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EXISTENCE RESULTS FOR EVOLUTION EQUATIONS VIA MONOTONE ITERATIVE TECHNIQUES

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Abstract. The aim of this paper is to find sufficient conditions for the existence of extremal solutions for discontinuous implicit initial value problems. We also prove the monotone dependence of the extremal solutions with respect to the data. An abstract monotone iterative technique and the theory of semigroups of operators are used.
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1 Introduction

The aim of this paper is to study the existence of extremal solutions for the time-dependent problems. We combine the monotone iterative technique (see [8,1,7,11,10,13,18]) and the theory of semigroups of operators (see [4,12,14,15]).

Let $X = (X, |\cdot|, \leq)$ be an ordered Banach space (OBS) with normal cone K, and let $T > 0$ be a given real number. Given $A : D(A) \subset X \to X$ a densely defined linear operator on X and a mapping $F : (0,T) \times X \to X$, we denote

$$Lu = u' + Au - F(t,u).$$

Given $u_0 \in X$ and a mapping $G : (0,T) \times X \times X \to X$, we consider the following initial value problem (IVP):

$$\begin{cases} Lu = G(t,u,Lu), \ t \in (0,T) \\ u(0) = u_0. \end{cases} \tag{1}$$

We also consider the special case of IVP (1):

$$\begin{cases} u' + Au = G(t,u), \ t \in (0,T) \\ u(0) = u_0. \end{cases} \tag{2}$$

Throughout this paper we shall consider that the following hypotheses are fulfilled.

(A1) $(-A)$ is the generator of a semigroup of linear contractive and positive operators on X, $\left(e^{-At}\right)_{t>0}$.

(G1) G is sup-measurable (i.e. $G\left(\cdot, u(\cdot), v(\cdot)\right) : (0,T) \to X$ is measurable for all measurable functions $u, v : (0,T) \to X$).

(H1) There exist $\underline{u}$ a mild lower solution and $\overline{u}$ a mild upper solution of (1) such that $\tilde{L}\underline{u} \leq \tilde{L}\overline{u}$.

Let us mention that the notion of mild lower (upper) solution and the operator $\tilde{L}$ will be defined in Section 2.

In addition, we assume that the following conditions hold.

Only for problem (1):

(F1) $F(\cdot, u)$ is measurable for all $u \in X$.

(F2) There exists $a \in L^1(0,T)$ such that for all $u_1, u_2 \in X$ and for almost all $t \in (0,T)$,

$$|F(t,u_1) - F(t,u_2)| \leq a(t)|u_1 - u_2|. \tag{3}$$

(F3) There exists $\omega \geq 0$ such that for almost every $t \in (0,T)$,

$$F(t,u_1) - F(t,u_2) \leq -\omega(u_1 - u_2) \text{ whenever } u_1 \leq u_2.$$

(G2) $u_1 \leq u_2$ and $v_1 \leq v_2$ imply $G(t,u_1,v_1) \leq G(t,u_2,v_2)$ for almost every $t \in (0,T)$.

(C1) The order cone K is regular.

Only for problem (2):

(G2)' There exists $\omega \geq 0$ such that for almost every $t \in (0,T)$,

$$G(t,u_1) - G(t,u_2) \leq -\omega(u_1 - u_2) \text{ whenever } u_1 \leq u_2.$$

(C1)' The order cone K is regular *or* $(-A)$ generates a compact semigroup.

Our main results are the following theorems.

Theorem 1.1 *The IVP (1) has minimal and maximal mild solutions in* $W = \left\{u \in \mathcal{D}(L) \ : \ \tilde{L}\underline{u} \leq \tilde{L}u \leq \tilde{L}\overline{u}\right\}$.

Theorem 1.2 *The extremal mild solutions of (1) in W are monotone increasing with respect to F, G and u_0.*

In the special case of IVP (2) we impose also the following hypotheses in order to obtain the convergence of monotone iterations.

(G3) For all monotone increasing sequences $y_n \to y^*$ and $z_n \to z^*$, the sequence $(G(t, y_n, z_n))_n$ converges to $G(t, y^*, z^*)$ for almost every $t \in (0, T)$.

