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Derived invariance of Clifford classes

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Abstract. We show that G-graded Rickard equivalences defined over small fields preserve Clifford classes associated to characters. These equivalences are compatible with operation on Clifford classes defined in terms of central simple crossed products.

1 Introduction

Clifford classes have been introduced by Turull in [12] as a tool in the investigation of Schur indices of irreducible complex characters of finite groups together with their Clifford theory. These classes arise from an equivalence relation between central simple G-acted algebras over a field F, where G is a finite group.

We have shown in [6] that the G-graded Morita equivalence between the corresponding skew group algebras gives rise to the same classes. In addition, the Morita equivalence relation can be defined between two central simple G-graded crossed products, an observation that has other advantages as well.

We recall our approach to Clifford classes in Section 2, and then we show how other notions and constructions from [13] extend from G-algebras to crossed products. In Section 3 we discuss the *interior subgroup* of $\mathrm{Cliff}(G,F)$, formed by classes of crossed products for which the 1-component is a central simple F-algebra, its action on $\mathrm{Cliff}(G,F)$, and its parametrization in terms of $H^2(G,F^\times)$ and $\mathrm{Br}(F)$. Section 4 deals with inflation, restriction (truncation) and induction of crossed products and of Clifford classes. In Section 5 we show that G-graded Rickard equivalences defined over small fields induce character correspondences that preserve Clifford classes and are compatible with the above-mentioned operations. We show that such Rickard equivalences defined over the p-adic number field $\mathbb{Q}_p$ exist in the case of blocks with cyclic defect groups (Section 6), by adapting Rouquier's proof [10] of Broué's abelian defect group conjecture for these blocks. This also give another explanation for the validity of Turull's conjecture [14, Theorem 2.2] for cyclic blocks.

Our general references are [8] for standard results on central simple algebras, [3] for block theory and [11] for Rickard equivalences. As this paper is a sequel of [6], we keep its conventions, and we refer the reader to it for unexplained facts concerning crossed products and Clifford classes.

2 Characters and Clifford classes

We fix a finite group G, a field F of characteristic zero, and let $\bar{F}$ be an algebraic closure of F. Let $R = \bigoplus_{g\in G} R_g$ be a finite-dimensional strongly G-graded F-algebra. Write $A = R_1$.

2.1. There is a natural action of the group G on the set of ideals of A and on the center $Z(A)$ of A. We say that R is a simple G-graded F-algebra if R_1 has no non-trivial proper G-invariant ideals. If in addition $Z(A)^G = F$, then we say that R is central simple.

The skew group algebra $R := A * G$ is called a *trivial* central simple G-graded F-algebra if $A = \mathrm{End}_F(N)$, where N is a (left) FG-module.

If R and S are strongly G-graded F-algebras, then we consider the diagonal subalgebra

$$\Delta(R \otimes_F S) := \bigoplus_{g\in G} R_g \otimes_F S_g$$

of $R \otimes_F S$.

2.2. Let R and S be strongly G-graded algebras. We say that there is a G-graded Morita equivalence between R and S if there are G-graded bimodules ${}_R M_S$ and ${}_S N_R$ and isomorphisms $M \otimes_S N \simeq R$ and $N \otimes_R M \simeq S$ of G-graded bimodules.

If V and V' are R-modules, then there is a natural action of G on $\mathrm{Hom}_{R_1}(V, V')$, and the graded Morita equivalence induces an isomorphism

$$\mathrm{Hom}_{R_1}(V, V') \simeq \mathrm{Hom}_{S_1}(N \otimes_R V, N \otimes_R V')$$

of FG-modules.

Theorem 2.3. *Let R and S be central simple G-graded F-algebras. Let $1 = e_1 + \cdots + e_n$ and $1 = f_1 + \cdots + f_m$ be decompositions into primitive central idempotents of A and B, respectively.*

The following statements are equivalent.

