MATHEMATICA, Tome 50 (73), N° 2, 2008, pp. 191–195

INDECOMPOSABLE MODULES OVER GROUP GRADED SKEW ALGEBRAS

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Abstract. G-graded skew algebras over a G-acted commutative ring have been introduced by E. Dade as a framework to combine Clifford theory and Galois theory. In this note we consider indecomposable graded modules over such algebras and their endomorphism rings.

MSC 2000. 20C20, 16W50.

Key words. Group graded algebras, skew algebras.

1. INTRODUCTION AND PRELIMINARIES

1.1. Let G be a group, $R = \bigoplus_{g\in G} R_g$ a G-graded ring (not necessarily strongly graded, and let $\mathcal{O}$ be a commutative G-ring. We assume that rings have identity elements, and that actions are on the left.

The following concept was introduced by E. Dade [1].

Definition 1.2. *The G-graded ring R is called a G-graded skew algebra over $\mathcal{O}$ if there is an identity preserving ring homomorphism*

$$\chi : \mathcal{O} \to R$$

satisfying

$$a\chi(r) = \chi({}^g r)a \in R_g$$

for all $r \in \mathcal{O}$, $g \in G$ and $a \in R$.

1.3. Note that R_1 becomes an $\mathcal{O}$-algebra since χ induces a ring homomorphism $\mathcal{O} \to Z(R_1)$, and moreover, R becomes an $(\mathcal{O},\mathcal{O})$-bimodule, where by definition

$$ras = \chi(r)a\chi(s)$$

for all $r, s \in \mathcal{O}$ and $a \in R$. By 1.2, we have that

$$ar = {}^g r \cdot a$$

for all $g \in G$, $r \in \mathcal{O}$ and $a \in R_g$.

1.4. Let us consider the case when A is strongly graded. Then there is a well-known action of G on the centralizer $C_R(R_1)$ of R_1 in R, defined as

This research has been supported by the Romanian PN-II-IDEI-PCE-2007-1 project ID_532, contract no. 29/01.10.2007.

follows. Let $g \in G$. Then, since $R_g R_{g^{-1}} = R_1$, there are elements $u_{g,i} \in R_g$ and $u'_{g,i} \in R_{g^{-1}}$ such that

$$\sum_i u_{g,i} u'_{g,i} = 1.$$

If $c \in C_R(R_1)$, define

$${}^g c = \sum_i u_{g,i} c u'_{g,i}.$$

This is independent on the choice of the elements $u_{g,i}$ and $u'_{g,i}$, and in fact, ${}^g c$ is the unique element of $C_R(R_1)$ satisfying

$$ac = {}^g c \cdot a$$

for all $a \in R_g$. Note that if $c \in C_R(R_1)_h$, then ${}^g c \in C_R(R_1)_{ghg^{-1}}$ for all $h \in G$.

In particular, the center $Z(R_1)$ of R_1 becomes a G-ring, and moreover, R is a G-graded skew algebra over $Z(R_1)$. In this situation, R is a G-graded skew algebra over $\mathcal{O}$ if and only if there is a homomorphism $\mathcal{O} \to Z(R_1)$ of G-rings.

1.5. The introduction of G-graded skew algebras is motivated in [1] by the new strengthenings of the Alperin-McKay conjecture due to G. Navarro involving Galois actions on characters of finite groups. Schur indices are also included in a conjecture formulated by A. Turull [6].

Dade showed in [1] that starting with a crossed product which is a G-graded skew algebra over a G-field, then the Clifford theoretical constructions performed with a split simple module lead to a crossed product skew algebra over the same G-field. In this note we extend this result to the situation of Clifford theory for indecomposable modules as considered in [3].

1.6. All the unexplained concepts and facts can be found in [4]. We recall here one more notion needed in the next section. By definition, the graded Jacobson radical $J_{\mathrm{gr}}(R)$ of R, is the intersection of the maximal graded left ideals of R. This is a graded ideal of R, with 1-component $J_{\mathrm{gr}}(R)_1$ coinciding with the Jacobson radical $J(R_1)$ of R_1. Moreover, we have that $J_{\mathrm{gr}}(R)_1 \subseteq J(R)$.

