Monotone Newton-type iterations for nonlinear equations

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1 Introduction

This paper presents in an abstract setting and with minimal assumptions a Newton-type iterative process for finding zeros of mappings. We work in the presence of lower and upper solutions. In particular, the class of mappings for which our results are applicable, include smooth and convex (or difference of two convex) operators.

The classical Newton method is very well known and widely used for finding zeros of mappings having continuous first and second order Fréchet derivatives [8]. If the starting point is sufficiently closed to the solution, this method provides a sequence which converges quadratically to the solution. The terms of the sequence are the solutions of the corresponding linear problems. Whenever the nonlinear mapping is convex, in this way one can provide lower estimates for the solution (see [3, 10]). It is worth to mention that, when apply to differential equations, this method is also known as quasilinearization method.

In the presence of lower and upper solutions, when the mapping is convex, the sequence starting from the lower solution is, in addition, monotone increasing (see [15, 11, 2]). In recent years these ideas have been extended and generalized to a variety of problems [10, 9, 12, 13, 1, 7, 6, 5]. A main new contribution is that the nonlinear part is allowed to be of DC-type (difference of two convex functions), but still it has to be twice continuously differentiable.

The problem to design a Newton algoritm for nonsmooth mappings is very important in applications. Some answers to this question contain [14, 16, 4]. In [14] is considered a class of functions admitting a certain type of approximation. Our results are for functions which can be ordered compared to some nonlinear mapping (see relation (2) below). We obtain two approximate sequences, one monotone increasing and the other one monotone decreasing, with quadratic order of convergence. In this way, we generalize similar results from [15, 11, 2], where the function is assumed to be differentiable and convex. Besides that we do not assume differentiability, our results cover more cases (e.g. convex, concave, DC-type). We consider as being very important the fact that our work pointed out that the quasilinearization method (as developed in [10]) can have a nonsmooth version.

2 Main results

We divided our main results in one lemma, two theorems and one corollary. Lemma 1 holds in ordered linear spaces (we do not need any topology) and provides two monotone sequences which are lower and, respectively, upper estimations for a possible solution. Theorem 1 is a convergence result, and Theorem 2 provides sufficient conditions that we are able to prove that the order of convergence is quadratic. In Corollary 1 the function is differentiable; however the result is an extension of some similar results from [15, 11, 2]. Also, Corollary 1 contains some ideas due to Lakshmikantham et al., so we can say that it represents an abstract version of the quasilinearization method.

Lemma 2.1 *Let X and Z be two ordered linear spaces, and $F : X \to Z$ be a mapping. We assume that there exist $\alpha_0, \beta_0 \in X$ such that*

$$\alpha_0 \le \beta_0 \text{ and } F(\beta_0) \le 0 \le F(\alpha_0). \tag{1}$$

For every $u, v \in X$ with $\alpha_0 \le u \le v \le \beta_0$ let $-A(u,v) : X \to Z$ be a linear and bijective mapping which has a positive inverse and

$$F(u) \le F(v) - A(u,v)(v-u). \tag{2}$$

We also suppose that for every $\alpha, \beta, u, v \in X$ with $\alpha_0 \le \alpha \le u \le v \le \beta \le \beta_0$,

$$-A(u,v)z \le -A(\alpha,\beta)z \text{ for all } z \in X,\ z \ge 0. \tag{3}$$

Then the sequences (α_n) and (β_n) given by the iterative schemes

$$F(\alpha_n) + A(\alpha_n,\beta_n)\,(\alpha_{n+1} - \alpha_n) = 0, \quad n \ge 0, \tag{4}$$

$$F(\beta_n) + A(\alpha_n,\beta_n)\,(\beta_{n+1} - \beta_n) = 0, \quad n \ge 0, \tag{5}$$

are well and uniquely defined in X and

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_n \le ... \le \beta_n \le ... \le \beta_2 \le \beta_1 \le \beta_0. \tag{6}$$

If in addition, there exists $w^* \in X$ such that $\alpha_0 \le w^* \le \beta_0$ and $F(w^*) = 0$ then

$$\alpha_n \le w^* \le \beta_n, \tag{7}$$

for all $n \in \mathbb{N}$.