(i) *There is a G-graded Morita equivalence between R and S.*

(ii) *$m = n$ and there is an isomorphism between the G-sets $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ such that e_1 corresponds to f_1, and moreover, there is an H-graded Morita equivalence between $R' := e_1 R e_1$ and $S' := f_1 S f_1$, where H is the stabilizer in G of e_1.*

(iii) *There exist FG-modules V and V' such that writing $T := \mathrm{End}_F(V) * G$ and $T' := \mathrm{End}_F(V') * G$, we have an isomorphism*

$$\Delta(R \otimes_F T) \simeq \Delta(S \otimes_F T')$$

of G-graded algebras.

Definition 2.4. Let R and S be central simple G-graded F-algebras. If the equivalent conditions of Theorem 2.3 hold, then we say that R and S are *equivalent*.

We denote by $\mathrm{Cliff}(G,F)$ the set of equivalence classes of central simple G-graded F-algebras, and by $[R]$ the class of R in $\mathrm{Cliff}(G,F)$.

The relationship between this definition and Turull's original definition is discussed in detail in [6].

Assume that the strongly G-graded F-algebra R is semisimple. Write $\bar{F}R := \bar{F} \otimes_F R$. Next we recall how to associate Clifford classes to irreducible characters of $\bar{F}R$.

Proposition 2.5. *Let V be a simple R-module, and let χ be the character of a simple submodule of the $\bar{F}R$-module $\bar{F} \otimes_F V$. Let*

$$E := \mathrm{End}_R(R \otimes_A V)^{\mathrm{op}}.$$

Then E is a central simple G-graded $F(\chi_A)$-algebra, where

$$F(\chi_A) = F(\{\chi(a) \mid a \in A\}) = F(\{\chi(a) \mid a \in A \cap Z(R)\}).$$

Definition 2.6. With the notation of the previous proposition, the Clifford class $[[\chi]]$ of χ is the Clifford class $[E]$ in $\mathrm{Cliff}(G,K)$, where $K := F(\chi_A)$.

The next result ([6, Theorem 3.4]) is a generalization of [12, Theorem 3.5], and examines what happens when the Clifford classes of two characters are equal.

Theorem 2.7. *Let R and S be strongly G-graded F-algebras, and write $A = R_1$ and $B = S_1$.*

Let χ be an irreducible character of $\bar{F}R$ and let η be an irreducible character of $\bar{F}S$. We assume that $F = F(\chi_A) = F(\eta_A)$, so the classes $[[\chi]]$ and $[[\eta]]$ belong to $\mathrm{Cliff}(G,F)$.

Assume that $[[\chi]] = [[\eta]]$. Then for any subgroup H of G, there is an isometry between $\mathrm{Char}(\bar{F}R_H|\chi_A)$ and $\mathrm{Char}(\bar{F}S_H|\chi_B)$. This correspondence commutes with induction, restriction and G-conjugation of characters, with multiplication with characters of $\bar{F}H$ and with $\mathrm{Gal}(\bar{F}/F)$-conjugation of characters.

Corresponding pairs of characters have the same fields of character values, the same Schur indices, and determine the same Clifford classes (and in particular the same elements in the respective Brauer groups).

3 Inertia groups and the interior subgroup

Note that $\mathrm{Cliff}(G,F)$ does not have a natural group structure. However, it contains a subgroup with respect to the operation given by taking the diagonal subalgebra, and this subgroup acts on $\mathrm{Cliff}(G,F)$.

3.1. Let G be a finite group and F a field of characteristic zero. Let R be a central simple G-graded F-algebra with $R_1 = A$. Let $1 = e_1 + \cdots + e_n$ be the decomposition into primitive central idempotents of A, and let $K = e_1 Z(A)$.

The subgroup $I := C_G(K)$ is called an *inertia group* of R. The set of inertia groups of R is a conjugacy class of subgroups of G, and this set is invariant under graded Morita equivalence. Moreover, K/F is a Galois extension with Galois group isomorphic to $C_G(e_1)/I$.

Definition 3.2. The *interior subgroup* of $\mathrm{Cliff}(G,F)$ is defined by

$$\mathrm{ICliff}(G,F) = \{[R] \in \mathrm{Cliff}(G,F) \mid Z(R_1) = F\}.$$

This generalizes the notion introduced in [13, Section 3] in the case of central simple G-acted algebras. From the definition and by [6, Lemma 2.9] it is not difficult to deduce the following result.

