CENTERS IN A QUADRATIC SYSTEM OBTAINED FROM A SCALAR THIRD ORDER DIFFERENTIAL EQUATION

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ABSTRACT. In this paper it is shown that $(0,0,0)$ is a center for

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -\frac{1}{a}\, z - a\,(2x+1)y - x(x+1),$$

and that $(-1,0,0)$ is a center for

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -a\,x\,z - \frac{1}{a}\, y - x(x+1),$$

(where $a > 0$) giving in this way a positive answer to questions arised in the paper *Analysis of a quadratic system obtained from a scalar third order differential equation*, Electron. J. Differential Equations **2010** no. 161 (2010).

1. Introduction and statement of the results

The starting point is the scalar third order differential equation

$$\dddot{x} + f(x)\ddot{x} + g(x)\dot{x} + h(x) = 0, \tag{1}$$

with f and g arbitrary polynomials of degree 1 and h a polynomial of degree 2. Without loss of generality we can take $h(x) = x(x+1)$ when h has two real zeros. We will associate to equation (1) the quadratic differential systems in $\mathbb{R}^3$

$$\dot{x} = y\ ,\ \dot{y} = z\ ,\ \dot{z} = -f(x)z - g(x)y - h(x)\ . \tag{2}$$

A *Hopf point* of (2) is a singularity that possesses two complex eigenvalues $\pm i$ with zero real part and one nonzero real eigenvalue. System (2) having a singular point of Hopf type at the origin has a local 2-dimensional *center manifold* $W^c(0)$. This manifold is invariant for (2) locally (only for sufficiently small $|x|$ and $|y|$) and for any $k \geq 1$ there exists $\tilde{h}$ of class C^k near the origin such that

$$\tilde{h}(0,0) = 0,\ D\tilde{h}(0,0) = 0,$$

$D\tilde{h}(x,y)$ being the Jacobian matrix of $\tilde{h}$, and

$$W^c(0) = \{(x, y, \tilde{h}(x,y)) \in \mathbb{R}^3 \,:\, (x,y) \,\text{in a small neighborhood of}\, (0,0)\}.$$

The center problem for system (2) at the Hopf type singularity consists in detecting when the singular point becomes either a *center* or a *focus* for the flow of system (2) restricted to the center manifold. We say that the singular point is a *center* of (2) if all the orbits on $W^c(0)$ near the origin are periodic, and a *focus* if they spiral around it. The classical procedure for the solution to the center problem can be found in [1, 4], while the projection method for the calculation of the Lyapunov constants is given in [7].

Recall here that the Lyapunov constants are the coefficients of the Taylor series of the diplacement map (Poincaré return map minus the identity), so that its vanishing is a necessary condition for having a center. But essentially, the main problem is the following. Let $\mathfrak{R} \subset \mathbb{R}[\lambda]$ be the ring of real polynomials whose variables are the coefficients $\lambda \in \mathbb{R}^p$ of some polynomial differential family (2). The Bautin ideal $\mathfrak{J}$ is the ideal of $\mathfrak{R}$ generated by all the Liapunov constants. Using Hilbert's basis theorem, it follows that $\mathfrak{J}$ is finitely generated. Thus, there are $\{B_1, B_2, \ldots, B_r\} \subset \mathfrak{J}$ such that $\mathfrak{J} = (B_1, B_2, \ldots, B_r)$. Such a set of generators is called a basis of $\mathfrak{J}$ when r is the minimum number of the ideal generators. In this case, we say that $r = \dim \mathfrak{J}$. For the concrete family of polynomial systems (2), an open problem nowadays is the determination of $\mathfrak{J}$

We recall that, by using the blow-up technique, the problem to characterize the local phase portrait near an isolated singular point of a planar vector field can be solved except when the singularity is monodromic, that is, it is either a focus or a center.

Some aspects of the dynamics of system (2) are studied in [5]. Under conditions on the set of parameters, the origin and the point $(-1, 0, 0)$ are Hopf points of system (2). The authors of [5] prove that the first three Lyapunov coefficients vanish at these Hopf points, hence they conjecture that they are centers. In this work, analyzing the vector field (2) with the techniques developed in [3], we show two families of centers, solving in this way the two conjectures formulated in [5]. We have the following result.