(G4) For all monotone decreasing sequences $y_n \to y^*$ and $z_n \to z^*$, the sequence $(G(t, y_n, z_n))_n$ converges to $G(t, y^*, z^*)$ for almost every $t \in (0, T)$.

The main result for IVP (2) is the following theorem.

Theorem 1.3 *If (G3) holds then (2) has a minimal mild solution u_* in W and the sequence $(u_n)_{n\geq 1}$ defined by*

$$u_1 = \underline{u},$$

u_{n+1} *is the mild solution of the IVP*

$$\begin{cases} u' + Au + \omega u = G(t, u_n) + \omega u_n, \\ u(0) = u_0 \end{cases} \tag{4}$$

is monotone increasing and converges to u_* in $C([0, T]; X)$.
If (G4) holds then (2) has a maximal mild solution u^* in W and the sequence $(v_n)_{n\geq 1}$ defined by

$$v_1 = \overline{u},$$

v_{n+1} *is the mild solution of the IVP*

$$\begin{cases} u' + Au + \omega u = G(t, v_n) + \omega v_n, \\ u(0) = u_0 \end{cases} \tag{5}$$

is monotone decreasing and converges to u^* in $C([0, T]; X)$.
Moreover, u_* and u^* are monotone increasing with respect to G and u_0.

Our work is especially related to [9] and [13].
S. Carl and S. Heikkila in [9] find sufficient conditions for the existence of extremal solutions for implicit IVP of type (1) within an order interval determined by a lower and an upper solution, and show that these extremal solutions are monotone with respect to the data. They mention that a crucial point in their treatment is the use of a partial ordering and a metric which depend on the operator L as well as on the initial condition. Due to an abstract monotone iterative technique developed by the same authors in [8] we avoid these tricks. They consider the case when the operator L is given

by $Lu = u' - A(t)u - C(t)$ and defined on the space of absolutely continuous functions $AC([0,T];X)$, where A is a Bochner integrable mapping from $[0,T]$ to the space of all bounded linear operators on X, and $C : [0,T] \to X$ is Bochner integrable, or $Lu = u' - q(t,u)$, where $q : [0,T] \times X \to X$ is a Caratheodory function which is quasimonotone increasing and Lipschitz continuous in its second argument.
In Theorem 1.3 we generalize and improve similar results obtained by X. Z. Liu, S. Sivaloganathan and S. Zhang in [13]. They study the IVP (2) when G is continuous and only when the order cone K is regular. Our results are valid also if K is not regular but the semigroup is compact. This fact allows us to treat the Cauchy-Dirichlet problem for a parabolic equation in the space $X = C_0(\Omega)$, since the order cone of $C_0(\Omega)$ is normal but is not regular. Our treatment is inspired by the abstract technique developed in [8]. We consider that it is more convenient since we work in the space $L^1(0,T;X)$ instead of $C([0,T];X)$.
Finally, let us mention that we extend the notion of lower (upper) solution as a C^1-function (like in [13]) or AC-function (like in [9]), to the notion of mild lower (upper) solution as a C-function (in Section 2).

2 Preliminaries

The purpose of this section is to define the notion of mild lower (upper) solution for the IVP (1) and to reduce the IVP (1) to a coincidence equation in ordered Banach spaces.

Lemma 2.1 [14] *For every $\omega \geq 0$, the linear operator $-(A+\omega I)$ is the generator of a continuous semigroup of linear nonexpansive and positive operators on X, denoted by $(S(t))_{t>0}$. Moreover, $S(t) = e^{-\omega t}e^{-At}$ for all $t > 0$. If the semigroup generated by $(-A)$ is compact then $(S(t))_{t>0}$ is also compact.*

We use the abstract Gronwall lemma which is due to Rus (see [16]).

Lemma 2.2 *Let X be an ordered metric space and $B : X \to X$ a monotone increasing and a Picard operator. If $u \leq Bu$ and $u^* = Bu^*$ then $u \leq u^*$.*

Let us mention that the operator B is Picard if it has a unique fixed point, which is the limit of the sequence $(B^n(u))_{n\geq 0}$ for every $u \in X$.
Other applications of this lemma are contained in [16,5,6].

