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A second order analysis of the periodic solutions for nonlinear periodic differential systems with a small parameter

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ABSTRACT

We deal with nonlinear T-periodic differential systems depending on a small parameter. The unperturbed system has an invariant manifold of periodic solutions. We provide the expressions of the bifurcation functions up to second order in the small parameter in order that their simple zeros are initial values of the periodic solutions that persist after the perturbation. In the end two applications are done. The key tool for proving the main result is the Lyapunov–Schmidt reduction method applied to the T-Poincaré–Andronov mapping.

1. Introduction

We want to study the existence of T-periodic solutions of the differential systems of the form,

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad (1)$$

where ε is a small parameter, $F_0, F_1, F_2 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $R : \mathbb{R} \times \Omega \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. We work in the hypothesis that there exists a k-dimensional submanifold of Ω ($k \le n$) whose points are initial values of T-periodic solutions of the unperturbed system

$$x'(t) = F_0(t, x). \quad (2)$$

Our objective is to study the periodic solutions of the unperturbed system (2) which can be continued to the perturbed system (1) for values of ε sufficiently small.

For $z \in \Omega$ we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^n$ the solution of (1) with $x(0, z, \varepsilon) = z$. From Theorem 8.3 of [1] we deduce that, whenever $t_{(z_0,0)} > T$ for some $z_0 \in \Omega$ there exists a neighborhood of $(z_0, 0)$ in $\Omega \times (-\varepsilon_f, \varepsilon_f)$ such that, for all (z, ε) in this neighborhood, $t_{(z,\varepsilon)} > T$. Under this assumption there exists an open subset D of Ω and a sufficiently small $\varepsilon_0 > 0$ such that, for all $(z, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$, the solution $x(\cdot, z, \varepsilon)$ is defined on the interval $[0, T]$. Hence, we can consider the function $f : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$, given by

$$f(z, \varepsilon) = x(T, z, \varepsilon) - z. \quad (3)$$

Then, every $(z_\varepsilon, \varepsilon)$ such that

$$f(z_\varepsilon, \varepsilon) = 0 \quad (4)$$

provides the periodic solution $x(\cdot, z_\varepsilon, \varepsilon)$ of (1).

The converse is also true, i.e. for every T-periodic solution of (1), if we denote by z_ε its value at $t = 0$ then (4) holds. Then, the problem of finding a T-periodic solution of (1), can be replaced by the problem of finding zeros of the finite-dimensional function $f(\cdot, \varepsilon)$ given by (3).

We denote the variational equation of (2) associated to one of its solutions $x(t, z, 0)$ with

$$y' = P(t, z)y, \quad (5)$$

where

$$P(t, z) = D_x F_0(t, x(t, z, 0)), \quad (6)$$

and with $Y(\cdot, z)$ some fundamental matrix solution of (5).

We denote the projection onto the first k coordinates by $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and the one onto the last $(n-k)$ coordinates by $\pi^\perp : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. For the n-dimensional function g of n variables $z = (\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we denote by $D_\beta(\pi g)$ the $k \times (n-k)$ matrix whose entries are the first order partial derivatives with respect to each component of $\beta \in \mathbb{R}^{n-k}$ of the first k components of g.

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The next theorem is our main result. In its proof we shall apply a version of the Lyapunov–Schmidt reduction method (provided in Theorem 4 of Section 2) to the function (3) after a suitable change of coordinates.

Theorem 1. *Let $\beta = (\beta_1, \ldots, \beta_{n-k}) : \overline{V} \to \mathbb{R}^{n-k}$ be a C^2 function, where $V \subset \mathbb{R}^k$ is open and bounded. We assume that*

(i) *$\mathcal{Z} = \left\{z_\alpha = (\alpha, \beta(\alpha)), \alpha \in \overline{V}\right\} \subset D$ and that for each $z_\alpha \in \mathcal{Z}$, the unique solution $x(t, z_\alpha, 0)$ of (2) with $x(0, z_\alpha, 0) = z_\alpha$ (denoted shortly $x_\alpha(t)$), is T-periodic; and*
(ii) *for each $z_\alpha \in \mathcal{Z}$, there exists a fundamental matrix solution $Y_\alpha(t) = Y(t, z_\alpha)$ of (5) such that the matrix $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$ has in the right up corner the null $k \times (n-k)$ matrix, while in the right down corner has the $(n-k) \times (n-k)$ matrix Δ_α, with* $\det(\Delta_\alpha) \neq 0$.