Proposition 3.3. (a) *Let* $[R] \in \mathrm{Cliff}(G,F)$. *Then* $[R] \in \mathrm{ICliff}(G,F)$ *if and only if* G *is the inertia group of* R.

(b) *Let* $[R], [S] \in \mathrm{Cliff}(G,F)$. *If either* $[R] \in \mathrm{ICliff}(G,F)$ *or* $[S] \in \mathrm{ICliff}(G,F)$, *then the product*

$$[R][S] := [\Delta(R \otimes S)]$$

is well defined. In particular, $\mathrm{ICliff}(G,F)$ *is a group acting on the set* $\mathrm{Cliff}(G,F)$, *and moreover, this action is compatible with field extensions of* F.

3.4. *Factorization*. Let $[R] \in \mathrm{Cliff}(G,F)$, write $R_1 := A$, and let $S \subseteq R$ be a strongly G-graded subalgebra of R such that $B := S_1$ is a central simple F-algebra. Then $C := C_A(B)$ becomes a G-acted algebra as follows. For each $g \in G$, there is an invertible element $s_g \in U(S) \cap S_g$; then define ${}^g c = s_g c s_g^{-1}$ for any $g \in G$ and $c \in C$. This clearly does not depend on the choice of s_g. We may form the skew group algebra $T := C * G$.

By using [8, Theorem 12.7] and the argument of [13, Theorem 4.1], it is easy to prove that $[T] \in \mathrm{Cliff}(G,F)$, and that the map

$$\Delta(S \otimes T) \to R, \quad bs_g \otimes cg \mapsto b \cdot {}^g c s_g,$$

where $g \in G$, $b \in B$, $s_g \in U(S) \cap S_g$ and $c \in C$, is an isomorphism of G-graded algebras, that is, $[R] = [S][T]$ in $\mathrm{Cliff}(G,F)$.

Proposition 3.5. *Let* $[R] \in \mathrm{ICliff}(G,F)$. *Then* R *determines a class*

$$\mathrm{cohi}(R) \in H^2(G, F^\times),$$

and the map

$$\mathrm{ICliff}(G,F) \to H^2(G,F^\times) \times \mathrm{Br}(F), \quad [R] \mapsto (\mathrm{cohi}(R), [R_1])$$

is an isomorphism of groups.

Proof. For each $g \in G$, let $r_g \in U(R) \cap R_g$. By the Skolem–Noether theorem, there is $u_g \in U(R_1)$ such that $r_g a r_g^{-1} = u_g a u_g^{-1}$ for all $a \in R_1$ and $g \in G$. It follows that $c_g := u_g^{-1} r_g \in C_R(R_1)_g \cap U(R)$, hence the family c_g, $g \in G$ determines a 2-cocycle $\gamma \in Z^2(G, F^\times)$. By definition, $\operatorname{cohi}(R)$ is the class $[\gamma] \in H^2(G, F^\times)$.

Clearly, the map sending $[R]$ to $(\operatorname{cohi}(R), [R_1])$ is a bijection with inverse given by

$$([\gamma], [A]) \mapsto [F_\gamma G \otimes_F A]$$

for all $[\gamma] \in H^2(G, F^\times)]$ and $[A] \in \operatorname{Br}(F)$. It is also easy to see that these maps are group homomorphisms. □

4 Operations with Clifford classes

4.1. *Inflation*. Let $\phi : G \to \overline{G}$ be a surjective group homomorphism, write $N = \operatorname{Ker} \phi$, and let $[S] \in \operatorname{Cliff}(\overline{G}, F)$. Consider the group algebra $R := S[N]$ of S and N. This F-algebra is strongly G-graded, with components $R_g = S_{\phi(g)}$, for all $g \in G$. We obtain a map

$$\operatorname{infl}_\phi : \operatorname{Cliff}(\overline{G}, F) \to \operatorname{Cliff}(G, F), \quad [S] \mapsto [S[\operatorname{Ker} \phi]],$$

since if there is a $\overline{G}$-graded Morita equivalence between S and S', then there is a G-graded Morita equivalence between $S[N]$ and $S'[N]$.