Lemma 2.3 *i) There exists a unique solution $u^* \in C([0,T];X)$ of*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[F(s,u(s)) + \omega u(s)]ds.$$

ii) If $\underline{u} \in C([0,T];X)$ is a solution of the inequality

$$u(t) \leq S(t)u_0 + \int_0^t S(t-s)[F(s,u(s)) + \omega u(s)]ds$$

then

$$\underline{u}(t) \leq u^*(t), \text{ for all } t \in [0,T].$$

Proof. Let us consider the operator

$$B : C([0,T];X) \to C([0,T];X) \tag{6}$$

$$Bu(t) = S(t)u_0 + \int_0^t S(t-s)[F(s,u(s)) + \omega u(s)]ds$$

and the Bielecki norm on the space $C([0,T];X)$,

$$||u||_b = max_{t\in[0,T]}|u(t)|e^{-2\int_0^t [a(s)+\omega]ds}.$$

By estimations

$$\begin{aligned} & |Bu_1(t) - Bu_2(t)| = \\ = & \left| \int_0^t S(t-s)[F(s,u_1(s)) - F(s,u_2(s)) + \omega u_1(s) - \omega u_2(s)] \right| \leq \\ \leq & \int_0^t [a(s)+\omega]|u_1(s) - u_2(s)|ds \leq \frac{1}{2} e^{2\int_0^t [a(s)+\omega]ds} ||u_1 - u_2||_b \end{aligned}$$

we obtain that

$$||Bu_1 - Bu_2||_b \leq \frac{1}{2}||u_1 - u_2||_b \ , \quad \forall u_1, u_2 \in C([0,T];X),$$

which means that B is a contraction on the Banach space $(C([0,T];X), ||\cdot||_b)$. We apply the Banach contraction mapping principle and deduce that B is a Picard operator (u^* is the unique fixed point).
The positivity of the semigroup $(S(t))$ and the hypothesis (F3) assure that B is monotone increasing.
We have that $u^* = Bu^*$, and $\underline{u} \leq B\underline{u}$. We apply the abstract lemma of Rus and obtain the conclusion. □

Notation 1.

$$\tilde{L} : \mathcal{D}(L) \subset C([0,T];X) \to L^1(0,T;X) \times X$$

$$\tilde{L}u = (w, w_0) \text{ if and only if}$$

$$u(t) = S(t)w_0 + \int_0^t S(t-s)[w(s) + F(s,u(s)) + \omega u(s)]ds.$$

Remark 1. If we apply Lemma 2.3 for $F+w$ instead of F, we see that $\mathcal{D}(L)$ it can be chosen such that $\tilde{L}$ is well defined and bijective.

Remark 2. $\tilde{L}u = (w, w_0)$ if and only if u is the mild solution of the IVP

$$\begin{cases} u' + Au = w(t) + F(t,u) \\ u(0) = w_0. \end{cases} \tag{7}$$

Let us notice also that $\tilde{L}u = (u' + Au - F(t,u),\ u(0))$ for all $u \in C([0,T];X))$ with $u(t) \in D(A)$ and there exists $u' \in L^1(0,T;X)$.
For all $u \in \mathcal{D}(L)$ we consider Lu given by $\tilde{L}u = (Lu, u(0))$.

Remark 3. If the function $u \in \mathcal{D}(L) \subset C([0,T];X)$ is such that $\tilde{L}u \leq (w, w_0)$ then

$$u(t) \leq S(t)w_0 + \int_0^t S(t-s)[w(s) + F(s,u(s)) + \omega u(s)]ds. \tag{8}$$

In order to prove this let us consider $(v, v_0) \in L^1(0,T;X) \times X$ such that $\tilde{L}u = (v, v_0)$. Then

$$v_0 \leq w_0 \quad \text{and} \quad v \leq w.$$

Using the definition of $\tilde{L}$ and the positivity of the semigroup $(S(t))$ we obtain the conclusion.