The corresponding bifurcation functions $f_1 : \overline{V} \to \mathbb{R}^k$ of first order in ε is

$$f_1(\alpha) = \pi \left(\int_0^T Y^{-1}(t, z_\alpha) F_1(t, x(t, z_\alpha, 0)) dt \right), \tag{7}$$

and $f_2 : \overline{V} \to \mathbb{R}^k$ of second order in ε is

$$\begin{aligned} f_2(\alpha) &= 2\pi g_2(\alpha) + 2\left(D_\beta(\pi g_1)(z_\alpha)\right)\gamma(\alpha) \\ &\quad + \sum_{i=1}^{n-k} \frac{\partial \beta_i}{\partial \varepsilon}(\alpha, 0) \frac{\partial}{\partial \beta_i}\left(D_\beta(\pi g_0)(z_\alpha)\right)\gamma(\alpha), \end{aligned} \tag{8}$$

where $\gamma : \overline{V} \to \mathbb{R}^{n-k}$ is defined by

$$\gamma(\alpha) = -\Delta_\alpha^{-1}(\pi^\perp g_1)(z_\alpha),$$

and

$$g_0(z) = Y^{-1}(T, z)\left(x(T, z, 0) - z\right),$$

$$g_1(z) = \int_0^T Y^{-1}(t, z) F_1(t, x(t, z, 0)) dt,$$

$$g_2(z) = \frac{1}{2} \int_0^T Y^{-1}(t, z) F_*(t, x(t, z, 0)) dt,$$

with

$$F_* = 2F_2 + 2(D_x F_1)\frac{\partial x}{\partial \varepsilon} + \sum_{i=1}^n \frac{\partial x_i}{\partial \varepsilon} \frac{\partial}{\partial x_i}(D_x F_0) \frac{\partial x}{\partial \varepsilon},$$

$$\frac{\partial x}{\partial \varepsilon} = Y(t, z) \int_0^t Y^{-1}(s, z) F_1(s, x(s, z, 0)) ds.$$

If there exists a zero $a \in V$ of $f_1(\alpha)$ such that its Jacobian $\det((D_\alpha f_1)(a)) \neq 0$, *then there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.*

If $f_1(\alpha) \equiv 0$ and there exists a zero $a \in V$ of $f_2(\alpha)$ such that its Jacobian $\det((D_\alpha f_2)(a)) \neq 0$, *then there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.*

We remark that in the expression of F_*, the notation $\frac{\partial}{\partial x_i}(D_x F_0)$ stands for the matrix-valued function whose entries are the first order partial derivatives with respect to x_i of the entries of the matrix-valued function $D_x F_0$. Each term in the sum that appeared in the expression of F_* is, then, a product between a scalar and an n-dimensional column vector obtained of applying an $n \times n$ matrix to an n-dimensional column vector. Similar notation for the k-dimensional column vector $f_2(\alpha)$.

Theorem 1 is proved in Section 3.

The first order bifurcation function $f_1(\alpha)$ was already obtained by Malkin [2] and Roseau [3], see also the book of Françoise [4]. For a shorter proof see [5]. What really is new in Theorem 1 is the expression of the bifurcation function $f_2(\alpha)$ corresponding to second order analysis in ε of the existence of T-periodic solutions of system (1). In the particular case that $k = n$ the expression of f_2 was found in [6], while when F_0 is identically zero (in particular this implies $k = n$), the expressions of the second order and third order bifurcation functions are known (see for instance [7] and the references therein). We remark that, in this particular case $F_0 \equiv 0$, the construction of these bifurcation functions are part of the averaging theory (for general results on averaging theory see for instance the books of Sanders et al. [8], Verhulst [9]). Sometimes, also in the general case $F_0 \not\equiv 0$ it is said that results like Theorem 1 describe the averaging method. Moreover, in the particular case $F_0 \equiv 0$ and for scalar equations (1) (i.e. for $n = 1$) the recursive expressions of the bifurcation functions up to any order are known, see [6,10].

We do two applications of the new averaging theorem at second order in ε, i.e. of Theorem 1. In the first application we consider a differential equation of order four of the form

$$\frac{d^4x}{dt^4} + \alpha x + \psi(x, t) = 0.$$

This class of equations have been studied in [11,12]. Here we will analyze the particular fourth order differential equation

$$\frac{d^4x}{dt^4} - x - \varepsilon(a + b\cos^2 t) - \varepsilon^2 \sin(x + t) = 0, \tag{9}$$

where $a, b \in \mathbb{R}$, or equivalently the first order differential system in $\mathbb{R}^4$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= x + \varepsilon(a + b\cos^2 t) + \varepsilon^2 \sin(x + t), \end{aligned} \tag{10}$$

where the dot denotes the derivative with respect to the time variable t. Our result on the periodic solutions of the fourth order differential equation (9) is the following.

Proposition 2. *For $|\varepsilon| \neq 0$ sufficiently small the differential system (10) has an arbitrary number of limit cycles bifurcating from the continuum of the periodic solutions of the 2-dimensional isochronous center that the system has for $\varepsilon = 0$.*

The proof of Proposition 2 is given in Section 4, and uses Theorem 1.