Proof. It is sufficient if we prove that

$$\left\{ \begin{array}{c} F(\beta_n) \le 0 \le F(\alpha_n) \text{ and} \\ \alpha_{n+1} \text{ and } \beta_{n+1} \text{ are well and uniquely defined} \\ \alpha_0 \le \alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n \le \beta_0 \end{array} \right. \tag{8}$$

and (7) for all $n \in \mathbb{N}$. We proceed by induction.
We start with (8).

First we prove that (8) holds for $n = 0$.
By hypothesis, $F(\beta_0) \leq 0 \leq F(\alpha_0)$.
Bijectivity of $-A(\alpha_0, \beta_0)$ assures that α_1 and β_1 are well and uniquely defined in D.
Now we show that

$$\alpha_0 \leq \alpha_1 \leq \beta_0. \tag{9}$$

We have that

$$F(\alpha_0) + A(\alpha_0, \beta_0)\,(\alpha_1 - \alpha_0) = 0.$$

Then, using that $F(\alpha_0) \geq 0$, we obtain

$$-A(\alpha_0, \beta_0)\alpha_0 \leq -A(\alpha_0, \beta_0)\alpha_1. \tag{10}$$

Using (2) and that $F(\beta_0) \leq 0$, we also obtain

$$\begin{aligned} -A(\alpha_0, \beta_0)\alpha_1 &= F(\alpha_0) - A(\alpha_0, \beta_0)\alpha_0 \\ &\leq F(\beta_0) - A(\alpha_0, \beta_0)\beta_0 \\ &\leq -A(\alpha_0, \beta_0)\beta_0. \end{aligned} \tag{11}$$

The inverse pozitivity of $-A(\alpha_0, \beta_0)$, (10) and (11) guarantee (9) holds.

In order to prove that

$$\alpha_0 \leq \beta_1 \leq \beta_0, \tag{12}$$

we proceed in similar manner and prove the following inequalities.

$$-A(\alpha_0, \beta_0)\beta_1 = F(\beta_0) - A(\alpha_0, \beta_0)\beta_0 \leq -A(\alpha_0, \beta_0)\beta_0. \tag{13}$$

$$\begin{aligned} -A(\alpha_0, \beta_0)\beta_1 &= F(\beta_0) - A(\alpha_0, \beta_0)\beta_0 \\ &\geq F(\alpha_0) - A(\alpha_0, \beta_0)\alpha_0 \\ &\geq -A(\alpha_0, \beta_0)\alpha_0. \end{aligned}$$

The inequality

$$\alpha_1 \leq \beta_1$$

is obtained by the following relations.

$$\begin{aligned} -A(\alpha_0, \beta_0)\alpha_1 &= F(\alpha_0) - A(\alpha_0, \beta_0)\alpha_0 \\ &\leq F(\beta_0) - A(\alpha_0, \beta_0)\beta_0 \\ &= -A(\alpha_0, \beta_0)\beta_1. \end{aligned}$$

Thus (8) holds for $n = 0$.

Let us assume now that (8) holds for $n = k \geq 0$ and prove for $n = k + 1$.

We prove now that

$$F(\beta_{k+1}) \leq 0 \leq F(\alpha_{k+1}).$$

This follows by the following relations which use (3).

$$\begin{aligned} F(\alpha_{k+1}) &\geq F(\alpha_k) + A(\alpha_k, \alpha_{k+1})(\alpha_{k+1} - \alpha_k) \\ &= \left(A(\alpha_k, \alpha_{k+1}) - A(\alpha_k, \beta_k)\right)(\alpha_{k+1} - \alpha_k) \\ &\geq 0. \end{aligned}$$

$$\begin{aligned} F(\beta_{k+1}) &\leq F(\beta_k) - A(\beta_{k+1}, \beta_k)(\beta_k - \beta_{k+1}) \\ &= \left(-A(\beta_{k+1}, \beta_k) + A(\alpha_k, \beta_k)\right)(\beta_k - \beta_{k+1}) \\ &\geq 0. \end{aligned}$$

The arguments used for the proof of the remaining part are similar to those used for the proof in the case $n = 0$.