Lemma 2.4 *$\tilde{L}u_1 \leq \tilde{L}u_2$ implies $u_1 \leq u_2$.*

Proof. Let $u_1, u_2 \in \mathcal{D}(L) \subset C([0,T];X)$ be such that $\tilde{L}u_1 \leq \tilde{L}u_2$. We denote $(w, w_0) = \tilde{L}u_2$. Then u_2 is the unique solution of the equation $\tilde{L}u = (w, w_0)$, and u_1 satisfies the inequality $\tilde{L}u \leq (w, w_0)$. We use Remark 3, apply Lemma 2.3 and obtain that $u_1 \leq u_2$. □

Definition 1. We say that $u \in \mathcal{D}(L) \subset C([0,T];X)$ is a *mild lower solution* of (1) if

$$\tilde{L}u(t) \leq (G(t, u(t), Lu(t)), u_0) \quad \text{for almost all } t \in (0,T).$$

If the reversed inequality holds then u is said to be a *mild upper solution*, or if equality holds, u is a *mild solution* of (1).

Remark 4. Let $u \in \mathcal{D}(L)$ be such that $u(t) \in D(A)$ for a.a. $t \in (0,T)$, there exists $u' \in L^1(0,T;X)$ and it satisfies the following inequalities

$$\begin{cases} u' + Au \leq g(t) \\ u(0) \leq u_0. \end{cases} \tag{9}$$

Then u is a mild lower solution of

$$\begin{cases} u' + Au = g(t) \\ u(0) = u_0. \end{cases} \tag{10}$$

Indeed, let us notice that there exist $v \in L^1(0,T;X)$ and $v_0 \in X$ such that u satisfies

$$\begin{cases} u' + Au = v(t) \\ u(0) = v_0. \end{cases} \tag{11}$$

Then, using (9),

$$v_0 \le u_0 \text{ and } v(t) \le g(t)$$

and, by Remark 2,

$$\tilde{L}u = (v, v_0).$$

Now the conclusion follows easily, since

$$\tilde{L}u(t) \le (g(t), u_0)\,.$$

Notation 2. For all $u \in \mathcal{D}(L)$ let us denote $Nu(t) = G(t, u(t), Lu(t))$ and $\tilde{N}u(t) = (Nu(t), u_0)$.

Notation 3. $W = \tilde{L}^{-1}\left([\tilde{L}\underline{u}, \tilde{L}\overline{u}]\right)$.

Lemma 2.5 *For every* $u \in W$, $Nu \in L^1(0,T;X)$.
The mappings $\tilde{L}, \tilde{N} : W \to L^1(0,T;X) \times X$ *are well-defined and*

$$\tilde{L}u_1 \le \tilde{L}u_2 \textit{ implies } \tilde{N}u_1 \le \tilde{N}u_2. \tag{12}$$

Proof. Let $u \in W$. By (G1), Nu is measurable and, from the definition of W, $\tilde{L}\underline{u} \le \tilde{L}u \le \tilde{L}\overline{u}$. By Lemma 2.3 we have $\underline{u} \le u \le \overline{u}$. Using also (G2), these imply that $N\underline{u}(t) \le Nu(t) \le N\overline{u}(t)$ for almost every $t \in (0,T)$. Then

$$L\underline{u}(t) \le N\underline{u}(t) \le Nu(t) \le N\overline{u}(t) \le L\overline{u}(t).$$

The cone K of X being normal, the norm is semi-monotone and we can prove the inequality

$$|Nu(t)| \le \delta |L\overline{u}(t) - L\underline{u}(t)| + |L\underline{u}(t)|$$

for almost every $t \in (0,T)$. Because the second member of this inequality is in L^1, we obtain that $Nu \in L^1(0,T;X)$.
Relation (12) follows easily by Lemma 2.3 and using (G2). $\square$

Remark 4. When (12) holds we say that $\tilde{N}$ is monotone increasing with respect to $\tilde{L}$ (see [7]).