In the second application we deal with the homogeneous polynomial differential system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2), \\ \dot{y} &= x(x^2 - y^2), \end{aligned} \tag{11}$$

of degree 3 that has the C^∞ flat first integral

$$H(x, y) = (x^2 + y^2)\exp\left(-\frac{2x^2}{x^2 + y^2}\right).$$

Proposition 3. *The homogeneous polynomial differential system (11) has a global center at the origin (i.e. all the solutions contained in $\mathbb{R}^2 \setminus \{(0, 0)\}$ are periodic). Let $P_i(x, y)$ and $Q_i(x, y)$ for $i = 1, 2$ be polynomials of degree at most 3. Then, for convenient polynomials P_i and Q_i, the polynomial differential system*

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon P_1(x, y) + \varepsilon^2 P_2(x, y), \\ \dot{y} &= x(x^2 - y^2) + \varepsilon Q_1(x, y) + \varepsilon^2 Q_2(x, y), \end{aligned} \tag{12}$$

has at first order averaging one limit cycle, and at second order averaging two limit cycles bifurcating from the periodic solutions of the global center (11).

As far as we know this is one of the first examples for which the limit cycles bifurcating from the periodic solutions of a 2-dimensional center of a polynomial differential system having a non-rational first integral have been studied. The other two examples that we know in this direction were given recently by Jiming Li [13] and Llibre et al. [14].

We remark that the result of Proposition 3 with the averaging method of first order is already contained in Theorem 2 of [14].

The proof of Proposition 3 is also given in Section 4. More precisely, we will apply Theorem 1, which gives a method to determine bifurcation of periodic solutions from a submanifold of isochronous periodic solutions. We note that the degenerate center of system (11) it is not isochronous because it cannot be linearized see [15] but, after a change of variables to polar coordinates (r,θ), it becomes isochronous with respect to the new time θ and Theorem 1 can be applied.

2. Lyapunov–Schmidt reduction for finite dimensional functions

The Lyapunov–Schmidt reduction method is the main tool that we shall use for proving our Theorem 1. Here we provide a version of it adapted to our necessities. For a general introduction to this reduction method see [16].

Theorem 4. *Let $g : D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ and $\beta=(\beta_1,\dots,\beta_{n-k}) : \overline{V}\to\mathbb{R}^{n-k}$ be C^2 functions, where D is an open subset of $\mathbb{R}^n$ and V is an open and bounded subset of $\mathbb{R}^k$. We assume that*

$$g(z,\varepsilon)=g_0(z)+\varepsilon g_1(z)+\varepsilon^2 g_2(z)+O(\varepsilon^3),$$

and that

(i) $\mathcal{Z}=\left\{z_\alpha=(\alpha,\beta(\alpha)),\alpha\in\overline{V}\right\}\subset D$ *and that for each* $z_\alpha\in\mathcal{Z}$, $g_0(z_\alpha)=0$, *and*

(ii) *the matrix* $G_\alpha=D_z g_0(z_\alpha)$ *has in the right up corner the null* $k\times(n-k)$ *matrix, while in the right down corner has the* $(n-k)\times(n-k)$ *matrix* Δ_α, *with* $\det(\Delta_\alpha)\neq 0$.

We consider the function $f_1 : \overline{V}\to\mathbb{R}^k$ defined by

$$f_1(\alpha)=\pi g_1(z_\alpha). \tag{13}$$

If there exists $a\in V$ with $f_1(a)=0$ and such that the Jacobian $\det((D_\alpha f_1)(a))\neq 0$, then there exists α_ε such that $g(z_{\alpha_\varepsilon},\varepsilon)=0$ and $z_{\alpha_\varepsilon}\to z_a$ as $\varepsilon\to 0$.

We consider the functions $\gamma : \overline{V}\to\mathbb{R}^{n-k}$ and $f_2 : \overline{V}\to\mathbb{R}^k$ defined by

$$\begin{aligned}\gamma(\alpha)&=-\Delta_\alpha^{-1}(\pi^\perp g_1)(z_\alpha),\\ f_2(\alpha)&=2(\pi g_2)(z_\alpha)+2D_\beta(\pi g_1)(z_\alpha)\gamma(\alpha)\\ &\quad+\sum_{i=1}^{n-k}\frac{\partial\beta_i}{\partial\varepsilon}(\alpha,0)\frac{\partial}{\partial\beta_i}D_\beta(\pi g_0)(z_\alpha)\gamma(\alpha).\end{aligned}\tag{14}$$

If $f_1(\alpha)\equiv 0$ and there exists $a\in V$ with $f_2(a)=0$ such that the Jacobian $\det((D_\alpha f_2)(a))\neq 0$, then there exists α_ε such that $g(z_{\alpha_\varepsilon},\varepsilon)=0$ and $z_{\alpha_\varepsilon}\to z_a$ as $\varepsilon\to 0$.

