STRONG SURJECTIONS AND NEARNESS

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Dedicated to Alfonso Vignoli on the ocasion of his 60th birthday

Summary: We show that the properties of being *strongly surjective* or *stable solvable* (in the sense of Furi-Martelli-Vignoli) carry over to maps which are *near* in the sense of Campanato.
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1. Introduction. M. Furi, M. Martelli and A. Vignoli in [3] introduced the notions of *strong surjection* and *stable solvable map* between two normed spaces E and F in order to define the spectrum for a nonlinear operator. Also, these concepts are related to that of *zero-epi map*, which is due to the same authors [4] and is very important in the study of solvability of nonlinear equations.
Near operators have been introduced by S. Campanato and also studied by A. Tarsia and S. Leonardi in [2,7,10,13,14] and have applications in nonlinear differential equations, too.
We prove that the property of being a strong surjection or stable solvable is preserved by nearness and notice that this can be used in order to prove existence results for differential equations in implicit form.

2. Main results. Let E be a normed space and F be a Banach space. A continuous map $f : E \to F$ is called a *strong surjection* if the equation $f(x) = h(x)$ has a solution for any continuous compact map $h : E \to F$.
A continuous map $f : E \to F$ is said to be *stable solvable* if the equation $f(x) = h(x)$ has a solution for any completely continuous map $h : E \to F$

with quasinorm $|h| = 0$. Recall that the quasinorm of a map h is defined by

$$|h| = \limsup_{||x|| \to \infty} \frac{||h(x)||}{||x||}.$$

We say that $g : E \to F$ is *near* $f : E \to F$ if there exist two positive constants α and k, with $k \in (0,1)$, such that we have

$$||f(x_1) - f(x_2) - \alpha[g(x_1) - g(x_2)]|| \leq k||f(x_1) - f(x_2)|| \tag{1}$$

for all $x_1, x_2 \in E$.
In order to prove our main results we shall give two lemmas.

Lemma 1 *Let $g : E \to F$ be near $f : E \to F$. If f is continuous, then g is continuous, too.*

Proof. Using that g is near f we obtain the following estimation.

$$\begin{aligned} & ||g(x_1) - g(x_2)|| = \\ = \; & \frac{1}{\alpha}||f(x_1) - f(x_2) - \alpha[g(x_1) - g(x_2)] - [f(x_1) - f(x_2)]|| \leq \\ \leq \; & (\frac{k}{\alpha} + 1)||f(x_1) - f(x_2)|| \end{aligned}$$

for all $x_1, x_2 \in E$.
Now, by the definition of the continuity we deduce that, if f is continuous, then g is also continuous. □
In what follows we shall denote by f_d^{-1} a right inverse for a surjective map f.

Lemma 2 *Let $g : E \to F$ be near a surjective map $f : E \to F$. The following statements are true.*
(i) $f(x) = f(\hat{x})$ implies that $g(x) = g(\hat{x})$
(ii) $s = (f - \alpha g) \circ f_d^{-1} : F \to F$ is a contraction and does not depend on the choice of the right inverse of f.

Proof. If we consider $x, \hat{x} \in E$ with $f(x) = f(\hat{x})$ and replace in (1) we obtain $||g(x) - g(\hat{x})|| \leq 0$, which means that $g(x) = g(\hat{x})$. From this we can deduce that s does not depend on the choice of f_d^{-1}.
The following estimation is obtained using (1) and express that s is a contraction.

$$\begin{aligned} & ||(f - \alpha g)(f_d^{-1}y_1) - (f - \alpha g)(f_d^{-1}y_2)|| \leq \\ \leq \; & k||f(f_d^{-1}y_1) - f(f_d^{-1}y_2)|| = k||y_1 - y_2|| \end{aligned}$$

for all $y_1, y_2 \in F$. $\square$

Remark. Relation (1) express also that the map $f - \alpha g$ is a contraction with respect to f. For other considerations in this direction we recommend [1,6,13].

Theorem 1 *Let $g : E \to F$ be near $f : E \to F$. If f is a strong surjection, then g is a strong surjection, too.*

Proof. By Lemma 1, g is a continuous map. Let $h : E \to F$ be continuous and compact. By Lemma 2 we have that $s = (f - \alpha g) \circ f_d^{-1} : F \to F$ is a contraction , where f_d^{-1} is a right inverse for f . In this situation we have that $(I - s)$ is a homeomorphism.
The map $(I - s)^{-1} \circ \alpha h : E \to F$ is continuous and compact. So, by hypothesis it has a coincidence point with the strong surjection f , i.e. $f(x) = (I - s)^{-1}(\alpha h(x))$. Let us denote $\hat{x} = f_d^{-1}(f(x))$ and notice that $f(x) = f(\hat{x})$ and $g(x) = g(\hat{x})$. The following implications are valid.

$$f(x) = (I - s)^{-1}(\alpha h(x)) \Longrightarrow (I - s)(f(x)) = \alpha h(x) \Longrightarrow$$
$$f(x) - (f - \alpha g)(\hat{x}) = \alpha h(x) \Longrightarrow g(x) = h(x)$$

This means that x is a coincidence point of g and h, where h is an arbitrary continuous and compact map. So, g is a strong surjection. $\square$

Theorem 2 *Let $g : E \to F$ be near $f : E \to F$. If f is stable solvable, then g is stable solvable, too.*

Proof. Let $h : E \to F$ be completely continuous with quasinorm $|h| = 0$. The arguments follow like in the previous theorem, noticing that $(I - s)^{-1} \circ \alpha h$ is completely continuous with quasinorm equal to 0. $\square$

3. An application. Let us consider two mappings $L, N : E \to F$ such that $L - N$ is a strong surjection. In applications (see [4,5,8,11]), usually, L is linear and bounded (in many cases, a differential operator) and N is completely continuous . In the case that $||(L - N)(x)|| \to \infty$ as $||x|| \to \infty$ there are some relations between the theory of strong surjections and the theory of zero-epi maps, or degree theory, or the theory of essential compact fields (see [3,4,7,9]). Using this, we can find many examples of strong surjections of the form $L - N$. One of them which is due to M. Furi,

M. Martelli and A. Vignoli [4] is the following.
Let $C_0^2[0,1]$ be the space of C^2-functions such that $x(0) = x(1) = 0$ and $L, N : C_0^2[0,1] \to C[0,1]$ be defined by $Lx(t) = x''(t)$ and $N(x)(t) = x^3(t)$. Then $L - N$ is a strong surjection.
We use our main results in order to state that a map of the (implicit) form

$$g : E \to F \ , \quad g(x) = G(Lx, N(x))$$

is a strong surjection provided that $L - N$ is a strong surjection and $G : F \times F \to F$ satisfies the following relation for some $\alpha > 0$ and $k \in (0,1)$ and for all $y_1, y_2, z_1, z_2 \in F$

$$||y_1 - z_1 - y_2 + z_2 - \alpha[G(y_1, z_1) - G(y_2, z_2)]|| \leq k||y_1 - z_1 - y_2 + z_2||. \quad (2)$$

For example, $g : C_0^2[0,1] \to C[0,1]$ defined by $g(x)(t) = \tilde{g}(x''(t), x^3(t))$ is a strong surjection if $\tilde{g} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (2) for all $y_1, y_2, z_1, z_2 \in \mathbb{R}$.

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