This map restricts to a group homomorphism $\operatorname{ICliff}(\overline{G}, F) \to \operatorname{ICliff}(G, F)$, it is compatible with the action of $\operatorname{ICliff}(G, F)$ on $\operatorname{Cliff}(G, F)$, and it is compatible with field extensions.

Note that this construction generalizes the one given in [13, Section 6]. Indeed, if B is a $\overline{G}$-acted algebra and $S = B * \overline{G}$ is the corresponding skew group algebra, then the skew group algebra $R = A * G$ corresponding to the G-acted algebra $A := \operatorname{infl}_\phi(B)$ coincides with $S[N]$.

4.2. *Restriction*. Let $[R] \in \operatorname{Cliff}(G, F)$ and let H be a subgroup of G. Then R_H is a strongly H-graded F-algebra, and truncation induces a map Res^G_H from

$$\operatorname{Cliff}(G, F)_H := \{[R] \in \operatorname{Cliff}(G, F) \mid Z(R_1)^H = F\}$$

to $\operatorname{Cliff}(H, F)$.

It is easy to see that this map sends $\operatorname{ICliff}(G, F)$ to $\operatorname{ICliff}(H, F)$, it is compatible with the action of $\operatorname{ICliff}(G, F)$ on $\operatorname{Cliff}(G, F)$, and it is compatible with field extensions.

4.3. *Induction*. A general notion of induction of crossed products was given by Klasen and Schmid [4]. We present here their construction in a slightly modified form, and with actions on the left.

Let R be a crossed product of the ring A with the group G. Let e be a central idempotent of A such that the centralizer $H := C_G(e)$ has finite index in G. Assume that the distinct G-conjugates of e are pairwise orthogonal, and their sum is 1. Then $S := eRe = eR_H$ is a crossed product of $B := eA$ with H, and we say that $R = \mathrm{Ind}_H^G S$ is *induced* from S and H. Note that $ReR = R$, hence R and S are Morita equivalent.

Conversely, let S be a crossed product of B and H, and assume that H has finite index in G. Then, by [4, Theorem 1], there exists a crossed product $R = \mathrm{Ind}_H^G(S)$, which is unique up to an isomorphism of G-graded rings.

To construct R, we start with a presentation $S = B_\beta^\tau H$ obtained by choosing an homogeneous invertible element $s_h \in U(S) \cap S_h$, for each $h \in H$. Then the map τ and the factor set β are given by

$$\tau : H \to \mathrm{Aut}(B), \quad \tau_x(b) = s_x b s_x^{-1},$$

and

$$\beta : H \times H \to U(B), \quad \beta(x, y) = s_x s_y s_{xy}^{-1},$$

for all $x, y \in H$ and $b \in B$. Let $\{t_i \mid i \in G/H\}$ be a system of representatives for the left cosets of H in G, with $t_H = 1$. For any $x \in G$ and $i \in G/H$ let $xi \in G/H$ and $x_i \in H$ be defined by the equality

$$xt_i = t_{xi} x_i.$$

Let $A = \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} B$, and define the multiplication in A by

$$(i \otimes b)(j \otimes b') = \delta_{ij} i \otimes bb',$$

for all $i, j \in G/H$ and $b, b' \in B$, where δ_{ij} is the Kronecker symbol. For $x, y \in G$ define $\sigma_x \in \mathrm{Aut}\, A$ and $\alpha(x, y) \in U(A)$ by

$$\sigma_x(i \otimes b) = xi \otimes \tau_{x_i}(b), \quad \alpha_{x,y} = \sum_{i \in G/H} xyi \otimes \beta(x_i, y_{xi}).$$

Finally, let R be the free left A-module with basis $\{r_x \mid x \in G\}$ and multiplication defined by

$$(ar_x)(a'r_y) = a\sigma_x(a')\alpha(x, y) r_{xy},$$

for all $x, y \in G$ and $a, a' \in A$. One observes immediately that [13, Definition 8.1] is a particular case of this construction. More precisely, we have the following.