Lemma 2.6 $\tilde{N}(W) \subset \tilde{L}(W) = \left[\tilde{L}\underline{u}, \tilde{L}\overline{u}\right]$.

Proof. The last equality follows by the bijectivity of $\tilde{L}$.
For every $u \in W$ we have $\tilde{L}\underline{u} \le \tilde{L}u \le \tilde{L}\overline{u}$. Then, using the definition of the mild lower and upper solution and (12), we obtain

$$\tilde{L}\underline{u} \le \tilde{N}\underline{u} \le \tilde{N}u \le \tilde{N}\overline{u} \le \tilde{L}\overline{u}. \ \square$$

3 Existence results

The proof of our main results is based on the next coincidence result which is established in [7] like a consequence of Proposition 3.4 from [8].

Proposition 3.1 *Let W be a nonempty set, Z be an ordered Banach space with a regular order cone and let $\tilde{L}, \tilde{N} : W \to Z$ be two mappings. Assume that the following conditions hold.*
(i) there exists $\underline{u}, \overline{u} \in W$ such that $\tilde{L}\underline{u} \le \tilde{N}\underline{u}$, $\tilde{L}\overline{u} \le \tilde{N}\overline{u}$ and $\tilde{L}\underline{u} \le \tilde{L}\overline{u}$;
(ii) $\tilde{L}u_1 \le \tilde{L}u_2$ implies $\tilde{N}u_1 \le \tilde{N}u_2$;
(iii) $\tilde{L}(W) = \left[\tilde{L}\underline{u}, \tilde{L}\overline{u}\right]$.
Then equation $\tilde{L}u = \tilde{N}u$ has a solution u_* with the property

$$Lu_* = \min\left\{Lw \in \left[\tilde{L}\underline{u}, \tilde{L}\overline{u}\right] \ : \ \tilde{L}w \ge \tilde{N}w\right\} \tag{13}$$

and a solution u^* with the property

$$Lu^* = \max\left\{Lw \in \left[\tilde{L}\underline{u}, \tilde{L}\overline{u}\right] \ : \ \tilde{L}w \le \tilde{N}w\right\}. \tag{14}$$

If, in adition, W is an ordered set and $\tilde{L}u_1 \le \tilde{L}u_2$ implies $u_1 \le u_2$, then u_* is the minimal solution and u^* is the maximal solution in W of $\tilde{L}u = \tilde{N}u$.

Proof of Theorem 1.1. If K is regular then the order cone of $L^1(0,T;X)$ is also regular. Then the ordered Banach space $Z = L^1(0,T;X) \times X$ (with the natural ordering, see [1]) has a regular order cone. The set W is given by Notation 3. Conditions (i), (ii) and (iii) are assured by hypothesis (H1), Lemma 2.5 and the bijectivity of $\tilde{L}$, respectively. W is an ordered space as a subset of $C([0,T];X)$. Finally, Lemma 2.4 assure that we can apply Proposition 3.1 and obtain the existence of extremal mild solutions for the IVP (1). $\square$

Proof of Theorem 1.2. Assume that conditions (F1),(F2),(F3) are valid for the functions $F, F_p : (0,T) \times X \to X$ and (G1) and (G2) are valid for $G, G_p : (0,T) \times X \times X \to X$. Assume also that $u_0, u_{0_p} \in X$ and that

$$u_0 \le u_{0_p}, \quad F(t,u) \le F_p(t,u) \ \text{ and } \ G(t,u,v) \le G_p(t,u,v) \tag{15}$$

for a.a. $t \in (0,T)$ and $u, v \in X$.
Moreover we assume the existence of $\underline{u}, \overline{u} \in W$ so that they are lower and upper mild solutions of both the IVP (1) and the IVP

$$\begin{cases} L_p u = G_p(t,u,L_p u), \ t \in (0,T) \\ u(0) = u_{0_p} \end{cases} \tag{16}$$

where $L_p u = u' + Au - F_p(t,u)$.
Thus problems (1) and (16) have minimal solutions u_*, u_{*_p} and maximal solutions u^*, u_p^*, respectively, in W. When $u \in W$ we denote