Relation (7) holds for $n = 0$ by hypothesis. Let us assume now that (7) holds for $n = k \geq 0$ and prove for $n = k + 1$. This follows by the inverse positivity of $-A(\alpha_k, \beta_k)$ and the following inequalities.

$$\begin{aligned} & -A(\alpha_k, \beta_k)\alpha_{k+1} = F(\alpha_k) - A(\alpha_k, \beta_k)\alpha_k \\ \leq\ & F(w^*) - A(\alpha_k, w^*)(w^* - \alpha_k) - A(\alpha_k, \beta_k)\alpha_k \\ =\ & -A(\alpha_k, \beta_k)w^* + \left(A(\alpha_k, \beta_k) - A(\alpha_k, w^*)\right)(w^* - \alpha_k) \\ \leq\ & -A(\alpha_k, \beta_k)w^*. \end{aligned}$$

$$\begin{aligned} & -A(\alpha_k, \beta_k)\beta_{k+1} = F(\beta_k) - A(\alpha_k, \beta_k)\beta_k \\ \geq\ & F(w^*) + A(w^*, \beta_k)(\beta_k - w^*) - A(\alpha_k, \beta_k)\beta_k \\ =\ & -A(\alpha_k, \beta_k)w^* + \left(A(w^*, \beta_k) - A(\alpha_k, \beta_k)\right)(\beta_k - w^*) \\ \geq\ & -A(\alpha_k, \beta_k)w^*. \ \square \end{aligned}$$

Theorem 2.1 *Let X be an ordered Banach space with regular positive cone, let Z be another ordered Banach space and let $F : X \to Z$ be a continuous mapping. We assume that all hypotheses of Lemma 2.1 are fulfilled and that $A(\alpha_0, \beta_0)$ is continuous.*
Then (α_n) and (β_n) given by (4) and (5) are well and uniquely defined monotone sequences such that they converge to the minimal and, respectively, the maximal solution in the order interval $[\alpha_0, \beta_0]$ of

$$F(u) = 0, \quad u \in D. \tag{14}$$

Proof. The convergence in the norm $||\cdot||_X$ of the sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ is assured by the regularity of the order cone of the Banach space X, since, by Lemma 2.1, they are monotone and ordered bounded. Let us denote by u_* and u^*, respectively, the limits of these sequences. Clearly,

$$\begin{aligned} \alpha_n \leq u_* \leq u^* \leq \beta_n, \\ \text{and } u_* \leq w^* \leq u^*, \end{aligned} \tag{15}$$

for every w^* such that $\alpha_0 \leq w^* \leq \beta_0$ and $F(w^*) = 0$.
It remains to prove that $F(u_*) = F(u^*) = 0$.
Let us denote $\Gamma_0 = -A(\alpha_0, \beta_0)$. This is a linear and bijective mapping between Banach spaces X and Z. Then Γ_0^{-1} is continuous. The following implications are assured by the continuity and positivity of Γ_0^{-1}, the regularity of the positive cone of X and the continuity of T.

$$\begin{aligned} 0 \leq F(\alpha_n) = -A(\alpha_n, \beta_n)(\alpha_{n+1} - \alpha_n) \leq \Gamma_0(\alpha_{n+1} - \alpha_n) \\ \Longrightarrow \Gamma_0^{-1} F(\alpha_n) \leq \alpha_{n+1} - \alpha_n \\ \Longrightarrow \Gamma_0^{-1} F(\alpha_n) \to 0, \quad n \to \infty \quad \Longrightarrow F(u_*) = 0. \end{aligned}$$