2. ENDOMORPHISM RINGS OF G-GRADED INDECOMPOSABLE MODULES

2.1. Let R be a G-graded skew algebra over $\mathcal{O}$, and let $M = \bigoplus_{x\in G} M_x$ be a G-graded (left) R-module. Then, by [1, Proposition 4.1], M has a structure of an $(\mathcal{O},\mathcal{O})$-bimodule by letting

$$mr = {}^g r \cdot m$$

for all $g \in G$, $m \in M_g$ and $r \in \mathcal{O}$.

The g-conjugate ${}^g M$ (also denoted by $M(g)$) of M coincides with M as an R-module, but has components

$$({}^g M)_x = M_{xg}$$

for all $x \in G$.

Let $r \in \mathcal{O}$, and let ${}^g m \in {}^g M_x$ denote the element $m \in M_{xg}$ regarded in ${}^g M$. Then mr belongs to M_{xg}, ${}^g(mr) \in {}^g M_x$, and we have that

$$ {}^g(mr) = {}^g m \cdot {}^g r. $$

The stabilizer of M in G is, by definition, the subgroup

$$ G_M = \{g \in G \mid M \simeq {}^g M \text{ as } G\text{-graded } R\text{-modules}\} $$

Finally, let $E := \mathrm{End}_R(M)^{\mathrm{op}}$, and for $f, f' \in E$ and $m \in M$, $mf = f(m)$ and $ff' = f' \circ f$. Then E is a G-graded ring such that M is a G-graded (R, E)-bimodule. The g-component of E is

$$ \begin{aligned} E_g &= \{f \in \mathrm{End}_R(M) \mid f(M_x) \subseteq M_{xg} \text{ for all } x \in G\} \\ &= \mathrm{Hom}_{R\text{-Gr}}(M, {}^g M). \end{aligned} $$

2.2. From now on we assume that G is a finite group acting on the commutative noetherian ring $\mathcal{O}$. Then the residue field $k = \mathcal{O}/J(\mathcal{O})$ is a G-field in a natural way. Moreover, we assume that $R/J(R)$ is finite dimensional over k.

We say that the G-graded R-module M is gr-indecomposable if it is not a direct sum of two nontrivial graded submodules.

THEOREM 2.3. *Let M be a gr-indecomposable R-module, free of finite rank over $\mathcal{O}$, and let $D := E/J_{\mathrm{gr}}(E)$. Then D is a k-skew crossed product of the division k-algebra $D_1 \simeq E_1/J(E_1)$ and G_M. The action of G_M on k coming from (1.3) is the same as the action coming from the G-graded skew $\mathcal{O}$-algebra structure of R.*

Proof. Consider the map

$$ \chi' : \mathcal{O} \to E_1 = \mathrm{End}_{R\text{-Gr}}(M)^{\mathrm{op}}, \quad \chi'(r)(m) = mr $$

for all $r \in \mathcal{O}$ and $m \in M$. Then for all $f \in E_g$ we have that

$$ f\chi'(r) = \chi'({}^g r)f $$

(see the proof of [1, Proposition 5.1]. It follows that E becomes a G-graded skew algebra over the given G-ring $\mathcal{O}$.

Since $E_1 \cap J_{\mathrm{gr}}(E) = J_{\mathrm{gr}}(E)_1 = J(E_1)$, we have that $D_1 \simeq E_1/J(E_1)$. Moreover, $E_1 = \mathrm{End}_{R\text{-Gr}}(M)^{\mathrm{op}}$ is a local ring since M is gr-indecomposable, and we get that D_1 is a division k-algebra.

Let $g \in G \backslash G_M$. Then M and ${}^g M$ are non-isomorphic gr-indecomposable R-modules, hence every grade-preserving map $f : M \to {}^g M$ generates a graded ideal of E. It follows that $E_g \subseteq J_{\mathrm{gr}}(E)$.