Proof. We consider the function

$$\pi^\perp g : \mathbb{R}^k\times\mathbb{R}^{n-k}\times[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}^{n-k},$$
$$(\alpha,\beta,\varepsilon)\mapsto\pi^\perp g(\alpha,\beta,\varepsilon).$$

Then, we have $\pi^\perp g(z_\alpha,0)=0$ and $D_\beta\left(\pi^\perp g\right)(z_\alpha,0)=\Delta_\alpha$. Since $\det(\Delta_\alpha)\neq 0$, we apply the Implicit Function Theorem and deduce that, for $|\varepsilon|$ sufficiently small, there exists a function $\bar\beta$ with

$$(\alpha,\varepsilon)\mapsto\bar\beta(\alpha,\varepsilon)\quad\text{such that } \bar\beta(\alpha,0)=\beta(\alpha)\quad\text{and}$$
$$\pi^\perp g(\alpha,\bar\beta(\alpha,\varepsilon),\varepsilon)=0.$$

Now we consider the function

$$\delta : \mathbb{R}^k\times[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}^k,\qquad \delta(\alpha,\varepsilon)=\pi g(\alpha,\bar\beta(\alpha,\varepsilon),\varepsilon).$$

We have

$$\delta(\alpha,0)=\pi g(z_\alpha,0)=0,$$
$$\frac{\partial\delta}{\partial\varepsilon}(\alpha,\varepsilon)=D_\beta(\pi g)(z_\alpha)\frac{\partial\beta}{\partial\varepsilon}(\alpha)+\frac{\partial(\pi g)}{\partial\varepsilon}(z_\alpha).$$

Using (ii) we see that $D_\beta(\pi g)(z_\alpha,0)=0_{k\times(n-k)}$, where $0_{k\times(n-k)}$ is the null $k\times(n-k)$ matrix. Hence $\partial\delta/\partial\varepsilon(\alpha,0)=f_1(\alpha)$. We claim that $\partial^2\delta/\partial\varepsilon^2(\alpha,0)=f_2(\alpha)$, such that we can write

$$\delta(\alpha,\varepsilon)=\varepsilon f_1(\alpha)+\frac{\varepsilon^2}{2}f_2(\alpha)+O(\varepsilon^3).$$

Applying the Implicit Function Theorem to $\delta(\alpha,\varepsilon)/\varepsilon$ in the case that f_1 has a simple zero, respectively to $\delta(\alpha,\varepsilon)/\varepsilon^2$ in the case that $f_1\equiv 0$ and f_2 has a simple zero, we obtain for $|\varepsilon|$ sufficiently small, the existence of $\alpha(\varepsilon)$ such that $\alpha(0)=a$ and $\delta(\alpha(\varepsilon),\varepsilon)=0$. Moreover, denoting $z_{\alpha_\varepsilon}=\left(\alpha(\varepsilon),\bar\beta(\alpha(\varepsilon),\varepsilon)\right)$ we have $g(z_{\alpha_\varepsilon},\varepsilon)=0$.

In order to justify the claim we write shortly the expressions

$$\frac{\partial\delta}{\partial\varepsilon}(\alpha,\varepsilon)=D_\beta(\pi g)\frac{\partial\beta}{\partial\varepsilon}+\frac{\partial(\pi g)}{\partial\varepsilon},$$

$$\begin{aligned}\frac{\partial^2\delta}{\partial\varepsilon^2}(\alpha,\varepsilon)&=\sum_{i=1}^{n-k}\frac{\partial\beta_i}{\partial\varepsilon}\frac{\partial}{\partial\beta_i}D_\beta(\pi g)\frac{\partial\beta}{\partial\varepsilon}+\frac{\partial}{\partial\varepsilon}D_\beta(\pi g)\frac{\partial\beta}{\partial\varepsilon}\\ &\quad+D_\beta(\pi g)\frac{\partial^2\beta}{\partial\varepsilon^2}+D_\beta\frac{\partial(\pi g)}{\partial\varepsilon}\frac{\partial\beta}{\partial\varepsilon}+\frac{\partial^2(\pi g)}{\partial\varepsilon^2}.\qquad\square\end{aligned}$$

Corollary 5. *Notice that the case $k=n$ is a trivial particular case of the above theorem. Indeed, when $k=n$, hypothesis* (i) *is satisfied if and only if $g_0\equiv 0$ and in this case hypothesis* (ii) *is automatically fulfilled. The bifurcations functions are $f_1(z)=g_1(z)$ and $f_2(z)=2g_2(z)$.*