Theorem 1. *Consider the following 4–parameter family of quadratic differential systems in* $\mathbb{R}^3$

$$\dot{x} = y \ , \ \dot{y} = z \ , \ \dot{z} = -f(x)z - g(x)y - h(x) \ , \tag{3}$$

where $f(x) = a_1 x + a_0$*,* $g(x) = b_1 x + b_0$ *and* $h(x) = x(x+1)$ *and the parameters* $(a_0, a_1, b_0, b_1) \in \mathbb{R}^4$*. We have the following center conditions:*

(i) *The point* $(0, 0, 0)$ *is a center of system (3) if* $b_0 > 0$*,* $a_0 = 1/b_0$*,* $a_1 = 0$ *and* $b_1 = 2b_0$*.*

(ii) *The point* $(-1, 0, 0)$ *is a center of system (3) if* $a_0 = b_1 = 0$*,* $b_0 = 1/a_1$ *and* $a_1 > 0$*.*

The proof of Theorem 1 is based on the properties of an inverse Jacobi last multiplier of system (3) as studied in [3]. Since this proof is based on the main results of the preprint [3], we write in the last section of this paper an appendix in order to give an alternative self-contained proof of Theorem 1.

We shortly present the properties of the inverse Jacobi last multiplier function developed in [3] below.

Let $D \subseteq \mathbb{R}^n$ be an open subset and $\mathcal{Y} = \sum_{i=1}^n f_i(x)\partial_{x_i}$ be a $C^1(D)$ vector field with $x = (x_1, \ldots, x_n) \in D$. A C^1 function $V : D \to \mathbb{R}$ is said to be an *inverse Jacobi last multiplier* of $\mathcal{Y}$ if it is not locally null and it satisfies the linear first order partial differential equation

$$\mathcal{Y}V = V \,\mathrm{div}\mathcal{Y},$$

where $\operatorname{div}\mathcal{Y} = \sum_{i=1}^{n} \partial f_i(x)/\partial x_i$ is the divergence of the vector field $\mathcal{Y}$. A good reference to the theory of inverse Jacobi last multipliers is [2]. See also [6] for a summary. The next result is a simple consequence of the main results proved in [3].

Corollary 2. [3] *Assume that the linear part of the vector field $\mathcal{Y}$ in $\mathbb{R}^3$ has the block diagonal representation*

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $B \in \mathbb{R}\backslash\{0\}$ and that there exists a smooth and nonflat inverse Jacobi last multiplier $V(x,y,z)$ of $\mathcal{Y}$ near the origin. If one has the following Taylor expansion

$$V(x,y,z) = z + \cdots , \quad \textit{where the dots indicate terms of order two or higher} ,$$

then the origin is a center for $\mathcal{Y}$.

2. Proof of Theorem 1

The singularities of (3) are $p_0 = (0,0,0)$ and $p_1 = (-1,0,0)$. In addition, these singular points are Hopf points in the following cases:

- Taking $b_0 > 0$, $a_0 = 1/b_0$, the origin of system (3) has associated eigenvalues $-1/b_0$ and $\pm i\sqrt{b_0}$.
- Taking $a_0 - a_1 < 0$ and $(a_0 - a_1)(b_0 - b_1) = -1$, the singularity $(-1,0,0)$ of system (3) has associated eigenvalues $a_1 - a_0$ and $\pm i/\sqrt{a_1 - a_0}$.

Under the parameter restrictions $b_0 > 0$, $a_0 = 1/b_0$, $a_1 = 0$ and $b_1 = 2b_0$ of statement (i), system (3) possesses the inverse Jacobi last multiplier

$$V(x,y,z) = z + b_0\, x(x+1).$$

This can be easily seen by verifying that this function is a solution of the linear partial differential equation

$$y\,\frac{\partial V}{\partial x} + z\,\frac{\partial V}{\partial y} - (z/b_0 + b_0\,(2x+1)y + x(x+1))\frac{\partial V}{\partial z} = -V/b_0.$$

We do the linear change of variables

$$(x,y,z) \to \frac{1}{1+b_0^3} \begin{pmatrix} -b_0 & 0 & b_0^3 \\ -b_0^{5/2} & -(1+b_0^3)b_0^{1/2} & -b_0^{3/2} \\ b_0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the rescaling of time $t \to -\sqrt{b_0}\, t$ bringing the linear part of (3) at the origin to its canonical form. In short, system (3) is written in the form

$$\begin{aligned} \dot{x} &= -y + \frac{1}{1+b_0^3}\left(b_0^{1/2}x^2 + 2b_0^2 xy - 2b_0^5 yz - b_0^{13/2} z^2\right) , \\ \dot{y} &= x + \frac{1}{1+b_0^3}\left(-b_0^{-1}x^2 - 2b_0^{1/2}xy + 2b_0^{7/2}yz + b_0^5 z^2\right) , \\ \dot{z} &= b_0^{-3/2}\, z + \frac{1}{1+b_0^3}\left(b_0^{-5/2}x^2 + 2b_0^{-1}xy - 2b_0^2 yz - b_0^{7/2} z^2\right) , \end{aligned} \tag{4}$$

having the inverse Jacobi last multiplier

$$V(x,y,z) = z + (x - b_0^3 z)^2/(b_0(1+b_0^3)).$$

Applying Corollary 2, the origin is a center of system (4) and consequently of system (3) proving statement (i).