Proposition 4.4. (1) *With the above notation, R is a crossed product of A and G, and* $R = \mathrm{Ind}_H^G(S)$.

(2) *If $S = B * H$ is a skew group algebra, then* $\mathrm{Ind}_H^G(S)$ *is a skew group algebra of A and G.* (*In this case we also say that* $A = \mathrm{Ind}_H^G B$ *as G-algebras.*)

(3) *Let* $R = \operatorname{Ind}_H^G(S)$ *and let* R' *be another* G*-graded crossed product. Then there is an isomorphism*

$$\Delta(\operatorname{Ind}_H^G(S) \otimes R') \simeq \operatorname{Ind}_H^G(\Delta(S \otimes R'_H))$$

of G*-graded algebras.*

Proof. For (1) we refer to [4, Theorem 1], and (2) follows from the construction. Let e be a central idempotent of R_1 such that $S = eRe = eR_H$. Then $e \otimes 1 \in R_1 \otimes R'_1$ is a central idempotent of $\Delta(R \otimes R')$, and we have that

$$(e \otimes 1)\Delta(R \otimes R')(e \otimes 1) = (e \otimes 1)\Delta(R \otimes R')_H = \Delta(S \otimes R'_H).$$

Hence the statement follows by the definition of the induction. □

Note also that induction of crossed products appears implicitly in [6, Theorem 2.13]. In particular, it says that any central simple G-graded F-algebra is induced from a uniquely determined (up to conjugacy) central simple H-graded F-algebra whose 1-component is a skew-field. The following connection with endomorphism rings of induced modules can be deduced without difficulty.

Proposition 4.5. *Let* R *be a strongly* G*-graded ring, and let* H *be a subgroup of* G*. Let* V *be a simple* R_1*-module such that its stabilizer* (*inertia group*)

$$G_V = \{g \in H \mid R_g \otimes_{R_1} V \simeq V \text{ as } R_1\text{-modules}\}$$

equals H*. Then the following statements hold.*

(1) *Let* $\tilde{V} = \bigoplus_{i \in G/H} R_{t_i} \otimes_{R_1} V$ *be the sum of distinct* G*-conjugates of* V*. Then*

$$\operatorname{End}_R(R \otimes_{R_1} \tilde{V}) \simeq \operatorname{Ind}_H^G(\operatorname{End}_{R_H}(R_H \otimes_{R_1} V))$$

as G*-graded crossed products.*

(2) *Let* U *be a simple* R_H*-module containing* V *as an* R_1*-submodule. Then* $\operatorname{Ind}_H^G U$ *is a simple* R*-module*, *and*

$$\operatorname{End}_{R_1}(\operatorname{Ind}_H^G U) \simeq \operatorname{Ind}_H^G(\operatorname{End}_{R_1}(U))$$

as G*-algebras.*

Finally, induction behaves well with respect to central simple crossed products.

Theorem 4.6. *Let* H *be a subgroup of the finite group* G*. Then induction of crossed products from* H *to* G *defines a map*

$$\operatorname{Ind}_H^G : \operatorname{Cliff}(H, F) \to \operatorname{Cliff}(G, F).$$

Proof. The proof of [13, Proposition 8.2] adapts easily to show that if S is a central simple H-graded F-algebra, then $\mathrm{Ind}_H^G(S)$ is central simple G-graded F-algebra. Now let S and S' be equivalent central simple H-graded F-algebras. Then there is an H-graded (S, S')-bimodule N inducing a Morita equivalence between S and S'. Let $R = \mathrm{Ind}_H^G(S)$ and $R' = \mathrm{Ind}_H^G(S')$. There are central idempotents $e \in R_1$ and $e' \in R_1'$ such that $S = eRe$ and $S' = e'R'e'$. Then the (R, R')-bimodule

$$M := Re \otimes_S N \otimes_{S'} e'R'$$

induces a G-graded Morita equivalence between R and R'. $\square$

Corollary 4.7. *Let F be a field of characteristic zero and let $\bar{F}$ be an algebraic closure of F. Let R a be strongly G-graded F-algebras, write $A = R_1$, and let H be a subgroup of G.*