3. Proof of Theorem 1

We need to study the zeros of the function (3), or, equivalently, of

$$g(z,\varepsilon)=Y^{-1}(T,z)f(z,\varepsilon)=Y^{-1}(T,z)\left(x(T,z,\varepsilon)-z\right).$$

To this function we shall apply Theorem 4. It is sufficient if we identify the functions g_0, g_1 and g_2 and we prove that they satisfy the hypotheses of Theorem 4. Of course, $g_0(z)=g(z,0)$ and we have that $g_0(z_\alpha)=0$, because $x(\cdot,z_\alpha,0)$ is T-periodic. We shall prove that

$$G_\alpha=D_z g_0(z_\alpha)=Y_\alpha^{-1}(0)-Y_\alpha^{-1}(T). \tag{15}$$

For this we need to know $(D_z x)(t,z,0)$. Since it is the matrix solution of (5) with $(D_z x)(0,z,0)=I_n$, we have that $(D_z x)(t,z,0)=Y(t,z)Y^{-1}(0,z)$. Moreover,

$$D_z f(z,0)=D_z x(T,z,0)-I_n=Y(T,z)Y^{-1}(0,z)-I_n$$

and

$$\begin{aligned}D_z g_0(z)&=Y^{-1}(0,z)-Y^{-1}(T,z)\\ &\quad+\left(\frac{\partial Y^{-1}}{\partial z_1}(T,z)f(z,0),\dots,\frac{\partial Y^{-1}}{\partial z_n}(T,z)f(z,0)\right),\end{aligned}$$

that, for $z_\alpha\in\mathcal{Z}$ reduces to (15).

We have

$$\frac{\partial g}{\partial\varepsilon}(z,0)=Y^{-1}(T,z)\frac{\partial x}{\partial\varepsilon}(T,z,0).$$

Taking the derivative with respect to ε in the relations

$$\begin{aligned}x'(t,z,\varepsilon)&=F_0(t,x(t,z,\varepsilon))+\varepsilon F_1(t,x(t,z,\varepsilon))\\ &\quad+\varepsilon^2F_2(t,x(t,z,\varepsilon))+O(\varepsilon^3),\end{aligned}\tag{16}$$
$$x(0,z,\varepsilon)=z,$$

one can see that the function $(\partial x/\partial\varepsilon)(\cdot,z,0)$ is the unique solution of the initial value problem

$$y'=D_xF_0(t,x(t,z,0))y+F_1(t,x(t,z,0)),\quad y(0)=0.$$

Hence

$$\frac{\partial x}{\partial \varepsilon}(t,z,0) = Y(t,z)\int_0^t Y^{-1}(s,z)F_1(s,x(s,z,0))ds.$$

Now we have

$$g_1(z) = \frac{\partial g}{\partial \varepsilon}(z,0) = \int_0^T Y^{-1}(s,z)F_1(s,x(s,z,0))ds.$$

Taking the second order derivative with respect to ε in the relations (16), one can see that the function $\left(\partial^2 x/\partial \varepsilon^2\right)(\cdot,z,0)$ is the unique solution of the initial value problem

$$y' = D_xF_0(t,x(t,z,0))y + F_*(t,x(t,z,0)), \qquad y(0)=0,$$

where the expression of F_* is given in the statement of the Theorem. Hence

$$\frac{\partial^2 x}{\partial \varepsilon^2}(t,z,0) = Y(t,z)\int_0^t Y^{-1}(s,z)F_*(s,x(s,z,0))ds.$$

Now we have

$$g_2(z) = \frac{1}{2}\frac{\partial^2 g}{\partial \varepsilon^2}(z,0) = \frac{1}{2}\int_0^T Y^{-1}(s,z)F_*(s,x(s,z,0))ds.$$

This completes the proof of Theorem 1.

Corollary 6 (*The Isochronous Case*). *We assume that there exists an open set V with $\overline{V} \subset D$ and such that for each $z \in \overline{V}$, $x(\cdot,z,0)$ is T-periodic, that is the hypothesis of the above theorem are fulfilled for $k=n$. In this case $g_0 \equiv 0$ and the bifurcation functions have simpler expressions $f_1(z) = g_1(z)$ and $f_2(z) = 2g_2(z)$, where g_1 and g_2 are calculated according to the formulas of Theorem 1.*