Similarly we can prove that $F(u^*) = 0$. □

Remark 1. The solution obtained in Theorem 2.1 is unique if in addition, there exists a mapping $B(v, u) : X \to Z$ for every $u, v \in X$ with $\alpha_0 \leq u \leq v \leq \beta_0$ such that

$$\begin{aligned} -B(v, u)(v - u) \leq 0 \quad \Longrightarrow \quad u = v \\ F(u) \geq F(v) - B(v, u)(v - u). \end{aligned}$$

Remark 2. In Theorem 2.1 we can replace the hypothesis

(H1) $-A(u, v) : X \to Z$ is a linear and bijective mapping which has a positive inverse

with
(H2) $-A(u, v) : X \to Z$ is a linear and continuous mapping such that it is also inverse positive, i.e. $0 \leq -A(u, v)z$ implies $0 \leq z$, and $A(\alpha_0, \beta_0)$ is bijective.

Proof. Let us notice that surjectivity of $A(u, v)$ was used only in order to prove that α_n and β_n are well defined. Thus we have to prove that the bijectivity of $A(\alpha_0, \beta_0)$ and the continuity of every $A(u, v)$ assure this fact.
We denote $-A(\alpha_0, \beta_0) = \Gamma_0$ and define a mapping $V : X \to X$ by $V(z) = z + \Gamma_0^{-1}(F(\alpha_n) + A(\alpha_n, \beta_n)z)$ for each fixed $n \geq 0$.
$V(0) = \Gamma_0^{-1} F(\alpha_n) \geq 0$,
$V(\beta_n - \alpha_n) = \beta_n - \alpha_n + \Gamma_0^{-1}(F(\alpha_n) + A(\alpha_n, \beta_n)(\beta_n - \alpha_n))$

$\leq \beta_n - \alpha_n + \Gamma_0^{-1} F(\beta_n) \leq \beta_n - \alpha_n$.
Let $z_1, z_2 \in X$ with $z_1 \leq z_2$

$$\begin{aligned} V(z_1) - V(z_2) &= z_1 - z_2 + \Gamma_0^{-1} A(\alpha_n, \beta_n)(z_1 - z_2) \\ &= z_1 - z_2 + \Gamma_0^{-1} \left(-A(\alpha_n, \beta_n)\right)(z_2 - z_1) \\ &\leq z_1 - z_2 + \Gamma_0^{-1} \Gamma_0 (z_2 - z_1) = 0 \end{aligned}$$

$V : [0, \beta_n - \alpha_n] \to [0, \beta_n - \alpha_n]$ is monotone increasing. By the theorem of Krasnoselskii, the regularity of the positive cone of X assure that V has a fixed point. This implies that there exists α_{n+1}.
The inverse positivity of $-A(\alpha_n, \beta_n)$ assures that α_n is uniquely defined. □

Theorem 2.2 *Let us assume that all hypotheses of Theorem (2.1) are satisfied. We also suppose that*
(i) $u_* = u^*$ *and* $A(u^*, u^*) : X \to Z$ *is continuous;*
(ii) for every $u, v \in X$ *with* $\alpha_0 \leq u \leq v \leq \beta_0$ *there exists a mapping* $B(v, u) : X \to Z$ *such that*

$$F(u) \geq F(v) - B(v, u)(v - u); \tag{16}$$

(iii) there exist $c_1, c_2 > 0$ *such that for every* $\alpha, u, \beta \in X$ *with* $\alpha_0 \leq \alpha \leq u \leq \beta \leq \beta_0$,

$$\left\| \left(B(w, \alpha) - A(\alpha, \beta)\right) z \right\| \leq \left(c_1 \left\| w - \alpha \right\| + c_2 \left\| \alpha - \beta \right\|\right) \left\| z \right\|, \tag{17}$$

for each $z \geq 0$.
Then the convergence of the sequences (α_n) *and* (β_n) *from Theorem 2.1 is quadratic.*

Proof. Let us denote

$$p_n = u^* - \alpha_n \text{ and } q_n = \beta_n - u^*.$$

Using (2), (3) and (16) we obtain the inequalities.