Let χ be an irreducible character of $\bar{F}R_H$ such that $F(\chi_A) = F$, so the class $[[\chi]]$ belongs to $\mathrm{Cliff}(H, F)$. *Assume that χ_A contains an irreducible character of $\bar{F}A$ which is stabilized by H. Then $\mathrm{Ind}_H^G \chi$ is an irreducible character of $\bar{F}R$, and*

$$[[\mathrm{Ind}_H^G \chi]] = \mathrm{Ind}_H^G[[\chi]] \in \mathrm{Cliff}(G, F).$$

5 Rickard equivalences

In this section we show that Clifford classes are invariant under derived equivalences. We adopt a setting slightly more general than that of [5].

5.1. Let $\mathscr{K}$ be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, and let $\mathcal{O}$ be the ring of integers in $\mathscr{K}$.

Fix a finite group G and let $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ be two G-graded crossed product $\mathcal{O}$-orders. We assume that R and S are symmetric $\mathcal{O}$-algebras, such that the symmetrizing forms of R and S are G-invariant symmetrizing forms for $A := R_1$ and $B := S_1$, respectively. Write $\mathscr{K}R = \mathscr{K} \otimes_{\mathcal{O}} R$, and assume that $\mathscr{K}R$ and $\mathscr{K}S$ (or equivalently $\mathscr{K}A$ and $\mathscr{K}B$) are semisimple $\mathscr{K}$-algebras.

We assume that there exists a finite extension $\hat{\mathscr{K}}$ of $\mathscr{K}$ such that $\hat{\mathscr{K}}$ is a splitting field of $\hat{\mathscr{K}}R_H$ and $\hat{\mathscr{K}}R_H$ for every subgroup H of G. Let $\hat{\mathcal{O}}$ be the ring of integers of $\hat{\mathscr{K}}$.

5.2. We say that there is a G-graded Rickard equivalence between R and S if there is a complex M of G-graded (R, S)-bimodules which are projective as R-modules and as right S-modules, such that

$$M \otimes_S M^\vee \simeq R$$

in the homotopy category of G-graded (R, R)-bimodules, and

$$M^\vee \otimes_R M \simeq S$$

in the homotopy category of G-graded (S, S)-bimodules, where $M^\vee$ is the $\mathcal{O}$-dual of M. In this case, the functors $M \otimes_S -$ and $M^\vee \otimes_R -$ are equivalences between the homotopy categories R-modules and S-modules. This is a G-graded equivalence, in the sense that it sends G-graded objects to G-graded objects, it commutes with grade shifting and with grade forgetting functors.

Theorem 5.3. *Assume that the complex M induces a G-graded Rickard equivalence between R and S. Then for each subgroup H of G there is an isometry between the $\hat{\mathcal{K}}$-characters of $\hat{\mathcal{K}}R_H$ and the $\hat{\mathcal{K}}$-characters of $\hat{\mathcal{K}}S_H$.*

These isometries are compatible with restriction, induction, G-conjugation and Galois conjugation of characters.

Moreover, corresponding characters have equal Clifford classes (and so in particular, equal Schur indices and determine the same element in the appropriate Brauer group), and the character correspondence commutes with induction and restriction of Clifford classes.

Proof. We know that for each subgroup H of G, the complex M_H induces an H-graded Rickard equivalence between R_H and S_H, and these equivalences commute with the induction, restriction and conjugation functors. By a well-known result of Broué, the Rickard equivalence is compatible with the extensions of $\mathcal{O}$ and $\mathcal{K}$, and induce an isometry of $\hat{\mathcal{K}}$-characters. The compatibility with Galois-conjugation follows from the fact that the complex M_H is Galois-invariant.