4. Proof of the two applications

Proof of Proposition 2. The linear part at the origin of the differential system (10) is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{pmatrix}, \tag{17}$$

and its eigenvalues are ± 1 and $\pm i$. Doing the change of variables $(x,y,z,w) \mapsto (X,Y,Z,W)$ given by

$$\begin{pmatrix} X\\ Y\\ Z\\ W\end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1\\ -1 & -1 & 1 & 1\\ 1 & 1 & 1 & 1\\ -1 & 1 & -1 & 1\end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix},$$

system (10) becomes

$$\begin{aligned}
\dot X &= -Y + \varepsilon(a + b\cos^2 t)\\
&\quad + \varepsilon^2 \sin((4t + X - Y + Z - W)/4),\\
\dot Y &= X + \varepsilon(a + b\cos^2 t)\\
&\quad + \varepsilon^2 \sin((4t + X - Y + Z - W)/4),\\
\dot Z &= Z + \varepsilon(a + b\cos^2 t)\\
&\quad + \varepsilon^2 \sin((4t + X - Y + Z - W)/4),\\
\dot W &= -W + \varepsilon(a + b\cos^2 t)\\
&\quad + \varepsilon^2 \sin((4t + X - Y + Z - W)/4).
\end{aligned} \tag{18}$$

Note that the differential of this system at the origin is the real normal Jordan form of the matrix (17).

Now we shall apply Theorem 1 to the differential system (18) taking

$$\begin{aligned}
&\mathbf{x} = (X,Y,Z,W),\\
&F_0(t,\mathbf{x}) = (-Y,X,Z,-W),\\
&F_1(t,\mathbf{x}) = (A,A,A,A),\\
&F_2(t,\mathbf{x},\varepsilon) = (B,B,B,B),\\
&\Omega = \mathbb{R}^4,
\end{aligned} \tag{19}$$

where $A = a + b\cos^2 t$, and $B = \sin((4t + X - Y + Z - W)/4)$.

Clearly system (18) with $\varepsilon = 0$ has a linear center at the origin in the (X,Y)-plane. We remark that all linear centers are isochronous. Using the notation from the Introduction (mainly the notation related with the statement of Theorem 1), the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ of this center with $\mathbf{z} = (X_0,Y_0,0,0)$ is

$$\begin{aligned}
&X(t) = X_0\cos t - Y_0\sin t,\\
&Y(t) = Y_0\cos t + X_0\sin t,\\
&Z(t) = 0,\\
&W(t) = 0,
\end{aligned} \tag{20}$$

with period $T = 2\pi$. The k, V and α of Theorem 1 are $k=2$,

$$V = \{(X,Y) : 0 < X^2 + Y^2 < \rho\},$$

for some real number $\rho > 0$, and $\alpha = (X_0,Y_0) \in V$.

For the function F_0 given in (19) and the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ given in (20) the fundamental matrix solution $M(t)$ of the differential system (5) such that $M(0)$ is the identity matrix of $\mathbb{R}^4$ is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & e^t & 0\\ 0 & 0 & 0 & e^{-t}\end{pmatrix}.$$

We remark that for system (18) with $\varepsilon = 0$ the fundamental matrix does not depend on the particular periodic orbit $\mathbf{x}(t,\mathbf{z})$; i.e. it is independent of the initial conditions $\mathbf{z}$. Therefore, an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1-e^{-2\pi} & 0\\ 0 & 0 & 0 & 1-e^{2\pi}\end{pmatrix}.$$

Consequently all the assumptions of Theorem 1 are satisfied. Hence the corresponding bifurcation function at first order $f_1(\alpha)$ is given by

$$f_1(\alpha) = \pi g_1(z_\alpha) = \pi\left(\int_0^{2\pi} M^{-1}(s)F_1(s,x(s,z,0))ds\right)$$

which is identically zero because we have that the following integrals

$$\int_0^{2\pi}(a + b\cos^2 s)(\cos s \pm \sin s)ds = 0.$$

Therefore, we must go to second order of bifurcation. In this case we must study the zeros in V of the system $f_2(\alpha) = 0$ of two equations and two unknowns, where f_2 is given by (8). More precisely, we have $f_2(\alpha) = (f_{2,1}(X_0,Y_0), f_{2,2}(X_0,Y_0))$ where

$$\begin{aligned}
f_{2,1} &= \int_0^{2\pi}(\cos t + \sin t)\\
&\quad \times \sin\left[t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t}{4}\right]dt,\\
f_{2,2} &= \int_0^{2\pi}(\cos t - \sin t)\\
&\quad \times \sin\left[t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t}{4}\right]dt.
\end{aligned}$$

After a tedious calculation (which can be checked using an algebraic processor) and the change of variables $(X_0, Y_0) \mapsto (r, s)$ given by

$$X_0 - Y_0 = 4r\cos s,$$
$$X_0 + Y_0 = -4r\sin s,$$

we obtain for

$$h_j(r,s) = f_{2,j}(2r(\cos s - \sin s), -2r(\cos s + \sin s)),$$

with $j = 1, 2$, that

$$h_1(r,s) = \pi[J_0(r) + J_2(r)(\cos 2s - \sin 2s)],$$
$$h_2(r,s) = -\pi[J_0(r) + J_2(r)(\cos 2s + \sin 2s)],$$

where $J_\mu(r)$ is the μ-th Bessel function of first kind (see [17]).