Under the parameter restrictions $a_0 = b_1 = 0$, $b_0 = 1/a_1$ and $a_1 > 0$ of statement (ii), system (3) possesses the inverse Jacobi last multiplier

$$V(x,y,z) = 1 + x + a_1 z.$$

This can be easily seen by verifying that this function is a solution of the linear partial differential equation

$$y\,\frac{\partial V}{\partial x} + z\,\frac{\partial V}{\partial y} - (a_1\,x\,z + y/a_1 + x(x+1))\frac{\partial V}{\partial z} = -a_1\,x\,V.$$

Firstly we translate the singular point $(-1,0,0)$ to the origin with the change $(x,y,z) \to (x+1,y,z)$. After, we do the linear change of variables

$$(x,y,z) \to \frac{1}{1+a_1^3}\begin{pmatrix} -a_1^2 & 0 & 1 \\ -a_1^{1/2} & a_1^{-1/2} + a_1^{5/2} & -a_1^{3/2} \\ a_1^2 & 0 & a_1^3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the rescaling of time $t \to a_1^{-1/2}\,t$ bringing the linear part of the system to canonical form. In short we obtain that system (3) becomes

$$\begin{aligned} \dot{x} &= -y + \frac{1}{a_1^{7/2}} z(a_1^3 x - z)\ , \\ \dot{y} &= x - a_1 x z + \frac{z^2}{a_1^2}\ , \\ \dot{z} &= \frac{z(a_1^2 + a_1^3 x - z)}{\sqrt{a_1}}, \end{aligned} \tag{5}$$

The origin of this system is trivially a center because $W^c(0) = \{z = 0\}$ and the system reduced to the center manifold is the linear center $\dot{x} = -y$, $\dot{y} = x$. Then statement (ii) is proved. Other proof of this fact follows applying Corollary 2, because an inverse Jacobi last multiplier of this system is $V(x,y,z) = z$.

3. Appendix

In this section we present an alternative proof of Theorem 1. We start with the explicit knowledge of an inverse Jacobi last multiplier $V(x,y,z)$ in all the cases, as stated before. Next the main idea is first to obtain from V the explicit expression of an analytic center manifold $W^c(0)$ and finally to check that the singularity of the reduced system to the center manifold is in fact a center. We only do this procedure in the proof of statement (i) of Theorem 1 because statement (ii) has been trivially proved without resorting to [3].

To prove statement (i) of Theorem 1 we must study the center problem at the origin for the equivalent system (4). Recall that this system admits the inverse Jacobi last multiplier $V(x,y,z) = b_0(1+b_0^3)z + (x - b_0^3 z)^2$. Therefore, $\{V(x,y,z) = 0\}$ defines an invariant algebraic surface of (4) which passes through the origin and is tangent to the plane $\{z = 0\}$ at this point. In particular, this means that this invariant surface is tangent to the center eigenspace, the (x,y)-plane, at the origin, hence in a neighborhood of the origin forms a local center manifold. Indeed, solving $V = 0$ for z and inserting into the first two equations in (4) we obtain the following expression of the reduced system (4) to the center manifold in local coordinates:

$$\begin{aligned} \dot{x} &= P(x,y) = -y + \frac{1}{4b_0^{7/2}(1+b_0^3)} f_+(x)\, g(x,y) \ , \\ \dot{y} &= Q(x,y) = x + \frac{1}{4b_0^5(1+b_0^3)} f_-(x)\, g(x,y) \ , \end{aligned}$$

where $f_\pm(x) = \pm 1 \pm b_0^3 \mp \sqrt{1+b_0^3}\,\sqrt{1+b_0^3-4b_0^2 x}$, and $g(x,y) = -1 - b_0^3 + 4b_0^2 x + \sqrt{1+b0^3}\,\sqrt{1+b_0^3-4b_0^2 x} + 4b_0^{7/2} y$. It is straightforward to check that the function $v(x,y) = \sqrt{1+b_0^3-4b0^2 x}$ is an inverse integrating factor of this reduced system, that is, the rescaled system $\dot{x} = P(x,y)/v(x,y)$, $\dot{y} = Q(x,y)/v(x,y)$ is hamiltonian. Since $v(x,y)$ is a non-vanishing analytic function near the origin, this implies that the reduced system to the center manifold possesses a local analytic first integral around the origin. Hence, the origin becomes a center.

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