If $g \in G_M$, then there exists an isomorphism $f : M \to {}^g M$, which gives an invertible element $\bar{f} \in U(D) \cap D_g$. Consequently, D is a crossed product of D_1 and G_M. Moreover, for any $r \in \mathcal{O}$, denoting by $\bar{r}$ the image of r in k, we have

$$ fr = {}^g r \cdot f, $$

hence

$$ \bar{f}\bar{r} = {}^g \bar{r} \cdot \bar{f}. $$

It follows that the action of G_M on k induced by the crossed product D coincides with the initial action. □

EXAMPLE 2.4. Here is a situation which motivates our assumptions. Let $\mathcal{O}$ be a complete local principal ideal domain with residue field k of characteristic $p > 0$. Let b be a block with defect group D of the block algebra $\mathcal{O}G$, and let c be the Brauer corresponding block of $\mathcal{O}N_G(D)$. Fix a root e of b, so e is a block of $kC_G(D)$ with defect group $Z(D)$. Let

$$N_G(D,e) = \{g \in N_G(D) \mid {}^g e = e\}$$

let

$$E_G(D,e) = N_G(D,e)/DC_G(D),$$

so $E_G(D,e)$ acts naturally on D.

Then the algebra $\mathcal{O}N_G(D)c$ is Morita equivalent to an $\hat{\mathcal{O}}$-skew crossed product of the group algebra $\hat{\mathcal{O}}D$ and $E_G(D,e)$, where $\hat{\mathcal{O}}$ is a separable cyclic extension of $\mathcal{O}$ on which $E_G(D,e)$ acts nontrivally in general.

This a generalization by Y. Fan and L. Puig of a result of B. Külshammer (see also [2] and the references given there).

COROLLARY 2.5. *Let R be an $\mathcal{O}$-skew crossed product, let N be anormal subgroup of G acting trivially on $\mathcal{O}$, and let U be an absolutely indecomposable R_N-module (i.e. $\mathrm{End}_{R_N}/J(\mathrm{End}_{R_N}(U) \simeq k)$. Denote $\bar{G} = G/N$, let $M = R \otimes_{R_N} U$ and let $\bar{D} = \bar{E}/J_{gr}(\bar{E})$, where $\bar{E} = \mathrm{End}_R(M)^{\mathrm{op}}$.*

Then $\bar{D}$ is a k-skew crossed product of k and $\bar{G}_M$, where the the action of $\bar{G}_M$ on k is induced by the action of G on $\mathcal{O}$.

Proof. Here we regard R and E as $\bar{G}$-graded skew algebras over $\mathcal{O}$. Since U is an indecomposable R_N-module and R is strongly graded, we have that M is a gr-indecomposable R-module,

$$\bar{G}_M = \{\bar{g} \in \bar{G} \mid R_{\bar{g}} \otimes_{R_N} U \simeq U \text{ in } R_N\text{-mod}\},$$

and the restriction to U of endomorphisms of M induce the isomorphisms

$$E_{\bar{1}} \simeq \mathrm{End}_{R_N}(U)^{\mathrm{op}}$$

and $D_{\bar{1}} \simeq k$. The statements now follow immediately from Theorem 2.3. □

2.6. We end by recalling the motivation for our interest in the algebra $D = E/J_{\mathrm{gr}}(E)$. Denote by $(R|M)$-mod the full subcategory of R-mod consisting of direct summands of finite direct copies of M, that is, $(R|M)$-mod is the smallest additive subcategory of R-mod containing M. By [4, Theorem 2.3.10], we have that the additive functor

$$D \otimes_E \mathrm{Hom}_R(R \otimes_{R_1} M, -) : (R|M)\text{-mod} \to D\text{-proj}$$

induces an isomorphism between the Grothendieck groups associated to these categories, and in particular, a bijection between the indecomposable summands of M and the indecomposable projective D-modules.

In the situation of the above corollary, the objects of the category $(R|R\otimes_{R_N} U)$-mod are called R-modules lying over U.

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Received April 30, 2008

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