Adding and subtracting the two equations $h_j(r,s) = 0$, for $j = 1, 2$, we obtain the system

$$\begin{aligned} p_1(r,s) &= J_2(r)\sin 2s = 0, \\ p_2(r,s) &= J_0(r) + J_2(r)\cos 2s = 0. \end{aligned} \tag{21}$$

It is known that the zeros of the functions $J_\mu(r)$ are distinct for different μ's, then either $s = 0$, or $s = \pi/2$. We are not interested in all the solutions of this system, we are only interested to show that it has as many solutions as we want satisfying the assumptions of Theorem 1. So, in what follows we only study the solutions with $s = 0$. Consequently, from the second equation of (21) we obtain

$$J_0(r) + J_2(r) = 0.$$

Since $J_0(r) + J_2(r) = 2J_1(r)/r$, and the function $J_1(r)$ has infinitely many positive zeros tending to be uniformly distributed when $r \to \infty$, because the asymptotic behavior of $J_1(r)$ is $\sqrt{2/(\pi r)}\cos(r - 3\pi/4)$, it follows that system (21) has infinitely many solutions of the form $(r_0, 0)$ being r_0 a positive zero of $J_1(r)$. Then, $(X_0, Y_0) = (2r_0, -2r_0)$ is a solution of the system $f_{2,j}(X_0, Y_0) = 0$ for $j = 1, 2$. Moreover, the determinant of $\partial(f_{2,1}, f_{2,2})/\partial(X_0, Y_0)$ at the point $(2r_0, -2r_0)$ is

$$\det(r_0) = \frac{\pi^2}{8} r_0^2 \mathcal{H}(3, -r_0^2/2)\left[\mathcal{H}(3, -r_0^2/2) - \mathcal{H}(2, -r_0^2/2)\right]$$

where $\mathcal{H}$ is the regularized hypergeometric function, see [17]. Using the formula

$$J_\mu(z) = \frac{z^2}{2^\mu(\mu+1)!}\mathcal{H}(\mu+1, -z^2/4),$$

we get

$$\det(r_0) = \frac{72\pi^2 J_2(r_0)^2}{r_0^2}.$$

Since the zeros of $J_1(r)$ and $J_2(r)$ are different, we get that $\det(r_0) \neq 0$. Hence, by Theorem 1 for each $(2r_0, -2r_0)$ contained in V we have a periodic orbit of system (18) with $|\varepsilon| \neq 0$ sufficiently small.

Finally, for a given positive integer N we can fix ρ in the definition of V in such a way that the interval $(0, \rho)$ contains exactly N zeros of the function $J_1(r)$. Then taking $|\varepsilon| \neq 0$ small enough, Theorem 1 guarantees the existence of N periodic solutions for system (18). Moreover, choosing $|\varepsilon| \neq 0$ smaller if necessary, since system (18) with $\varepsilon = 0$ has its periodic solutions strongly stable and unstable in the directions Z and W respectively, it follows that the N periodic solutions for system (18) obtained using Theorem 1 are limit cycles; i.e. they are isolated in the set of all periodic solutions. This completes the proof of the proposition. □

Proof of Proposition 3. First we show that the homogeneous polynomial differential system (11) has a global center at the origin. In polar coordinates (r, θ) defined by $x = r\cos\theta, y = r\sin\theta$, system (11) becomes

$$\dot{r} = -r^3\sin 2\theta,$$
$$\dot{\theta} = r^2.$$

Of course, to study this system is equivalent to study the differential equation

$$\frac{dr}{d\theta} = -r\sin 2\theta, \tag{22}$$

whose solution $r(\theta, z)$ satisfying $r(0, z) = z$ is

$$r(\theta, z) = z\exp\left(-\sin^2\theta\right). \tag{23}$$

Therefore all the solutions of the differential equation (22) and consequently all the solutions of the homogeneous polynomial differential system (11) are periodic with the exception of the origin which is a singular point. Hence it is proved that the origin of system is a global center.

Now we want to study the limit cycles of the perturbed system (12) for $|\varepsilon| \neq 0$ sufficiently small, which bifurcate from the periodic solutions of the center of system (11).

