = \text{Pic}(B)^H$. The exact sequence from Theorem 7.5 specializes to

$$1 \rightarrow \{b \in B \mid b^q = b\} \rightarrow \text{Pic}^H(A) \rightarrow \text{Pic}(B) \rightarrow H^2(H, Z(B)).$$

Suppose that H is a finite dimensional commutative Hopf algebra. Then H^* is a cocommutative Hopf algebra. If A is an H^* -Galois object, then A is an H -module algebra, with left H -action $h(a) = \langle a_{[1]}, h \rangle a_{[0]}$. Furthermore $\text{Alg}(H^*, k) = G(H)$, the group of grouplike elements of H . Take $g \in G(H)$; (31) can then be rewritten as

$$(32) \quad a \cdot a' = ag(a').$$

Example 9.4. In [12], forms of the cyclic group algebra have been studied. One of the examples is the following quotient of the trigonometric Hopf algebra over \mathbb{Q} :

$$H = \mathbb{Q}[c, s]/(c^2 + s^2 - 1, sc).$$

H is a form of the group algebra over the cyclic group of order 4, that is, $H \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}C_4$. The grouplike elements of $H \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[c, s]/(c^2 + s^2 - 1, sc)$ are $g_i = (c + is)^i$, $i = 0, \dots, 3$. It is easy to see that $g_1, g_3 \notin H$ and $g_0 = 1, g_2 = c^2 - s^2 \in H$, hence

$$G(H) = \{1, g_2 = c^2 - s^2\}.$$

An example of an H^* -Galois object is given in [12, Remark p. 135]: $A = \mathbb{Q}(\mu)$, with $\mu = \sqrt[4]{2}$, and H -action given by the formulas

$$\begin{aligned} c(1) &= 1 & c(\mu) &= 0 & c(\mu^2) &= -\mu^2 & c(\mu^3) &= 0 \\ s(1) &= 0 & s(\mu) &= -\mu & s(\mu^2) &= 0 & s(\mu^3) &= \mu^3 \end{aligned}$$

Since $G(H) = \{1, g_2\}$, it follows from Proposition 9.1 that $\text{Pic}^H(A)$ is the cyclic group of order 2. Using (32), we can describe its nontrivial element $[P]$. First observe that the action of g_2 on A is given by the formula $g_2(\mu^i) = (-1)^i \mu$. Then $P = A$ as a left A -module and a left H -module, with right A -action given by

$$a \cdot \mu^i = (-1)^i \mu^i a.$$

REFERENCES

- [1] T. Aubriot, On the classification of Galois objects over the quantum group of a nondegenerate bilinear form, *Manuscripta Math.* **122** (2007), 119–135.
- [2] T. Aubriot, Classification des objets galoisiens de $U_q(\mathfrak{g})$ à homotopie près, *Comm. Algebra* **35** (2007), 3919–3936.
- [3] M. Beattie and A. del Río, The Picard group of a category of graded modules, *Comm. Algebra* **24** (1996), 4397–4414.
- [4] M. Beattie and A. del Río, Graded equivalences and Picard groups, *J. Pure Appl. Algebra* **141** (1999), 131–152.
- [5] S. Caenepeel, Brauer groups, Hopf algebras and Galois theory, *K-Monographs Math.* **4**, Kluwer Academic Publishers, Dordrecht, 1998.
- [6] S. Caenepeel, S. Crivei, A. Marcus and M. Takeuchi, Morita equivalences induced by bimodules over Hopf-Galois extensions, *J. Algebra* **314** (2007), 267–302.
- [7] S. Caenepeel, G. Militaru and S. Zhu, “Frobenius and separable functors for generalized module categories and nonlinear equations”, *Lect. Notes Math.* **1787**, Springer-Verlag, Berlin, 2002.
- [8] E.C. Dade, Extending irreducible modules, *J. Algebra* **72** (1981), 374–403.
- [9] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, “Hopf Algebras. An Introduction”, *Monographs Textbooks Pure Appl. Math.* **235**, Marcel Dekker, New York, 2001.
- [10] C. Faith, “Algebra: Rings, Modules, Categories”, Springer-Verlag, Berlin, 1973.
- [11] J. Haefner and A. del Río, Actions of Picard groups on graded rings, *J. Algebra* **218** (1999), 573–607.
- [12] R. Hagemüller, B. Pareigis, Hopf algebra forms of the multiplicative group and other groups, *Manuscripta Math.* **55** (1986), 121–136.
- [13] A. Marcus, Equivalences induced by graded bimodules, *Comm. Algebra* **26** (1998), 713–731.
- [14] A. Marcus, On Picard groups and graded rings, *Comm. Algebra* **26** (1998), 2211–2219.
- [15] G. Militaru and D. Ştefan, Extending modules for Hopf Galois extensions, *Comm. Algebra* **22** (1994), 5657–5678.

- [16] S. Montgomery, “Hopf algebras and their actions on rings”, American Mathematical Society, Providence, 1993.
- [17] P. Schauenburg, Hopf bigalois extensions, *Comm. Algebra* **24** (1996), 3797–3825.
- [18] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, *Israel J. Math.* **72** (1990), 167–195.
- [19] H.-J. Schneider, Representation theory of Hopf Galois extensions, *Israel J. Math.* **72** (1990), 196–231.
- [20] M. E. Sweedler, Cohomology of algebras over Hopf algebras, *Trans. Amer. Math. Soc.* **133** (1968), 205–239.
- [21] K.-H. Ulbrich, On modules induced or coinduced from Hopf subalgebras, *Math. Scand.* **67** (1990), 177–182.

FACULTY OF ENGINEERING, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2, B-1050 BRUSSELS, BELGIUM

E-mail address: `scaenepe@vub.ac.be`

URL: `http://homepages.vub.ac.be/~scaenepe/`

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, BABEȘ-BOLYAI UNIVERSITY, STR. MIHAIL KOGĂLNICEANU 1, RO-400084 CLUJ-NAPOCA, ROMANIA

E-mail address: `marcus@math.ubbcluj.ro`