$$\begin{aligned} & -A(\alpha_n, \beta_n)(u^* - \alpha_{n+1}) = \\ = \ & -A(\alpha_n, \beta_n)(u^* - \alpha_n) - F(\alpha_n) + F(u^*) \\ \geq \ & \left(-A(\alpha_n, \beta_n) + A(\alpha_n, u^*)\right)(u^* - \alpha_n) \\ \geq \ & 0. \end{aligned}$$

$$\begin{aligned} & -A(\alpha_n, \beta_n) p_{n+1} = -A(\alpha_n, \beta_n) p_n - F(\alpha_n) + F(u^*) \\ \leq \ & \left(-A(\alpha_n, \beta_n) + B(u^*, \alpha_n)\right)(p_n). \end{aligned}$$

We write $\Gamma_n = -A(\alpha_n, \beta_n)$, $\Gamma_* = -A(u^*, u^*)$. Using (3) we have that $\Gamma_* z \leq \Gamma_n z$ for all $z \geq 0$, which asures that $\Gamma_n^{-1} y \leq \Gamma_*^{-1} y$ for all $y \geq 0$.

We use this fact and the positivity of Γ_n^{-1} in order to justify the following implications.

$$0 \leq \Gamma_n p_{n+1} \leq \left(-A(\alpha_n, \beta_n) + B(u^*, \alpha_n)\right)(p_n) \quad \Longrightarrow$$

$$\begin{aligned} 0 \leq p_{n+1} &\leq \Gamma_n^{-1}\left(-A(\alpha_n, \beta_n) + B(u^*, \alpha_n)\right)(p_n) \\ &\leq \Gamma_*^{-1}\left(-A(\alpha_n, \beta_n) + B(u^*, \alpha_n)\right)(p_n). \end{aligned}$$

The positive cone of X is regular, thus the norm of X is semimonotone. Hence, using (17) and the continuity of Γ_*^{-1}, we obtain.

$$\begin{aligned} ||p_{n+1}|| &\leq \delta c_\Gamma ||\left(-A(\alpha_n, \beta_n) + B(u^*, \alpha_n)\right)(p_n)|| \\ &\leq \delta c_\Gamma c_1 ||p_n||^2 + \delta c_\Gamma c_2 ||p_n|| \cdot ||\beta_n - \alpha_n|| \\ &\leq a||p_n||^2 + b||q_n||^2. \end{aligned}$$

The proof of quadratic convergence is therefore complete. $\square$

Corollary 2.1 *Let X and Z be two ordered Banach spaces such that the positive cone of X is regular and $F : X \to Z$ be a mapping. We assume that there exist $\alpha_0, \beta_0 \in X$ such that*

$$\alpha_0 \leq \beta_0 \text{ and } F(\beta_0) \leq 0 \leq F(\alpha_0).$$

Let $F_1, F_2 : X \to Z$ be G-differentiable. Suppose F_1, F_2 are continuous and and convex on $[\alpha_0, \beta_0]$, $F = F_1 - F_2$ and $F_1'(\alpha_0) - F_2'(\beta_0)$ is bijective. In addition, suppose that for every $u, v \in X$ with $\alpha_0 \leq u \leq v \leq \beta_0$, the map $-\left(F_1'(u) - F_2'(v)\right) : X \to Z$ is inverse positive.
Then the sequences (α_n) and (β_n) given by the iterative schemes

$$\begin{aligned} F(\alpha_n) + \left(F_1'(\alpha_n) - F_2'(\beta_n)\right)(\alpha_{n+1} - \alpha_n) = 0, \quad n \geq 0, \\ F(\beta_n) + \left(F_1'(\alpha_n) - F_2'(\beta_n)\right)(\beta_{n+1} - \beta_n) = 0, \quad n \geq 0, \end{aligned}$$

are well and uniquely defined and

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0.$$

Moreover, they converge superlinearly to the extremal solutions of

$$F(u) = 0, \quad \alpha_0 \leq u \leq \beta_0.$$

If in addition F_1' and F_2' are Lipschitz in $[\alpha_0, \beta_0]$ then the order of convergence is quadratic.

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