We write the polynomial $P_i(x, y)$ and $Q_i(x, y)$ of degree 3 of system (11) as

$$\begin{aligned} P_1 &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 \\ &\quad + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ P_2 &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ &\quad + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \\ Q_1 &= A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2 + A_{30}x^3 \\ &\quad + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3, \\ Q_2 &= B_{00} + B_{10}x + B_{01}y + B_{20}x^2 + B_{11}xy + B_{02}y^2 \\ &\quad + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3. \end{aligned}$$

Doing change to polar coordinates to system (12), we obtain that it can be written as

$$\frac{dr}{d\theta} = -r\sin 2\theta + \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + O(\varepsilon^3), \tag{24}$$

where

$$\begin{aligned} F_1(\theta, r) &= A_1(\theta)/r^2 + A_2(\theta)/r + A_3(\theta) + A_4(\theta)r, \\ F_2(\theta, r) &= B_1(\theta)/r^5 + B_2(\theta)/r^4 + B_3(\theta)/r^3 + B_4(\theta)/r^2 \\ &\quad + B_5(\theta)/r + B_6(\theta) + B_7(\theta)r, \end{aligned}$$

where A_i and B_i are trigonometric polynomials. Now we shall apply Corollary 6 to the differential equation (24) taking $k = n = 1$ and

$$\mathbf{x} = r, \quad t = \theta, \quad F_0(\theta, \mathbf{x}) = -r\sin 2\theta, \quad \Omega = (0, \infty). \tag{25}$$

Clearly the differential equation (24) is $T = 2\pi$ periodic in the variable θ. Moreover this equation for $\varepsilon = 0$ has all its solutions 2π-periodic and given by (23). The V and α of Theorem 1 are $V = \{r : 0 < r < \rho\}$, for some real number $\rho > 0$, and $\alpha = z \in V$.

For the function F_0 given in (25) and the periodic solution $r(\theta, z)$ given in (23) the 1×1 fundamental matrix $M(\theta)$ of the differential equation (24) with $\varepsilon = 0$ such that $M(0) = (1)$ is $M(\theta) = (e^{-\sin^2\theta})$. We remark that for system (25) the fundamental matrix does not depend on the particular periodic orbit $r(\theta, z)$; i.e. it is independent of the initial condition z. Therefore $M^{-1}(\theta) = (e^{\sin^2\theta})$.

Since all the assumptions of Theorem 1 are satisfied, by Corollary 6 we must study the zeros in V of the function $g_1(z)$, where g_1 is given by

$$g_1(z) = \int_0^{2\pi} M^{-1}(t)F_1(t, x(t, z, 0))dt,$$
$$= \frac{\pi}{2z}((a_{30} + 2b_{03} + b_{21})z^2 + 2e(b_{01}(J_0(1) + 2J_1(1)) + a_{10}J_1(1))).$$

In order to obtain the bifurcation function of second order we impose that first order vanish identically. Hence we must to impose $b_{01} = a_{10} = 0$ and $b_{21} = -a_{30} - 2b_{03}$. In this case we have that $g_1(z) = 0$. To determine the bifurcation function of second order we must to compute

$$\frac{\partial x}{\partial \varepsilon} = M(t) \int_0^t M^{-1}(t)F_1(t, x(t, z, 0))dt.$$

The computation of this integral gives some terms that cannot be expressed by means of elementary functions or in function of the error function erf(z) where the error function is the integral of the Gaussian distribution given by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

We avoid this terms fixing the some of the arbitrary parameters in the following form $a_{02} = -3a_{20} + 3b_{11}$, $a_{20} = b_{11}$ and $a_{00} = 0$. Moreover the second order derivative of F_0 is identically zero. Therefore we have that

$$F_* = 2F_2 + 2D_xF_1 \cdot \frac{\partial x}{\partial \varepsilon},$$

and the bifurcation function of second order is given by

$$g_2(z) = \frac{1}{2} \int_0^T M^{-1}(t)F_*(t, x(t, z, 0))dt,$$
$$= -\frac{\pi}{256z^3}\Big\{(16a_{03}a_{12} + 32a_{03}a_{30} - 128A_{30} - 112a_{03}b_{03} - 256B_{03} - 32a_{12}b_{12} - 64a_{30}b_{12} - 32b_{03}b_{12} - 128B_{21} - 16a_{12}b_{30} - 32a_{30}b_{30} - 144b_{03}b_{30})z^4 + 128ez^2\big[-2(A_{10} + B_{01} + a_{01}(a_{12} - 4(a_{30} + b_{03})) + 2a_{12}b_{10} - 7a_{30}b_{10} - 8b_{03}b_{10} - b_{11}b_{20})J_0(1) + (4A_{10} + 5a_{01}a_{12} - 17a_{01}a_{30} - 4B_{01} - 22a_{01}b_{03} + 8a_{12}b_{10} - 33a_{30}b_{10} - 37b_{03}b_{10} - a_{11}b_{11} + b_{02}b_{11} - 3b_{11}b_{20})J_1(1)\big] + 256b_{00}b_{11}e^2J_2(2)\Big\}.$$

Consequently this bifurcating function can have two zeros and two limit cycles can bifurcate from the periodic solutions of the degenerate center and this completes the proof of the Proposition. □

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