$$\tilde{N}_p u = \big(G_p(t,u,Lu), u_{0_p}\big)$$

and consider $\tilde{L}_p$ defined on $\mathcal{D}(L)$ like in Notation 1, with F_p instead of F.
For $u \in \mathcal{D}(L)$ we denote $(w, w_0) = \tilde{L}u$ and $(w_p, w_{0_p}) = \tilde{L}_p$ and obtain that $w_0 = w_{0_p} = u(0)$ and, by the definition of $\tilde{L}$, $w + F(t,u) = w_p + F_p(t,u)$. Then, using (15), $w_p \le w$. Hence $\tilde{L}u \ge \tilde{L}_p u$.
It follows from this relation, from $\tilde{L}_p u_{*_p} = \tilde{N}_p u_{*_p}$ and (15) that $\tilde{L}u_{*_p} \ge \tilde{N}u_{*_p}$. This and (13) imply that $\tilde{L}u_* \le \tilde{L}u_{*_p}$, so that, by applying Lemma 2.4,

$$u_* \le u_{*_p}.$$

Similarly, it can be shown, by applying the formula (14) that

$$u^* \le u_p^*. \ \square$$

In order to prove Theorem 1.3 we shall also use the operator $\tilde{L}$ given by Notation 1, but with $F(t,u) = -\omega u$ and the operator $\tilde{N}$ given by Notation 2, but with $Nu(t) = G(t,u) + \omega u$. The set W is given by Notation 3 with $\tilde{L}$ described above. We shall also need the following lemma.

Lemma 3.1 *(i) $\tilde{L}^{-1}$ is Lipschitz continuous.*
(ii) if the semigroup generated by $(-A)$ is also compact then $\tilde{L}^{-1}$ is completely continuous.

Proof. (i) Let $(w, w_0) \in L^1(0,T;X) \times X$ and denote $u = \tilde{L}^{-1}(w, w_0)$. Using that $S(t)$ is contractive and the definition of $\tilde{L}$ we obtain the following relations

$$|u(t)| \le |w_0| + \int_0^t |w(s)|ds \le |w_0| + ||w||_{L^1}, \text{ for all } t \in [0,T],$$

thus

$$||u||_C \le |w_0| + ||w||_{L^1}.$$

(ii) Let M be a bounded subset of $L^1(0,T;X) \times X$. Of course, $\tilde{L}^{-1}(M)$ denotes the set of all mild solutions of the IVP (7) when (w, w_0) varies in M. The conclusion is assured by the fact that the semigroup $S(t)$ is also compact (see [17]). $\square$

Proof of Theorem 1.3. We shall prove only in the case that condition (G3) holds. If we assume that (G4) is valid, the proof is similar. The last

statement follows like a direct consequence of Theorem 1.2.
A mild solution of the IVP (2) is a solution of the abstract equation $\tilde{L}u = \tilde{N}u$, $\underline{u}$ satisfies $\tilde{L}u \leq \tilde{N}u$ and $\overline{u}$ is such that $\tilde{L}u \geq \tilde{N}u$.
Let us consider the sequence (u_n) given by (4). Using Lemma 2.6, we obtain by induction that $u_n \in W$ for all $n \geq 1$, or, equivalently, $\tilde{L}u_n \in [\tilde{L}\underline{u}, \tilde{L}\overline{u}]$ for all $n \geq 1$. It is easy to see that the following relations hold

$$Lu_1 = L\underline{u} \leq Nu_1 = N\underline{u} = Lu_2 \leq Nu_2 = Lu_3 \leq Nu_3 \leq \dots .$$

Then (u_n) is a sequence such that (Lu_n) and (Nu_n) are increasing.

1) If K is regular then the order cone of $L^1(0,T;X)$ is also regular. This assures that $(\tilde{N}u_n)$ and $(\tilde{L}u_n)$ converge in L^1.