Let χ be an irreducible character of $\hat{\mathcal{K}}R_H$, and let V be a simple $\mathcal{K}R_H$-module such that χ is a constituent of the character of $\hat{\mathcal{K}} \otimes_{\mathcal{K}} V$. Let W be the simple $\mathcal{K}R_H$-module corresponding to V under the Rickard equivalence. Then the irreducible character η of $\hat{\mathcal{K}}S_H$ corresponding to χ is a constituent of $\hat{\mathcal{K}} \otimes_{\mathcal{K}} W$. The character χ determines the Clifford class

$$[[\chi]] = [\mathrm{End}_{R_H}(R_H \otimes_A V)^{\mathrm{opp}}] \in \mathrm{Cliff}(H, \mathcal{K}(\chi_A)),$$

while η determines the Clifford class

$$[[\eta]] = [\mathrm{End}_{S_H}(S_H \otimes_B W)^{\mathrm{opp}}] \in \mathrm{Cliff}(H, \mathcal{K}(\eta_B)).$$

The endomorphism algebra of a $\mathcal{K}R_H$-module is the same when regarded in the category of $\mathcal{K}R_H$-modules and in the homotopy category of $\mathcal{K}R_H$-modules. By Clifford theory, the restriction of V to A is a direct sum of H-conjugate simple A-modules. Since the Rickard equivalences obtained from M commute with induction, restriction and conjugation, we deduce that there is an isomorphism

$$\mathrm{End}_{R_H}(R_H \otimes_A V) \simeq \mathrm{End}_{S_H}(S_H \otimes_B W)$$

of H-graded $\mathcal{K}$-algebras. Consequently, $\mathcal{K}(\chi_A) = \mathcal{K}(\eta_B)$ and $[[\chi]] = [[\eta]]$.

Let θ be an irreducible character of A contained in χ_A, and assume that H stabilizes θ. Then, by Corollary 4.7, and since V is θ-quasihomogeneous, we have

$$\mathrm{Ind}_H^G[[\chi]] = [[\mathrm{Ind}_H^G \chi]] = [\mathrm{Ind}_H^G(\mathrm{End}_{\mathscr{K}A}(V))]$$
$$= [\mathrm{End}_{\mathscr{K}A}(\mathrm{Ind}_H^G V)] \in \mathrm{Cliff}(H, \mathscr{K}(\chi_A)).$$

The correspondent ρ of θ under the G-graded Rickard equivalence is also stabilized by H, and we similarly have

$$\mathrm{Ind}_H^G[[\eta]] = [[\mathrm{Ind}_H^G \eta]] = [\mathrm{Ind}_H^G(\mathrm{End}_{\mathscr{K}B}(W))] = [\mathrm{End}_{\mathscr{K}B}(\mathrm{Ind}_H^G W)].$$

Since $\mathrm{Ind}_H^G V = \mathscr{K}R \otimes_{\mathscr{K}R_H} V$ corresponds to $\mathrm{Ind}_H^G W$ under the G-graded Rickard equivalence, we deduce that

$$[[\mathrm{Ind}_H^G \chi]] = [[\mathrm{Ind}_H^G \eta]].$$

Finally, let ϕ be an irreducible character of $\overline{\mathscr{K}}R$ lying over θ, and assume that for any $g \in G$ there exists $h \in H$ such that ${}^g\theta = {}^h\theta$. Then $[[\phi]] \in \mathrm{Cliff}(G, \mathscr{K}(\chi_A))_H$, and

$$[[\mathrm{Res}_H^G \phi]] = \mathrm{Res}_H^G[[\phi]] \in \mathrm{Cliff}(H, \mathscr{K}(\chi_A)),$$

where $\mathrm{Res}_H^G \phi$ denotes the restriction of ϕ to $\overline{\mathscr{K}}R_H$. The correspondent ψ of ϕ is an irreducible character of $\overline{\mathscr{K}}S$ lying over the correspondent ρ of θ. Arguments as above show that

$$[[\mathrm{Res}_H^G \phi]] = [[\mathrm{Res}_H^G \psi]] \in \mathrm{Cliff}(H, \mathscr{K}(\chi_A)). \quad \square$$

6 Turull's conjecture and blocks with cyclic defect groups

Let G be a finite group, p a prime number and D a p-subgroup of G. Denote by $\mathrm{Irr}(G, D)$ the union of the sets $\mathrm{Irr}(B)$ of ordinary irreducible characters belonging to p-blocks B of G having defect group D. The notation Irr_0 means characters of height zero. Also let $\mathbb{Q}_p$ denote the field of p-adic numbers, and $\overline{\mathbb{Q}}_p$ its algebraic closure.