2) If the semigroup generated by $-A$ is compact we shall use the complete continuity of $\tilde{L}^{-1}$. The sequence $(\tilde{L}u_n)$ being bounded, it follows by the complete continuity of $\tilde{L}^{-1}$, that (u_n) has a convergent subsequence (u_{n_k}). Using (F2), the sequence $(Nu_{n_k}(t))$ is convergent for almost every $t \in (0,T)$. But for each $t \in (0,T)$, this is a subsequence of the monotone increasing sequence $(Nu_n(t))$. Hence the whole sequence is convergent since the cone K of X is normal. Also (Nu_n) is bounded in $L^1(0,T;X)$. Then, by the monotone convergence theorem, (Nu_n) is convergent in L^1. Using (4), (Lu_n) is also convergent in L^1.

Now, in both cases, let us denote by $\tilde{L}u_*$ the limit of (Lu_n), which is also the limit of (Nu_n). Of course, $u_* \in W$ and, from the continuity of $\tilde{L}^{-1}$ we have that (u_n) is a convergent sequence and u_* is its limit. Using (G3), we obtain that the limit of $(\tilde{N}u_n)$ is $\tilde{N}u_*$. Now it is clear that $u_* \in W$ is a mild solution of (2).

Let us consider $v^* \in W$ an arbitrary mild solution of (2). It is clear that $\tilde{L}u_1 \leq \tilde{L}v^*$. Then $\tilde{L}u_2 = \tilde{N}u_1 \leq \tilde{L}v^*$. It can be proved by induction that $\tilde{L}u_n \leq \tilde{L}v^*$. By Lemma 1.2 (ii), $u_n \leq v^*$ and passing to the limit, $u_* \leq v^*$. Hence u_* is the minimal mild solution in W. □

4 An example

Let us consider Ω a bounded open subset of R^n. By $C_0(\Omega)$ we denote the space of all continuous scalar valued functions on $\overline{\Omega}$ which are 0 on the boundary $\partial\Omega$ of Ω; i.e. $C_0(\Omega) = \{u \in C(\overline{\Omega}) \ : \ u|_{\partial\Omega} = 0\}$. We consider the Laplacian Δ_0 with maximal distributional domain, i.e. $D(\Delta_0) = \{f \in C_0(\Omega) \ : \ \Delta f \in C_0(\Omega)\}$, $\Delta_0 f = \Delta f$.
We say that Ω is *regular in the sense of Wiener* (see [3]) if and only if the

Dirichlet problem

$$\begin{cases} u \in C(\overline{\Omega}), \ u|_{\partial\Omega} = \varphi \\ \Delta u = 0 \ \text{ in } \ \mathcal{D}'(\Omega) \end{cases} \tag{17}$$

has a solution for all $\varphi \in C(\partial\Omega)$.

Lemma 4.1 [2,3] *If Ω is regular in the sense of Wiener then Δ_0 generates a compact, contractive and positive semigroup on $C_0(\Omega)$.*

We shall study the existence of extremal solutions of the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(u), \quad \text{a.e. in } \Omega \times (0,T) \\ u(x,0) = u_0(x), \quad \text{in } \Omega \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T) \end{cases} \tag{18}$$

where $u_0 \in C_0(\Omega)$ and $f : R \to R$ is a function such that $f(0) = 0$ and

(f1) $f \in C(R)$;

(f2) there exists $\omega \geq 0$ such that

$$f(u_1) - f(u_2) \leq -\omega(u_1 - u_2), \text{ whenever } u_1 \leq u_2.$$

Theorem 4.1 *If, in adition, hypothesis (H1) is fulfilled for the problem (18) then there exist extremal solutions in W of (18). Moreover, there exists a monotone increasing (decreasing) sequence which converges in $C([0,T];X)$ to the minimal (maximal) solution of (18).*

Proof. Let us consider the ordered Banach space $X = C_0(\Omega)$ and the Nemytzki operator $F : X \to X$, $F(u)(x) = f(u(x))$. Our assumptions assure that F satisfies all the hypotheses of Theorem 1.3. The order cone of X is normal and $-A = \Delta_0$ generates a compact semigroup (by Lemma 4.1). Hence all conditions of Theorem 1.3 are fulfilled. □

5 References

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