Turull [14] formulated the following conjecture strengthening Navarro's conjecture B [7, Section 1].

Conjecture 6.1. There exists a bijection

$$f : \mathrm{Irr}_0(G, D) \to \mathrm{Irr}_0(N_G(D), D)$$

having the following properties:

(1) if $\chi \in \mathrm{Irr}_0(B)$, and B has defect group D, then $f(\chi) \in \mathrm{Irr}_0(b)$, where b is the Brauer correspondent block of $N_G(D)$ of the block B;

(2) f commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; so, in particular, $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(f(\chi))$ for every $\chi \in \mathrm{Irr}_0(G, D)$;

(3) for every $\chi \in \mathrm{Irr}_0(G, D)$, we have $m_p(f(\chi)) = m_p(\chi)$, so that $f(\chi)$ and χ have the same p-local Schur index.

Turull also proved in [14, Theorem 2.2] that the above conjecture holds for blocks with cyclic defect groups. In this section we point out that this result is also a consequence of Rouquier's work [10] on Broué's conjecture. We mention here that a detailed discussion of the connections between Broué's conjecture and the Alperin–McKay, Dade, Isaacs–Navarro conjectures can be found in [15]. Uno states in [15, Section 4] that Rouquier's complex is invariant under a certain Galois action. What we need here is a splendid tilting complex defined over small fields.

6.2. We adopt the setting of [3, Chapter VII]. Let $\mathscr{K}$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ the ring of integers in $\mathscr{K}$, and k the residue field of $\mathcal{O}$. Assume that D is a cyclic subgroup of G of order $p^a > 1$. For $0 \leqslant i \leqslant a$, let D_i be the unique subgroup of D of index p^i, and let $N_i = N_G(D_i)$. Let B be a block of $\mathcal{O}G$, and let B_i be the block of $\mathcal{O}N_i$ corresponding to B.

6.3. Under the assumption that the p-modular system $(\mathscr{K}, \mathcal{O}, k)$ is 'big enough', a result of Rouquier ([10, Theorem 10.1]) states that the block algebras B and B_0 are splendidly Rickard equivalent. The main ingredients of his proof are the following:

(1) the Morita stable equivalence between $B = B_a$ and B_{a-1} given by restriction and induction;

(2) the structure of the block algebra B_0 having normal defect group D;

(3) the construction of a Rickard tilting complex, based on the information on modules and characters encoded in the Brauer tree;

(4) induction on i, $i = a, a-1, \dots, 0$.

Note that (1) and (4) do not require assumptions on the size of the p-modular system $(\mathscr{K}, \mathcal{O}, k)$, and also the result on structure of blocks with normal defect groups generalizes to small fields (see [2, Theorem 1.17], or [1] for an alternative proof). The Brauer tree of B is defined if $\mathscr{K}$ satisfies condition $(*)$ of [3, p. 276]. However, this condition can be avoided as well.

Proposition 6.4. *The blocks B and B_0 are splendidly Rickard equivalent.*

Proof. There is a totally ramified extension $\mathscr{K}'$ of $\mathscr{K}$ for which condition $(*)$ of [3, p. 276] is satisfied. Let $\mathcal{O}'$ be the ring of integers in $\mathscr{K}'$. Then the residue field of $\mathcal{O}'$ is k. Let B' be the block of $\mathcal{O}'G$ corresponding to B, and B_0' the block of $\mathcal{O}'N_0$ corresponding to B_0. By [10, Theorem 10.1], there is a splendid tilting complex X' of (B', B_0')-bimodules. Then $k \otimes_{\mathcal{O}'} X'$ is a splendid tilting complex of (kB', kB_0')-bimodules. By [9, Theorem 5.2], there is a splendid tilting complex X of (B, B_0)-bimodules such that

$$k \otimes_{\mathcal{O}} X \simeq k \otimes_{\mathcal{O}'} X',$$

and X is unique up to isomorphism. $\square$

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